# Shift Invariant Data Decomposition

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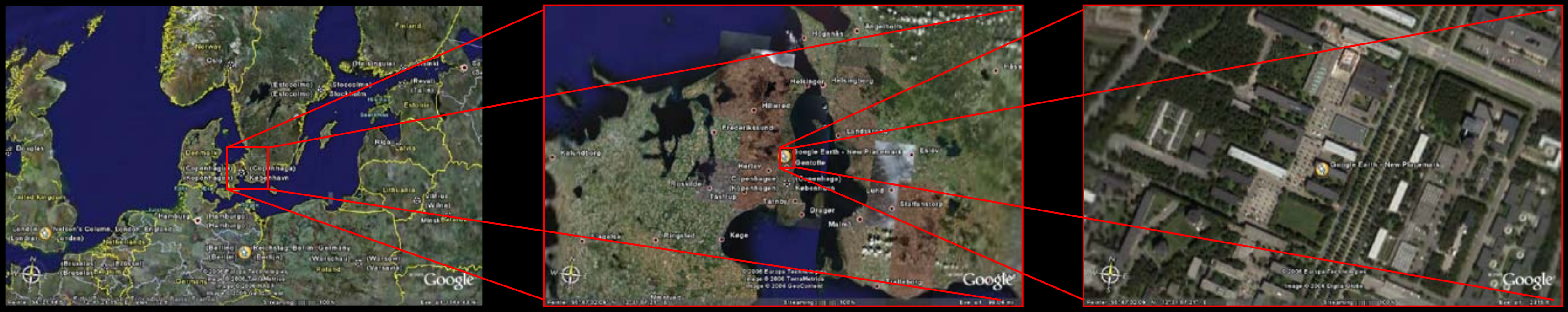

# Collaborators


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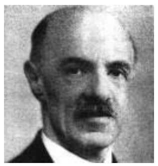
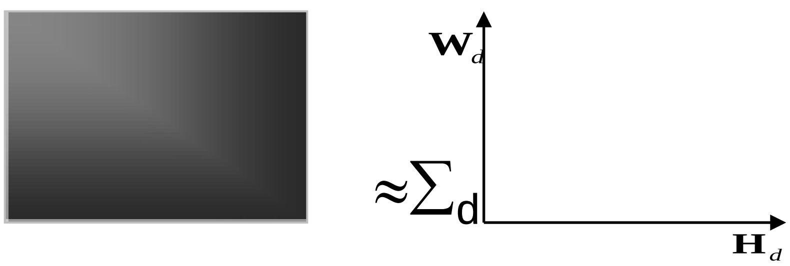
# Outline

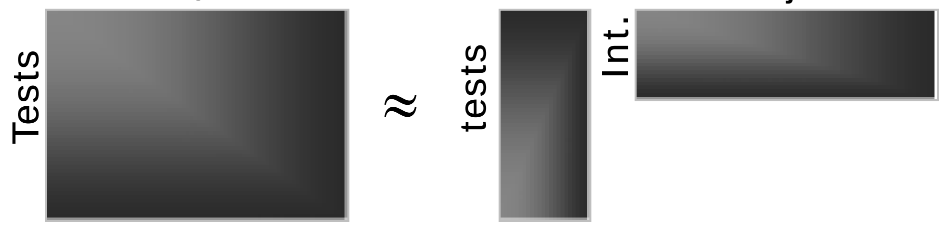
- Shifted Non-negative Matrix Factorization
- Shifted Independent Component analysis
- Generalization to tensors (i.e., the PARAFAC model)
- Shift Invariant Sparse Coding

# Factor Analysis



Spearman ~1900




$$\mathbf{V} \approx \mathbf{W}\mathbf{H}$$

$$\mathbf{V}^{\text{tests x subjects}} \approx \mathbf{W}^{\text{tests x intelligences}}\mathbf{H}^{\text{intelligencesxsubject}}$$

Independent Component Analysis (ICA)
rows of **H** statistically independent

(P. Common, Bell & Sejnowski ~1995)

Non-negative Matrix Factorization (NMF):

$$\mathbf{V}_{n,m}, \mathbf{W}_{n,d}, \mathbf{H}_{d,m} \geq 0$$

(~1970 Lawson, ~1995 Paatero, ~2000 Lee & Seung)

# Multiplicative updates

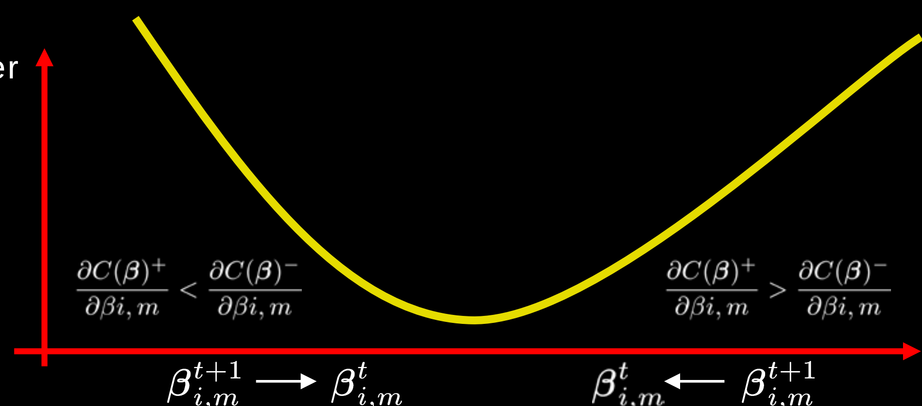
$$\frac{\partial C(\boldsymbol{\beta})}{\partial \beta i,m} = \frac{\partial C(\boldsymbol{\beta})^{+}}{\partial \beta i,m} - \frac{\partial C(\boldsymbol{\beta})^{-}}{\partial \beta i,m}$$

$$\boldsymbol{\beta}_{i,m} \quad = \quad \boldsymbol{\beta}_{i,m} - \mu_{i,m}\frac{\partial C(\boldsymbol{\beta})}{\partial \beta i,m}, \mu_{i,m} = \frac{\boldsymbol{\beta}_{i,m}}{\frac{\partial C(\boldsymbol{\beta})^{+}}{\partial \beta i,m}}$$

$$\boldsymbol{\beta}_{i,m} \quad = \quad \boldsymbol{\beta}_{i,m} - \frac{\boldsymbol{\beta}_{i,m}}{\frac{\partial C(\boldsymbol{\beta})^{+}}{\partial \beta i,m}}\left(\frac{\partial C(\boldsymbol{\beta})^{+}}{\partial \beta i,m} - \frac{\partial C(\boldsymbol{\beta})^{-}}{\partial \beta i,m}\right) = \boldsymbol{\beta}_{i,m}\frac{\frac{\partial C(\boldsymbol{\beta})^{-}}{\partial \beta i,m}}{\frac{\partial C(\boldsymbol{\beta})^{+}}{\partial \beta i,m}}$$

Step size parameter

$$\boldsymbol{\beta}_{i,m}^{t+1} \leftarrow \boldsymbol{\beta}_{i,m}^{t}\left(\frac{\frac{\partial C(\boldsymbol{\beta})^{-}}{\partial \boldsymbol{\beta}_{i,m}^{t}}}{\frac{\partial C(\boldsymbol{\beta})^{+}}{\partial \boldsymbol{\beta}_{i,m}^{t}}}\right)^{\alpha}$$



# Non-negative matrix factorization

$$\mathbf{V}_{i,j} \geq 0 \quad , \quad \mathbf{W}_{i,d} \geq 0 \quad \textit{and} \quad \mathbf{H}_{d,j} \geq 0$$

$$C_{LS} = \frac{1}{2}\|\mathbf{V} - \mathbf{W}\mathbf{H}\|_F^2 = \frac{1}{2}\sum_{i,j}(\mathbf{V}_{i,j} - (\mathbf{W}\mathbf{H})_{i,j})^2$$

$$\mathbf{W}_{i,d} \leftarrow \mathbf{W}_{i,d}\frac{(\mathbf{V}\mathbf{H}^{\mathrm{T}})_{i,d}}{(\mathbf{W}\mathbf{H}\mathbf{H}^{\mathrm{T}})_{i,d}}$$

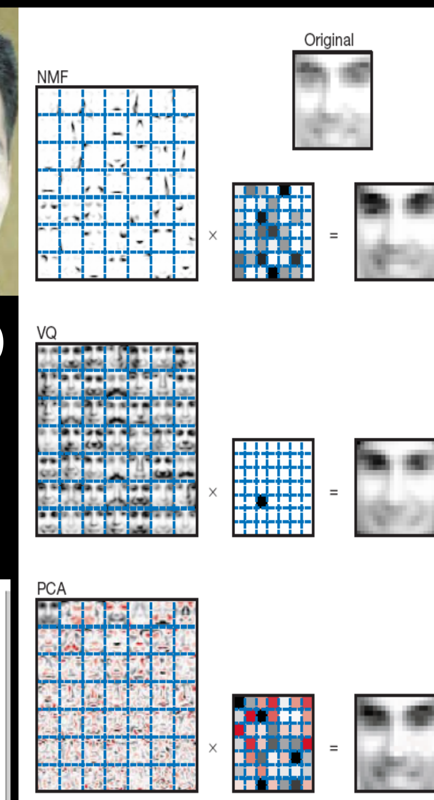
$$\mathbf{H}_{d,j} \leftarrow \mathbf{H}_{d,j}\frac{(\mathbf{W}^{\mathrm{T}}\mathbf{V})_{d,j}}{(\mathbf{W}^{\mathrm{T}}\mathbf{W}\mathbf{H})_{d,j}}$$

$$C_{KL} = \sum_{i,j}\mathbf{V}_{i,j}\log\frac{\mathbf{V}_{i,j}}{(\mathbf{W}\mathbf{H})_{i,j}} - \mathbf{V}_{i,j} + (\mathbf{W}\mathbf{H})_{i,j}$$

$$\mathbf{W}_{i,d} \leftarrow \mathbf{W}_{i,d}\frac{\sum_j \frac{\mathbf{V}_{i,j}}{(\mathbf{W}\mathbf{H})_{i,j}}\mathbf{H}_{d,j}}{\sum_j \mathbf{H}_{d,j}}$$

$$\mathbf{H}_{d,j} \leftarrow \mathbf{H}_{d,j}\frac{\sum_i \mathbf{W}_{i,d}\frac{\mathbf{V}_{i,j}}{(\mathbf{W}\mathbf{H})_{i,j}}}{\sum_i \mathbf{W}_{i,d}}$$

(Lee & Seung - 2001)




**Learning the parts of objects by non-negative matrix factorization**

Daniel D. Lee* & H. Sebastian Seung*†

* Bell Laboratories, Lucent Technologies, Murray Hill, New Jersey 07974, USA
† Department of Brain and Cognitive Sciences, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

Is perception of the whole based on perception of its parts? There

(Some other approaches:
Active Set, projected gradient,
barrier functions, exponentiation)

NMF gives Part based representation
(Lee & Seung – Nature 1999)

# Maximum likelihood (ML) ICA approach

Equivalently we derive two step procedure

**[A',S',H'] = SVD(X)** → Shift Invariant Subspace Analysis (SISA)

**W' = A'S'**

Notice decomposition then ambiguous since

$$\mathbf{WH} = (\mathbf{W'Q}^{-1})(\mathbf{QH'}) = \mathbf{W'H'}$$

Thus ICA forms objective for ambiguity **Q** that minimizes: → Shifted Independent Component Analysis (SICA)

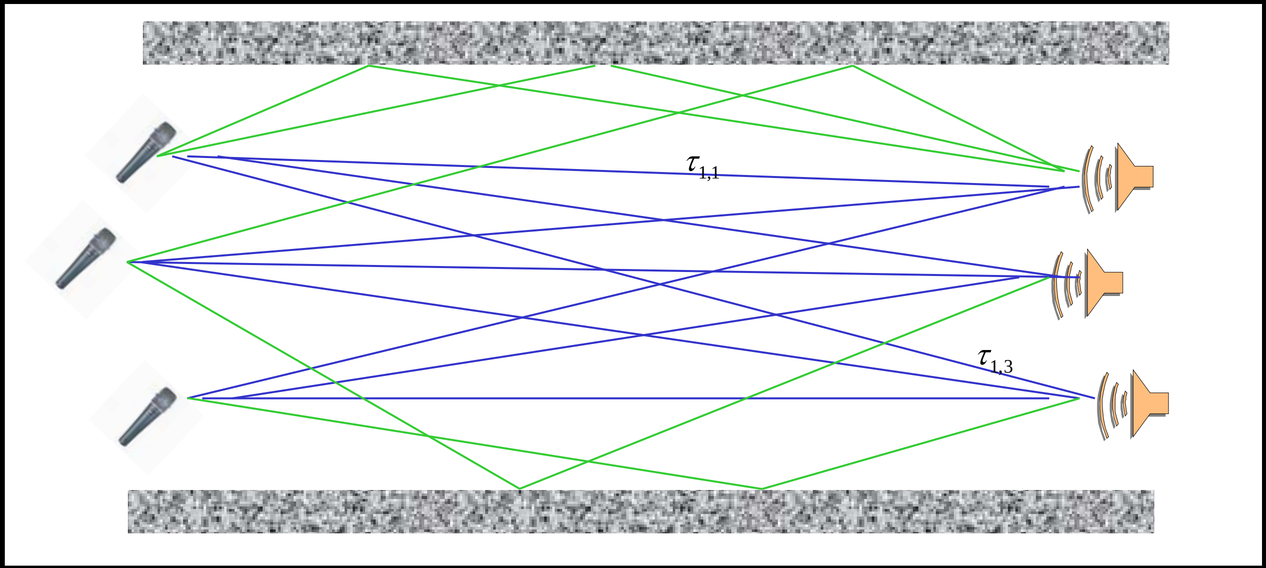
ICA-ML

$$p(\mathbf{H}|\mathbf{Q}) = \prod_m p(\mathbf{H}_m|\mathbf{Q}) = \prod_m |det(\mathbf{Q})| p(\mathbf{Q}\mathbf{H}_m)$$

Assume Independence

Change of variable principle

# The shift problem




Convolutive ICA/NMF
(echo effects, Smaragdis 2003)

Shifted ICA/NMF
(One specific delay between each sensor and source)

$$\mathbf{V}_{n,m} = \sum_{d,\tau} \mathbf{W}^{\tau}_{n,d} \mathbf{H}_{d,m-\tau} + \mathbf{E}_{n,m}$$

$$\mathbf{V}_{n,m} = \sum_{d} \mathbf{W}_{n,d} \mathbf{H}_{d,m-\boldsymbol{\tau}_{n,d}} + \mathbf{E}_{n,m}$$

# History of shift


- Bell & Sejnowski 1995
  (Sketched how to handle time delays in networks)
- Torkkola 1996
  (Further developed Bell and Sejnowski's work)

- Emile & Comon 1998
  (delay in model based on equally mixed sources formed by moving averages)

- Hong and Harshman 2003 – shifted factor analysis
  (a procedure based on exhaustive search over integer shifts – model conjectured unique)

- Yeredor 2003 (Solved the ICA model with shifts by joint diagonalization (sources=sensors) of the source cross spectra based on the AC-DC algorithm with non-integer shifts for the 2x2 system)

- Yeredor 2005 (extension to complex signals)

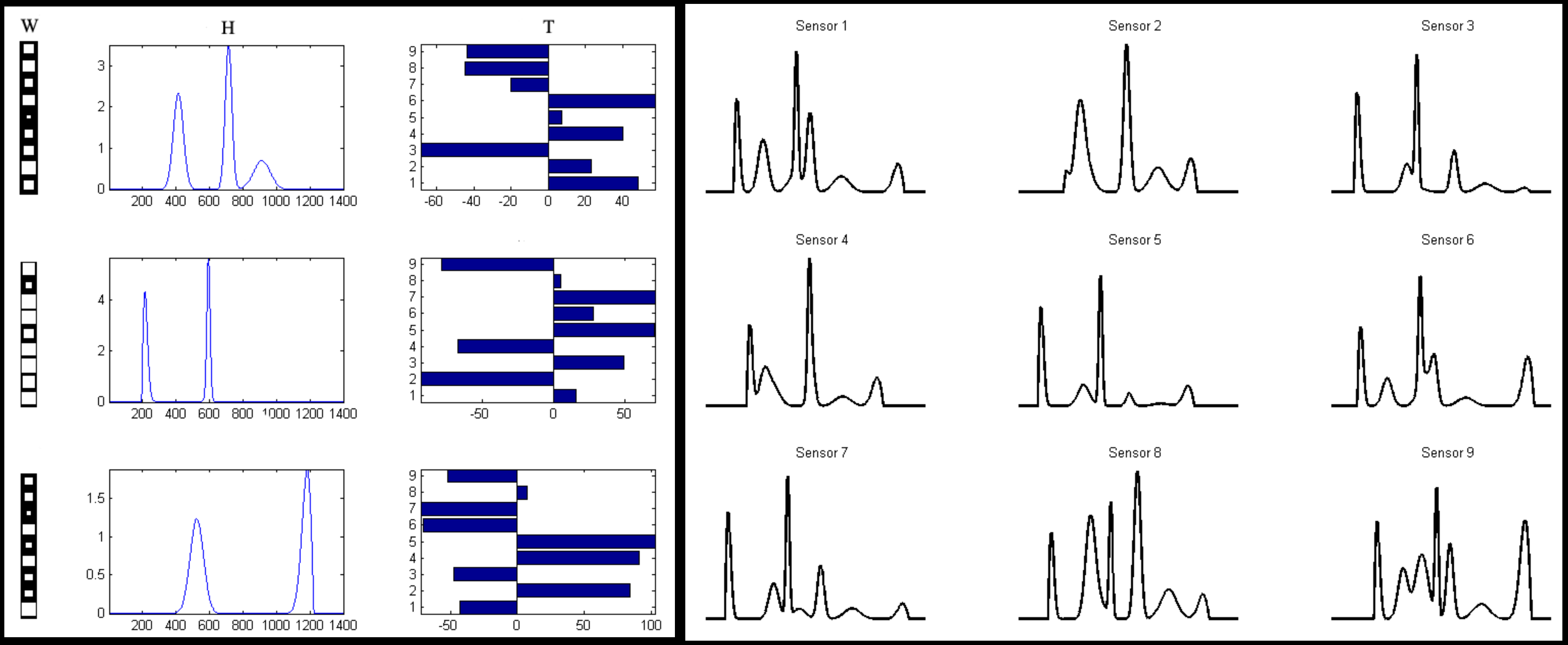
# Why shifted ICA/NMF

Causes of shifts for instance

Doppler effect, Time of arrival differences

- Magnetic resonance spectra
  (Du et al, 2005)
- Astronomical spectrometers (red shift)
  (Pauca et al. 2006)
- Fluorescence spectra
  (Gobinet et al. 2004)
- PET imaging
  (Kim et al. 2001, Lee et al., 2001, Bödvarsson et al. 2007)
- Sound recording (delays between source and sensor due to propagation delay)

# Generative model



# Notation and LS-objective

- $\mathbf{U}$ and $\widetilde{\mathbf{U}}$ denotes same matrix in time and frequency domain respectively.
- $\widetilde{\mathbf{U}}^H$ denotes the conjugate transpose of $\widetilde{\mathbf{U}}$
- $\mathbf{U}_{d,m-\tau} \sim \widetilde{\mathbf{U}}_{d,f} e^{-i2\pi\frac{f-1}{M}\tau}$
- $\mathbf{U} \bullet \mathbf{V}$ denotes the direct product, i.e. element-wise multiplication.
- $(e^{-i2\pi\frac{f-1}{M}\boldsymbol{\tau}})_{n,d} = e^{-i2\pi\frac{f-1}{M}\boldsymbol{\tau}_{n,d}}$.
- $\widetilde{\mathbf{U}}^{(f)} = \mathbf{U} \bullet e^{-i2\pi\frac{f-1}{M}\boldsymbol{\tau}}$
- $\mathbf{U}_d$ $d^{th}$ column, $\mathbf{U}_{n,:}$ $n^{\text{th}}$ row and $\mathbf{U}_{n,d}$ a given element of $\mathbf{U}$.

$$\boxed{\begin{aligned} C_{LS}(W,H) &= \frac{1}{2}\sum_{n,m}(\mathbf{V}_{n,m} - \sum_d \mathbf{W}_{n,d}\mathbf{H}_{d,m-\boldsymbol{\tau}_{n,d}})^2 \\ &= \frac{1}{2M}\|\widetilde{\mathbf{V}}_f - (\mathbf{W} \bullet e^{-i2\pi\frac{f-1}{M}\boldsymbol{\tau}})\widetilde{\mathbf{H}}_f\|_F^2 = \frac{1}{2M}\|\widetilde{\mathbf{V}}_f - \mathbf{W}^{(f)}\widetilde{\mathbf{H}}_f\|_F^2 \end{aligned}}$$

Follows from Parsevals identity with the above notation

# **W** update

Let $\widetilde{\mathbf{H}}^{(n)}_{d,f} = \widetilde{\mathbf{H}}_{d,f} e^{-i2\pi \frac{f-1}{M} \boldsymbol{\tau}_{n,d}}$ denote the delayed version of the source signal $\widetilde{\mathbf{H}}_{d,f}$ to the $n^{\text{th}}$ channel. The shift ICA/NMF model can then be stated as

$$\mathbf{V}_{n,:} = \mathbf{W}_{n,:}\mathbf{H}^{(n)} + \mathbf{E}_{n,:}.$$

This is the regular ICA/NMF problem which can be solved by the least squares ICA/NMF-update

$$\begin{aligned} SISA : \mathbf{W}_{n,:} &= \mathbf{V}_{n,:}/(\mathbf{H}^{(n)}\mathbf{H}^{(n)^T}), \\ NMF : \mathbf{W}_{n,d} &= \mathbf{W}_{n,d}\frac{\mathbf{V}_{n,:}\mathbf{H}^{(n)^T}_{d,:}}{\mathbf{W}_{n,:}\mathbf{H}^{(n)}\mathbf{H}^{(n)^T}_{d,:}}. \end{aligned}$$

# H update

$$C_{LS} = \frac{1}{2M}\|\widetilde{\mathbf{V}}_f - \widetilde{\mathbf{W}}^{(f)}\widetilde{\mathbf{H}}_f\|_F^2$$

$$\mathbf{G}_f = \frac{\partial C_{LS}}{\partial \widetilde{\mathbf{H}}_f} = -\frac{1}{M}\widetilde{\mathbf{W}}^{(f)^H}(\widetilde{\mathbf{X}}_f - \widetilde{\mathbf{W}}^{(f)}\widetilde{\mathbf{H}}_f)$$

## ShiftNMF

$$\begin{aligned}\widetilde{\mathbf{G}}_f^+ &= \frac{1}{M}\widetilde{\mathbf{W}}^{(f)^H}\widetilde{\mathbf{W}}^{(f)}\widetilde{\mathbf{H}}_f \\ \widetilde{\mathbf{G}}_f^- &= \frac{1}{M}\widetilde{\mathbf{W}}^{(f)^H}\widetilde{\mathbf{X}}_f\end{aligned}$$

$$\mathbf{H}_{d,n} = \mathbf{H}_{d,n}\left(\frac{\mathbf{G}_{d,n}^-}{\mathbf{G}_{d,n}^+}\right)^\alpha$$

## SISA

$$\mathbf{G}_f = (\widetilde{\mathbf{W}}^{(f)^\top}\widetilde{\mathbf{W}}^{(f)})\backslash\widetilde{\mathbf{X}}_f$$

# Update of the shifts (τ)

$$C_{LS} = \frac{1}{2M}\sum_f (\widetilde{\mathbf{V}}_f - \ldots (\mathbf{W} \bullet e^{-i2\pi\frac{f-1}{M}\boldsymbol{\tau}})\widetilde{\mathbf{H}}_f)$$

**Warning! Prone to local minima**

wher… …be estimated
by the N…

# Update of shifts (τ) based on Cross-correlation

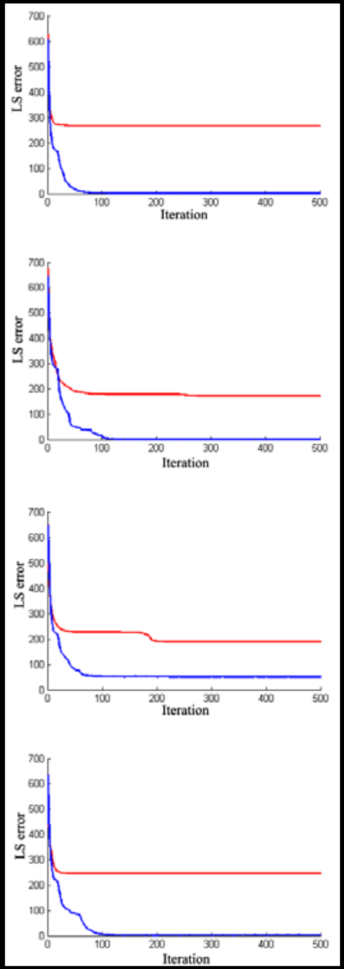
$$
\begin{aligned}
\mathbf{R}_{n,m} &= \mathbf{V}_{n,m} - \sum_{d\neq d'} \mathbf{W}_{n,d}\mathbf{H}_{d,m-\boldsymbol{\tau}_{n,d}} \\
C_{LS} &= \frac{1}{2}\sum_{n,m}(\mathbf{V}_{n,m} - \sum_{d} \mathbf{W}_{n,d}\mathbf{H}_{d,m-\boldsymbol{\tau}_{n,d}})^2 \\
&= \frac{1}{2}\sum_{n,m}(\mathbf{R}_{n.m} - \mathbf{W}_{n,d'}\mathbf{H}_{d',m-\boldsymbol{\tau}_{n,d'}})^2 \\
&= \frac{1}{2}\|\mathbf{R}\|^2 - \sum_{n}\mathbf{W}_{n,d'}\sum_{m}\mathbf{R}_{n,m}\mathbf{H}_{d',m-\boldsymbol{\tau}_{n,d'}} + \frac{1}{2}\|\mathbf{W}\mathbf{H}\|^2
\end{aligned}
$$

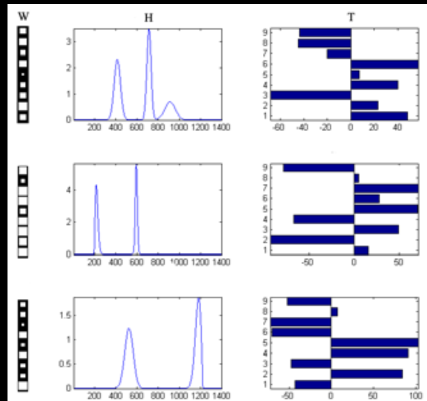
Independent of τ   Cross correlation R and H   Independent of τ

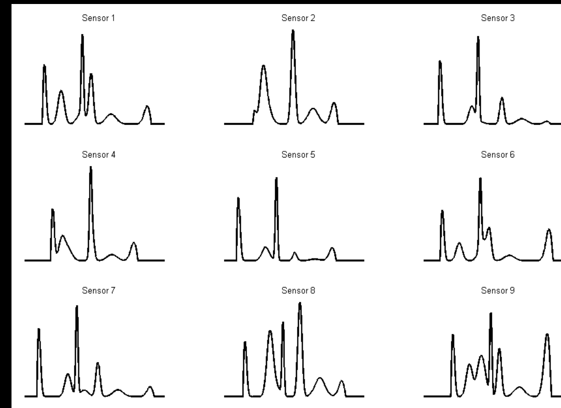
$$
\widetilde{\mathbf{c}}_f = \widetilde{\mathbf{R}}^*_{n,f}\widetilde{\mathbf{H}}_{d',f} \quad t = \arg\max_{m} \mathbf{c}_m, \quad \boldsymbol{\tau}_{n,d'} = t-(M+1).
$$

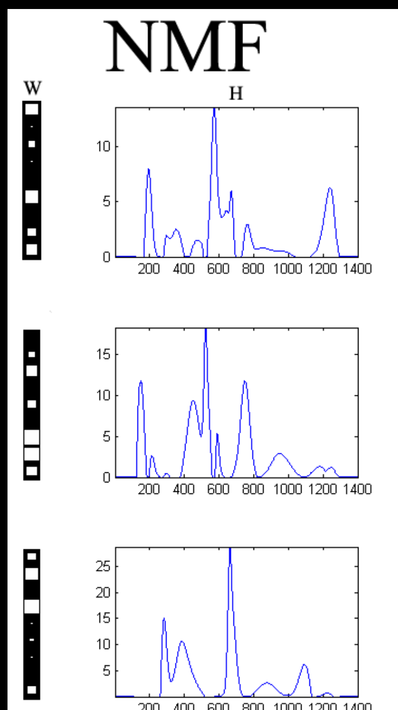
The value of $\mathbf{W}_{n,d'}$ corresponding to this delay is given by

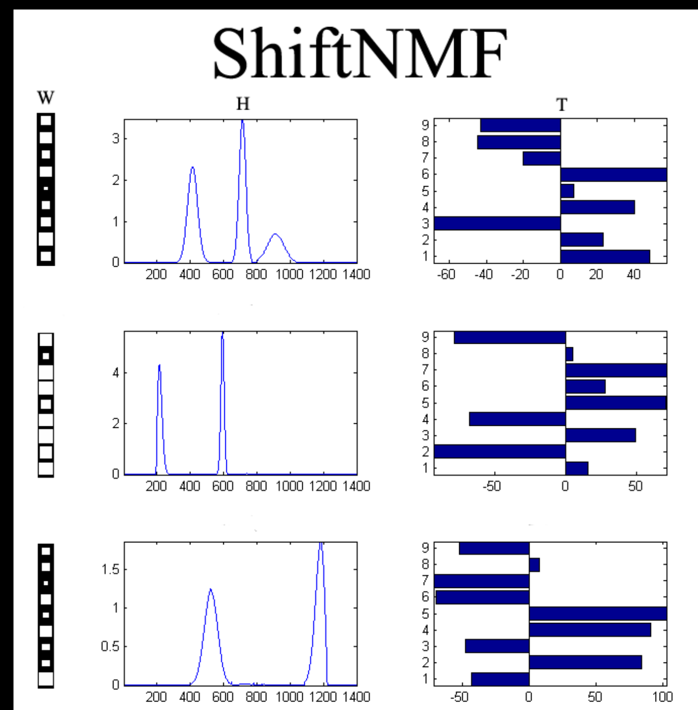
$$
\mathbf{W}_{n,d'} = \frac{\mathbf{c}_t}{\mathbf{H}_{d',:}\mathbf{H}^T_{d',:}}.
$$



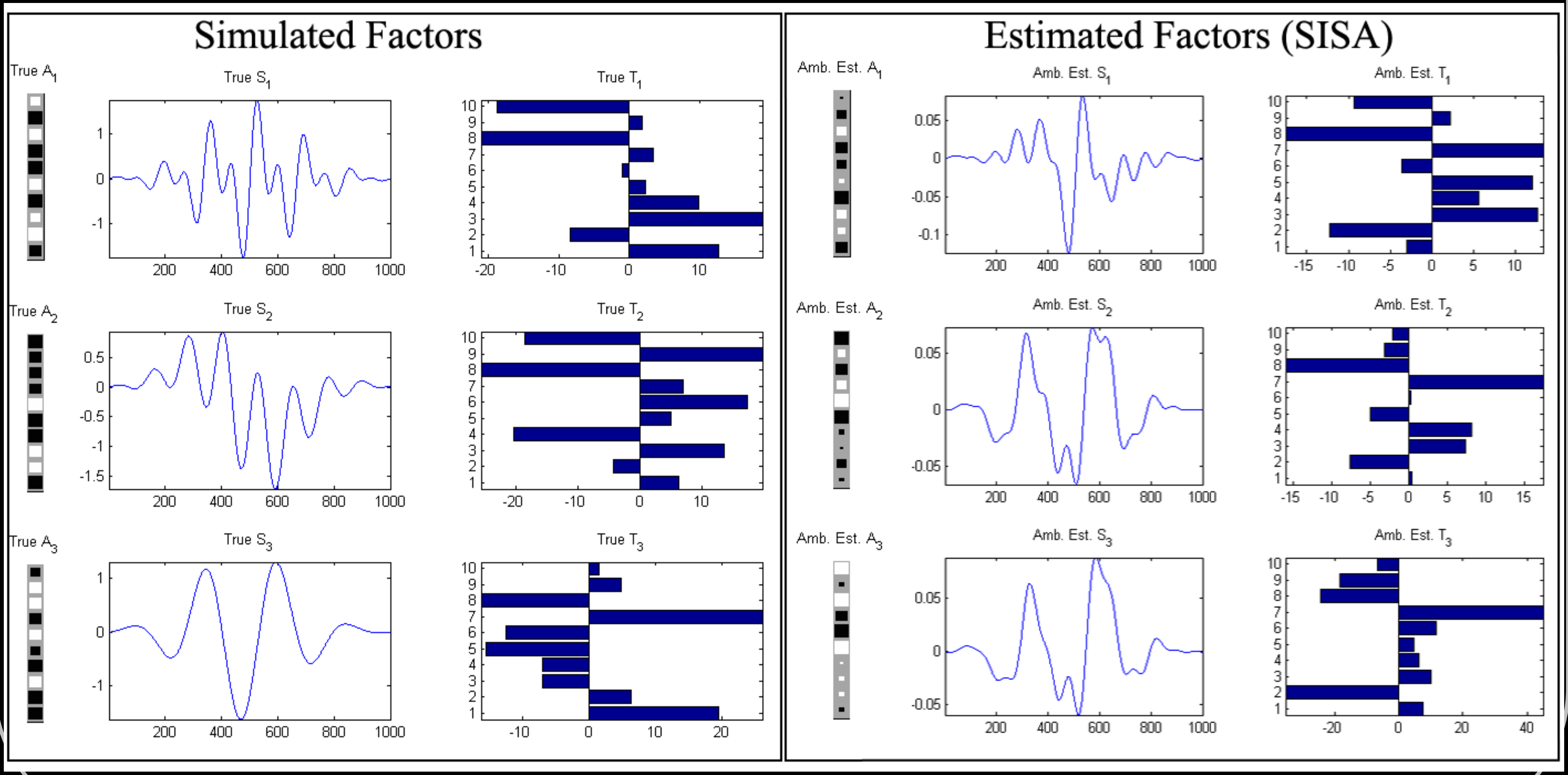

W
H
T


Sensor 1
Sensor 2
Sensor 3
Sensor 4
Sensor 5
Sensor 6
Sensor 7
Sensor 8
Sensor 9


NMF
W
H


ShiftNMF
W
H
T

# Shift Invariant Subspace Analysis



# Shifted Independent Component Analysis

Define, $\widetilde{\mathbf{U}}_f = \widetilde{\mathbf{Q}}^{(f)}\widetilde{\mathbf{H}}_f$, i.e. the sources at frequency $f$ when transformed according to the rotation and shift ambiguity described in the previous section. The ambiguity can be resolved by maximizing the log-likelihood assuming the (non-gaussian) Laplace distribution $p(\widetilde{\mathbf{U}}_f) \propto e^{-|\widetilde{\mathbf{U}}_{d,f}|}$, i.e.

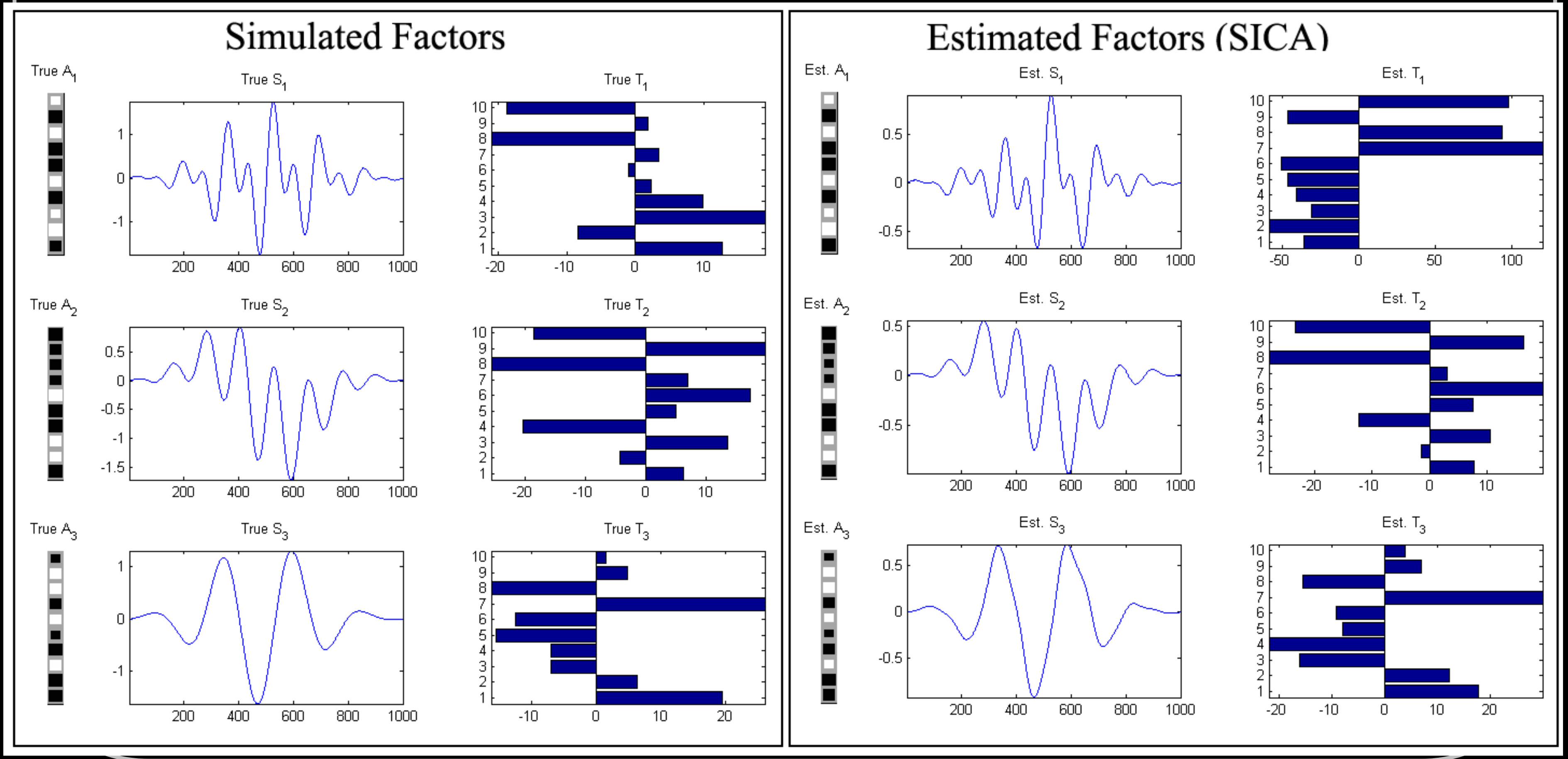
$$p(\widetilde{\mathbf{H}}_f|\mathbf{Q},\widehat{\boldsymbol{\tau}}) = \prod_f p(\widetilde{\mathbf{H}}_f|\mathbf{Q},\widehat{\boldsymbol{\tau}}) = \prod_f |det(\widetilde{\mathbf{Q}}^{(f)})| p(\widetilde{\mathbf{Q}}^{(f)}\widetilde{\mathbf{H}}_f) \tag{1}$$

Such that the log-likelihood as a function of $\mathbf{Q}$ and $\widehat{\boldsymbol{\tau}}$ becomes

$$\mathcal{L}(\mathbf{Q},\widehat{\boldsymbol{\tau}}) = \sum_f \ln|\det(\widetilde{\mathbf{Q}}^{(f)})| - \sum_d |\widetilde{\mathbf{Q}}^{(f)}\widetilde{\mathbf{H}}_f|_d \tag{2}$$

By maximizing $\mathcal{L}(\mathbf{W},\widehat{\boldsymbol{\tau}})$ $\mathbf{W}$ and $\widehat{\boldsymbol{\tau}}$ is estimated and a new unambiguous $\mathbf{H}$ solution found by $\widetilde{\mathbf{H}}_f = \widetilde{\mathbf{Q}}^{(f)}\widetilde{\mathbf{H}}_f$. The corresponding mixing and delays can be estimated alternating between the $\mathbf{W}$ and $\boldsymbol{\tau}$ update. We initialized $\mathbf{W}$ as $\mathbf{W} = \mathbf{W}\mathbf{Q}^{-1}$ and $\boldsymbol{\tau}_{i,d}$ by the cross-correlation procedure.

# Shifted Independent Component Analysis



# Algorithm assumptions

- Sources **H** and measured signal **V** have to be periodic

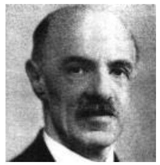
  If signals are not periodic a window function can be employed. However this is not trivial to implement in the $\tau$-update and slows down the algorithm significantly. Zero padding is simple and fast but introduces a bias towards small delays.

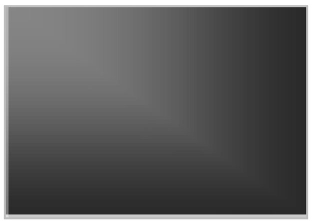
- Noise **E** assumed homoscedatic (normal) iid.

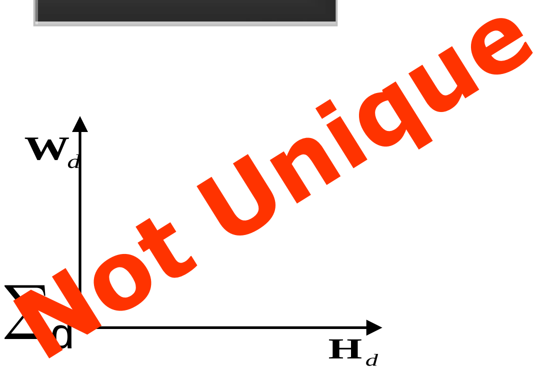
  If non-homoscedatic use weighted least squares. Algorithm works for Least squares due to Parseval's identity. No such identity exists for other types than the least squares objective.

# Extensions to tensors

Factor Analysis

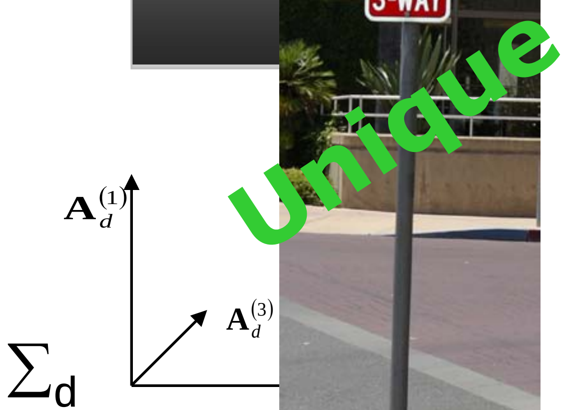







$$\mathbf{V}_{i_1 i_2} \approx \sum_{d=1}^{D} \mathbf{W}_{i_1 d} \mathbf{H}_{i_2 d}$$

PARAFA … mp


$$\mathcal{V}_{i_1 i_2 i_3} \approx \sum_{d=1}^{D} \mathbf{A}_{i_1 d}^{(1)} \ldots$$

# The Candecomp/PARAFAC (CP) model

$$\mathcal{X}_{i,j,k} = \sum_{d}^{D} \mathbf{A}_{i,d}\mathbf{B}_{j,d}\mathbf{C}_{k,d} + \mathcal{E}_{i,j,k}$$

$$\begin{aligned}
\mathbf{X}_{(1)} &= \mathbf{A}(\mathbf{C} \odot \mathbf{B})^T + \mathbf{E}_{(1)} \quad &\Rightarrow \quad \mathbf{A} \leftarrow \mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})^{T^\dagger} \\
\mathbf{X}_{(2)} &= \mathbf{B}(\mathbf{C} \odot \mathbf{A})^T + \mathbf{E}_{(2)} \quad &\Rightarrow \quad \mathbf{B} \leftarrow \mathbf{X}_{(2)}(\mathbf{C} \odot \mathbf{A})^{T^\dagger} \\
\mathbf{X}_{(3)} &= \mathbf{C}(\mathbf{B} \odot \mathbf{A})^T + \mathbf{E}_{(3)} \quad &\Rightarrow \quad \mathbf{C} \leftarrow \mathbf{X}_{(3)}(\mathbf{B} \odot \mathbf{A})^{T^\dagger}
\end{aligned}$$

$$(\mathbf{C} \odot \mathbf{B}) = (\mathbf{C}_1 \otimes \mathbf{B}_1 \ \mathbf{C}_2 \otimes \mathbf{B}_2 \ \ldots \ \mathbf{C}_D \otimes \mathbf{B}_D) \quad X_{(1)} = X^{I \times JK}, \quad X_{(2)} = X^{J \times IK}, \quad X_{(3)} = X^{K \times IJ}$$

The CP model is unique if

$$k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} \geq 2D + 2$$

where $k_{\mathbf{A}}$ is the k-rank denoting the smallest subset of columns of $\mathbf{A}$ that is guaranteed to be linearly independent. Thus, $k_{\mathbf{A}} \leq rank(\mathbf{A})$.

# Shifted CP model

$$\mathcal{X}_{i,j,k} = \sum_{d}^{D} \mathbf{A}_{i,d} \mathbf{B}_{j-\boldsymbol{\tau}_{k,d},d} \mathbf{C}_{k,d} + \mathcal{E}_{i,j,k}$$

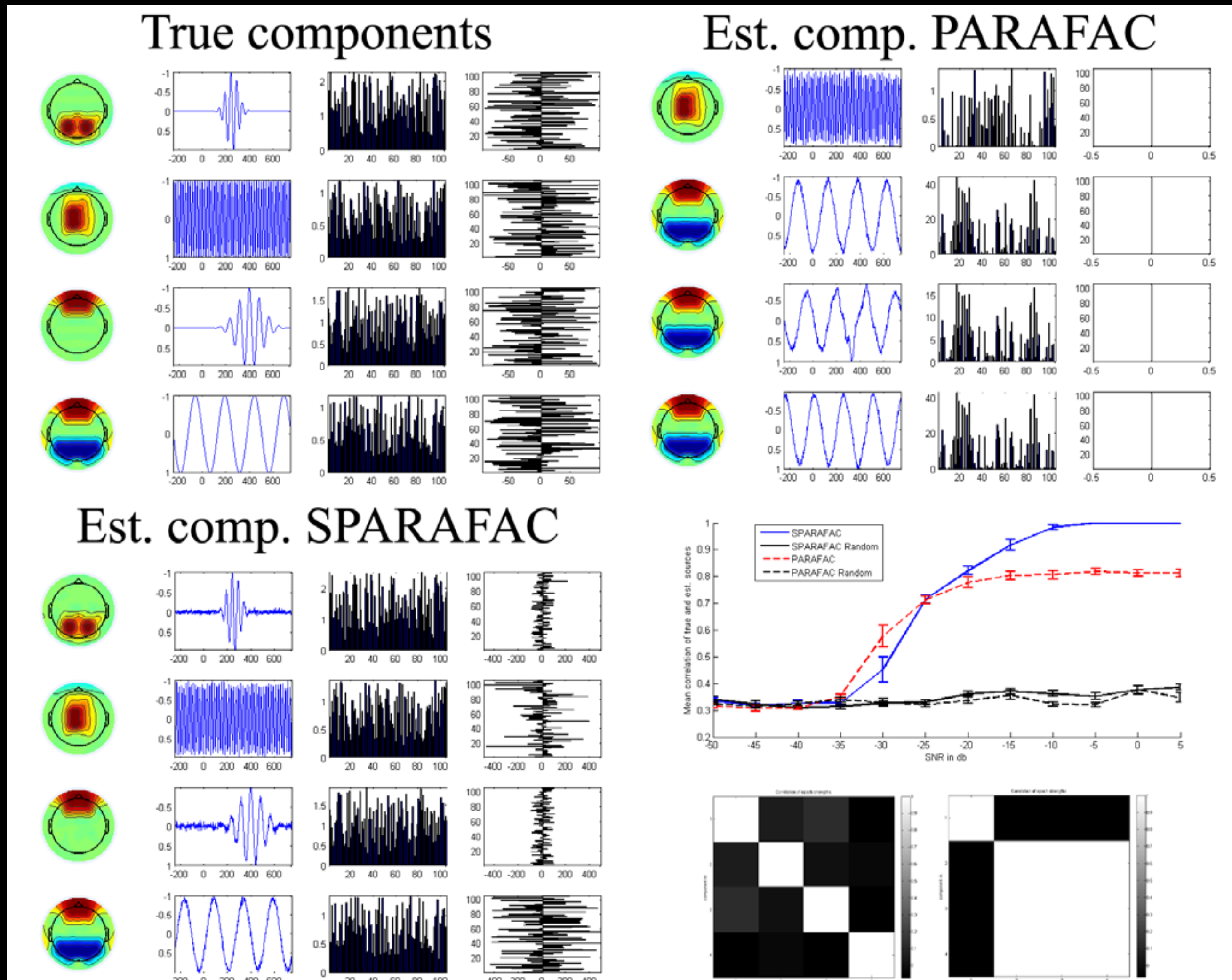
$$\begin{aligned}
\mathbf{X}_{(1)} &= \mathbf{A}\mathbf{Z}^T + \mathbf{E}_{(1)} & \Rightarrow \quad & \mathbf{A} \leftarrow \mathbf{X}_{(1)} \mathbf{Z}^{T^\dagger} \\
\widetilde{\mathbf{X}}_{(2)_{f,:}} &= \widetilde{\mathbf{B}}_{f,:}(\widetilde{\mathbf{C}}^{(f)} \odot \mathbf{A})^T + \widetilde{\mathbf{E}}_{(2)_{f,:}} & \Rightarrow \quad & \widetilde{\mathbf{B}}_{f,:} \leftarrow \widetilde{\mathbf{X}}_{(2)_{f,:}} (\widetilde{\mathbf{C}}^{(f)} \odot \mathbf{A})^{T^\dagger} \\
\mathbf{X}_{(3)_{k,:}} &= \mathbf{C}_{k,:}(\mathbf{B}^{(k)} \odot \mathbf{A})^T + \mathbf{E}_{(3)_{k,:}} & \Rightarrow \quad & \mathbf{C}_{k,:} \leftarrow \mathbf{X}_{(3)_{k,:}} (\mathbf{B}^{(k)} \odot \mathbf{A})^{T^\dagger}
\end{aligned}$$

$$\mathbf{r}_j^{(k,d')} = \sum_i \mathcal{R}_{i,j,k}^{d'} \mathbf{A}_{i,d'} \quad \widetilde{\mathbf{c}}_{k,d'}(f) = \widetilde{\mathbf{r}}_f^{(k,d')^*} \widetilde{\mathbf{B}}_{f,d'}.$$

$$t_{k,d'} = \arg\max_t |\mathbf{c}_{k,d'}(t)| \quad \boldsymbol{\tau}_{k,d'} = t_{k,d'} - (J+1).$$

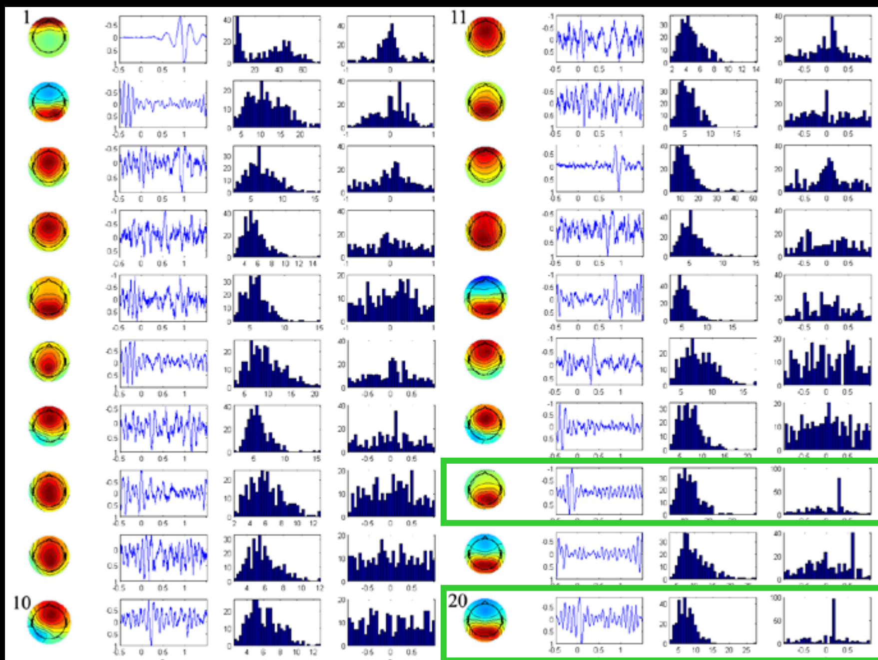
$$\mathbf{C}_{k,d'} = \frac{\mathbf{c}_{k,d'}(t_{k,d'})}{\mathbf{B}_{d'}^T \mathbf{B}_{d'}}.$$

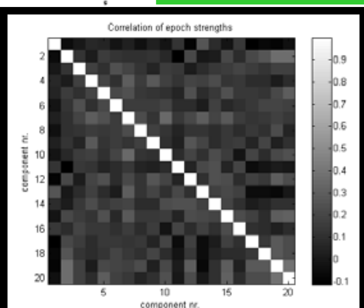
# Synthetic EEG data



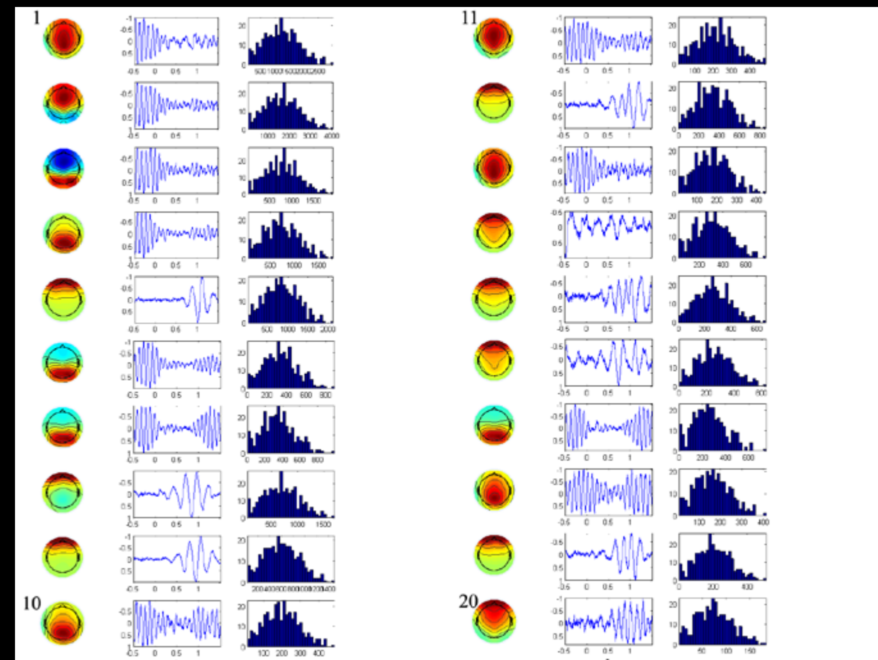
# EEG data from Visual Paradigm

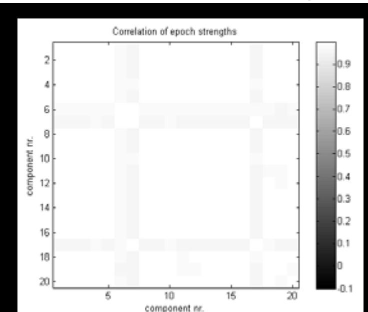
## Shifted CP





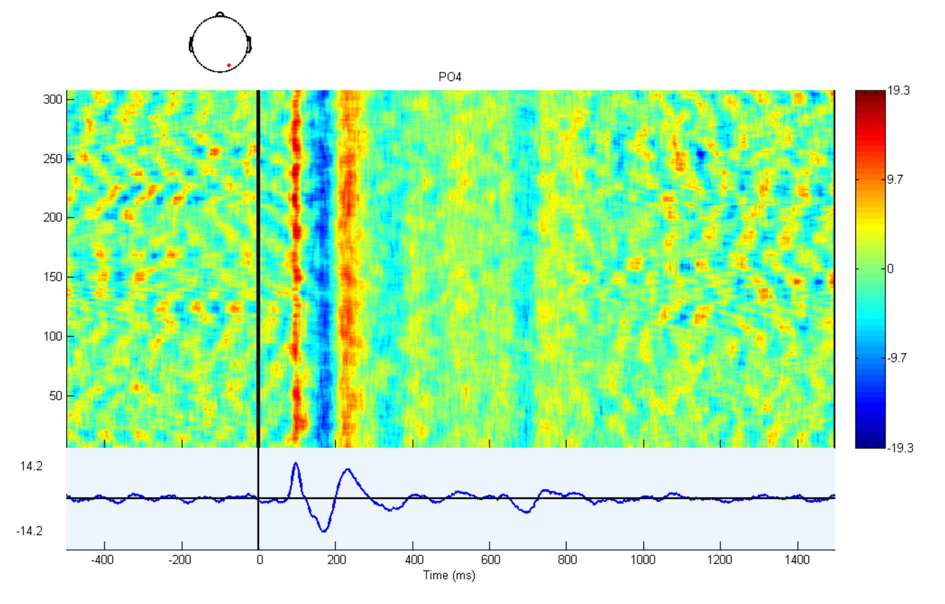

## CP

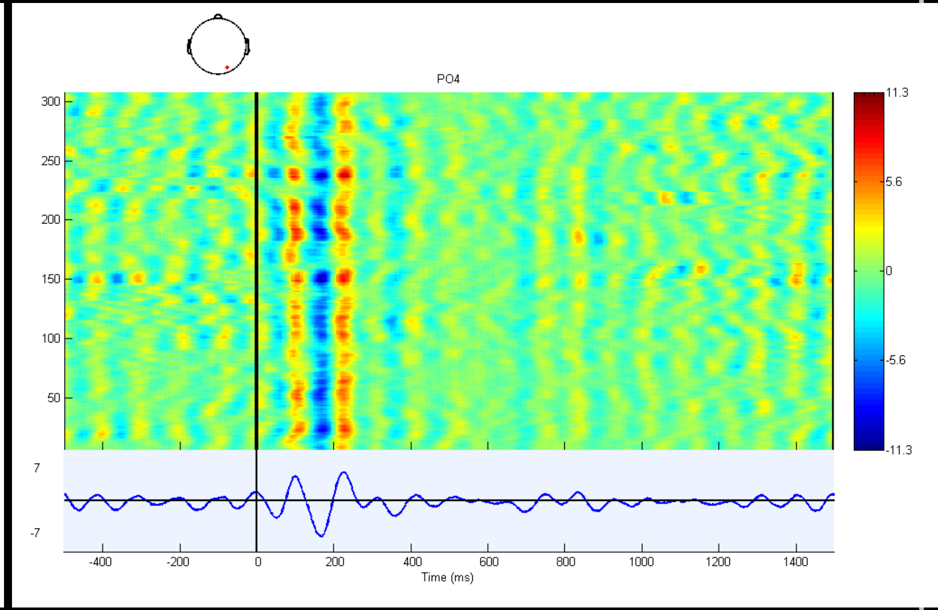


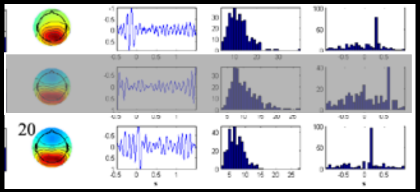


# True Evoked Potential (EP)

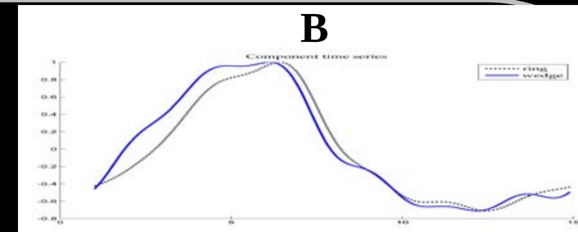
# Reconstructed EP component 18 and 20



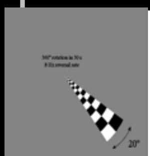



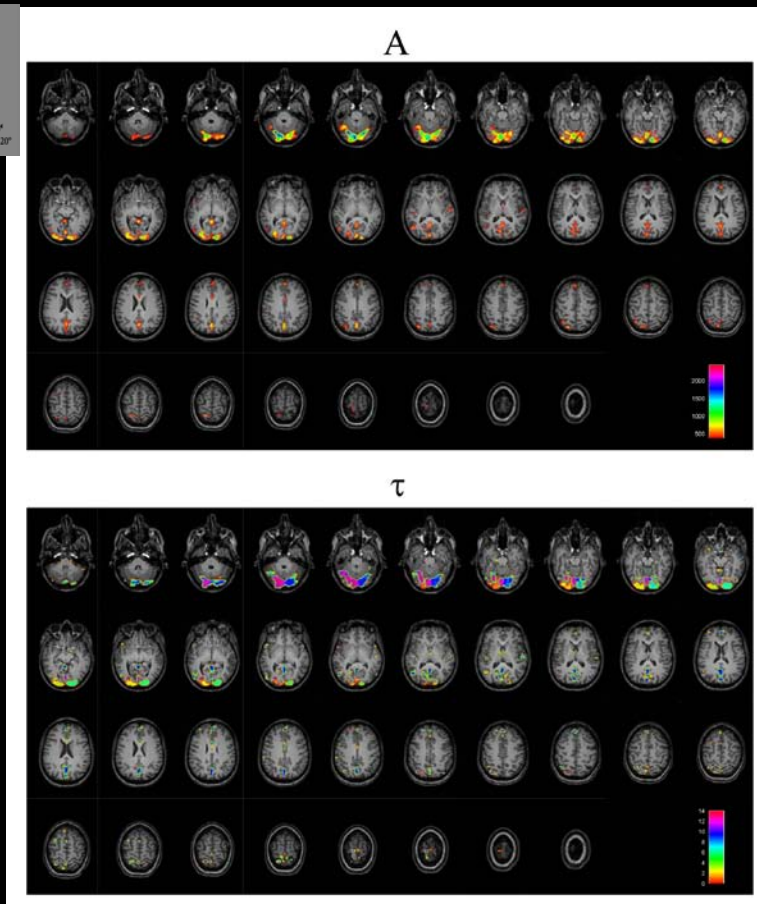



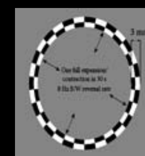
# fMRI data visual paradigm

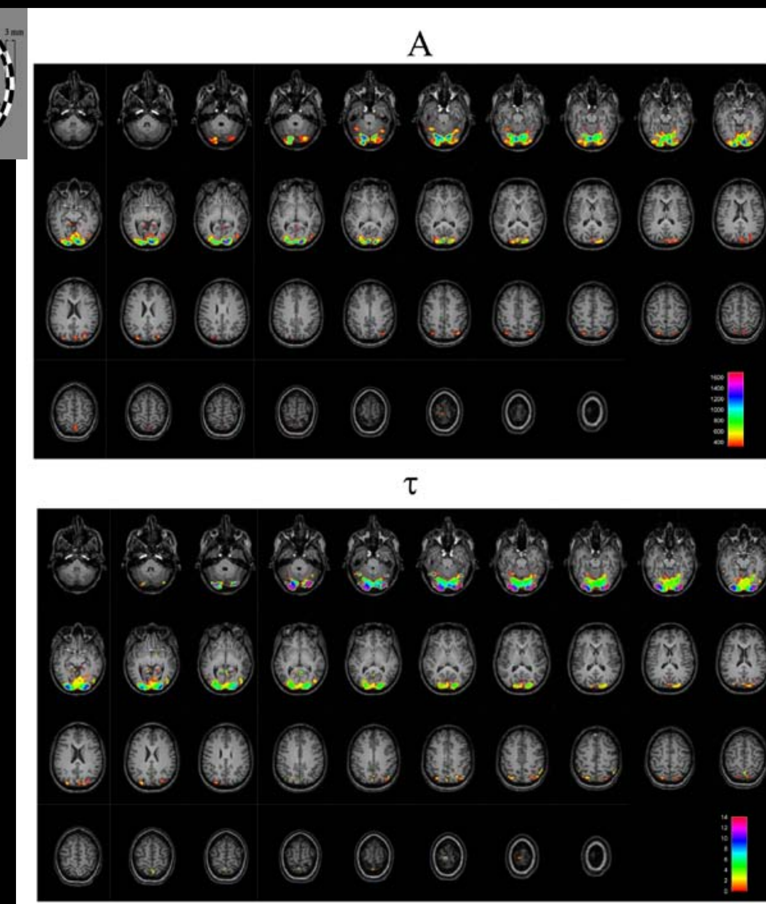



$$\mathcal{X}_{i,j,k} = \sum_{d}^{D} \mathbf{A}_{i,d}\mathbf{B}_{j-\boldsymbol{\tau}_{k,d},d}\mathbf{C}_{k,d} + \mathcal{E}_{i,j,k}$$








(Kristoffer Hougaard Madsen)

# Shift invariant sparse coding

## (Solving the shift problem using sparse coding)

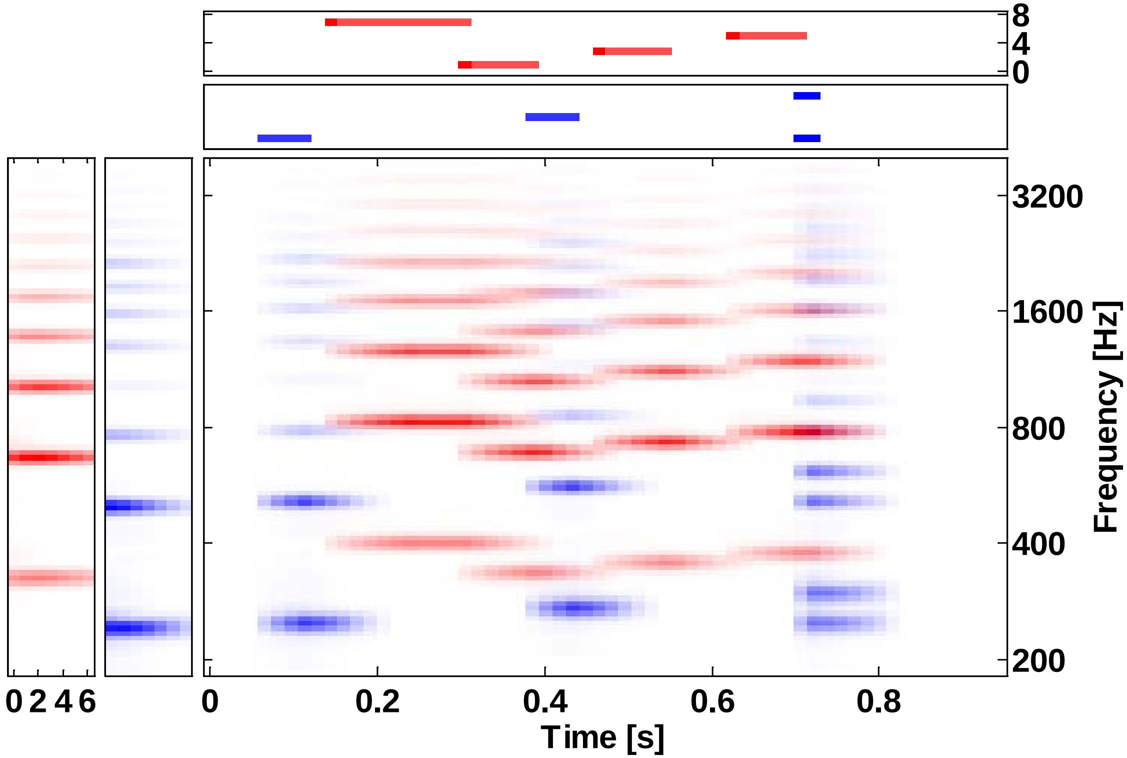
$$X(x,y) \approx \sum_{d} \sum_{u,v} \alpha_d(u,v)\Phi_d(x-u,y-v)$$

**Non-negative matrix 2D de-convolution introduced in:**

**Schmidt and Mørup, 2006d**

**Extended to Sparse coding**

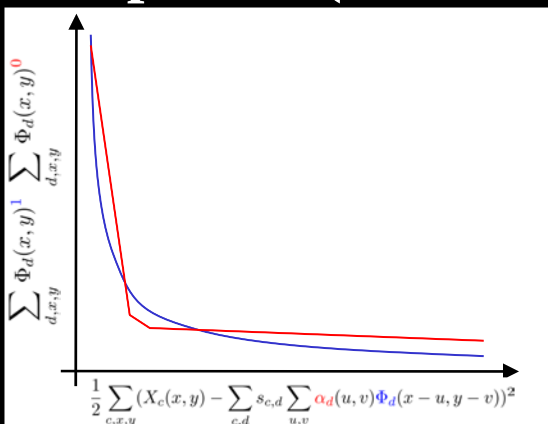
**Mørup et al. 2007b**



Objective function:

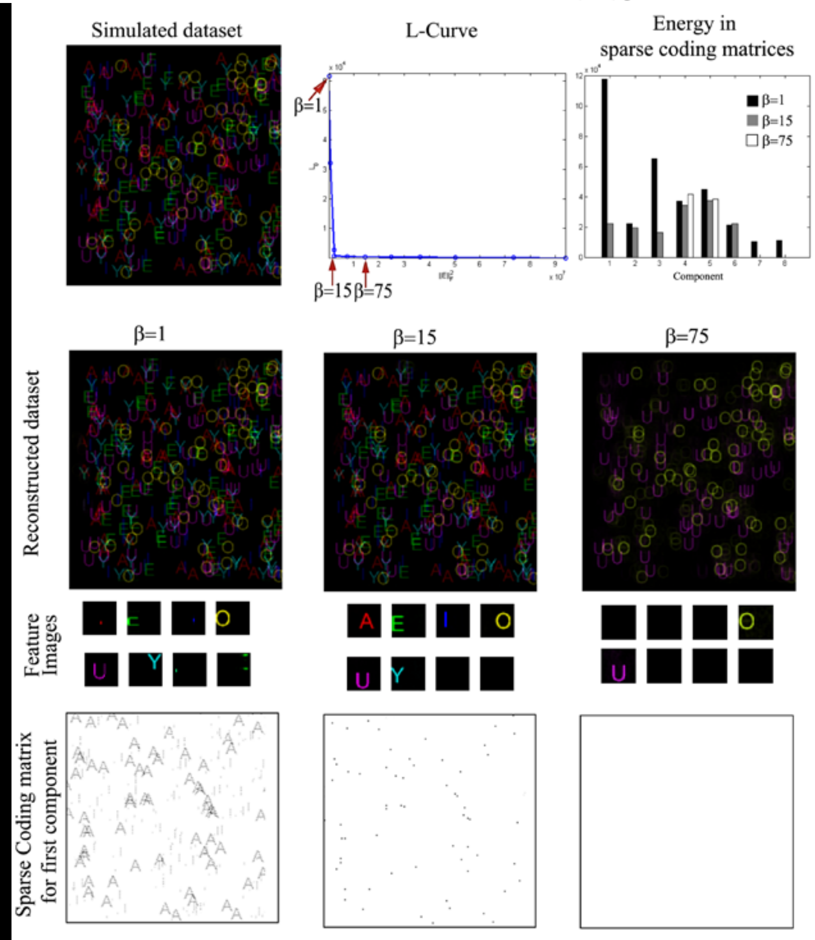
$$\frac{1}{2}\underbrace{\sum_{c,x,y}\Big(X_c(x,y)-\sum_{c,d}s_{c,d}\sum_{u,v}{\color{red}\alpha_d}(u,v){\color{blue}\Phi_d}(x-u,y-v)\Big)^2}_{\textbf{Reconstruction}}+\beta\underbrace{\sum_{d,x,y}\Phi_d(x,y)}_{\mathbf{L_1\ norm}}$$

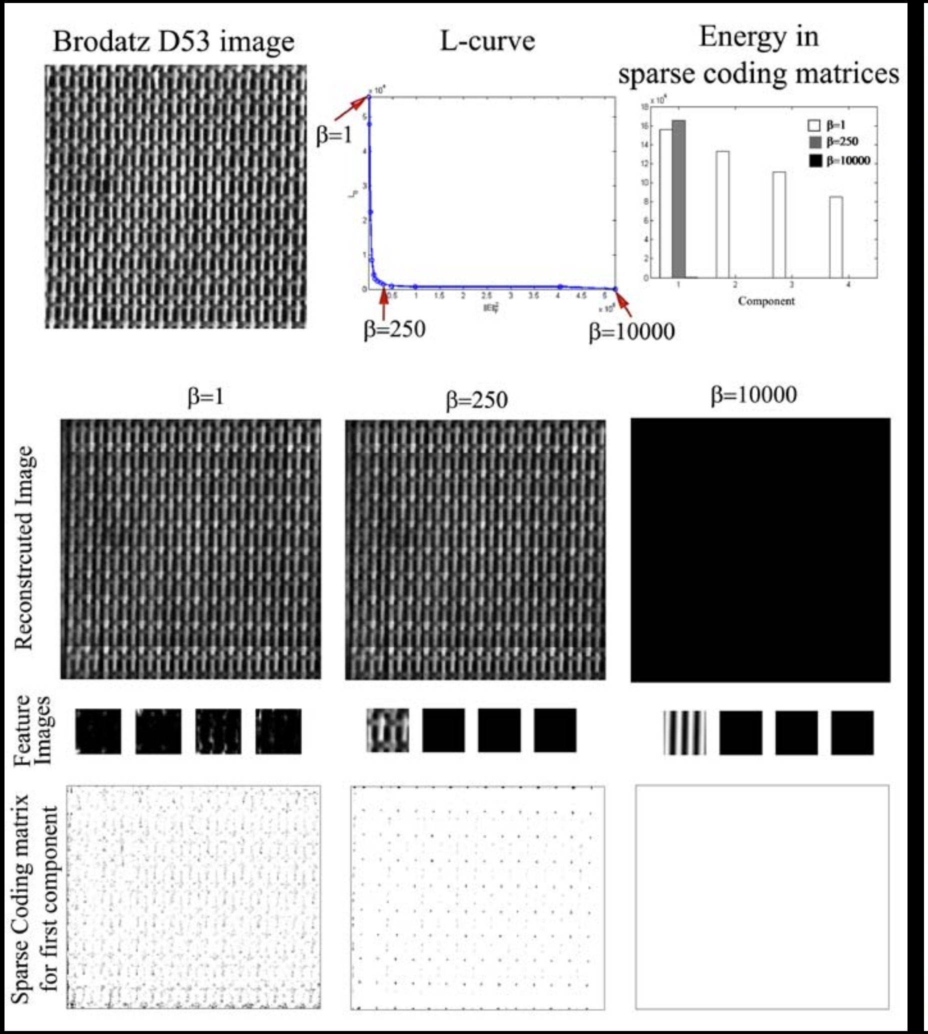
**Solved by multiplicative updates (i.e., X, s, α, Φ≥0)**

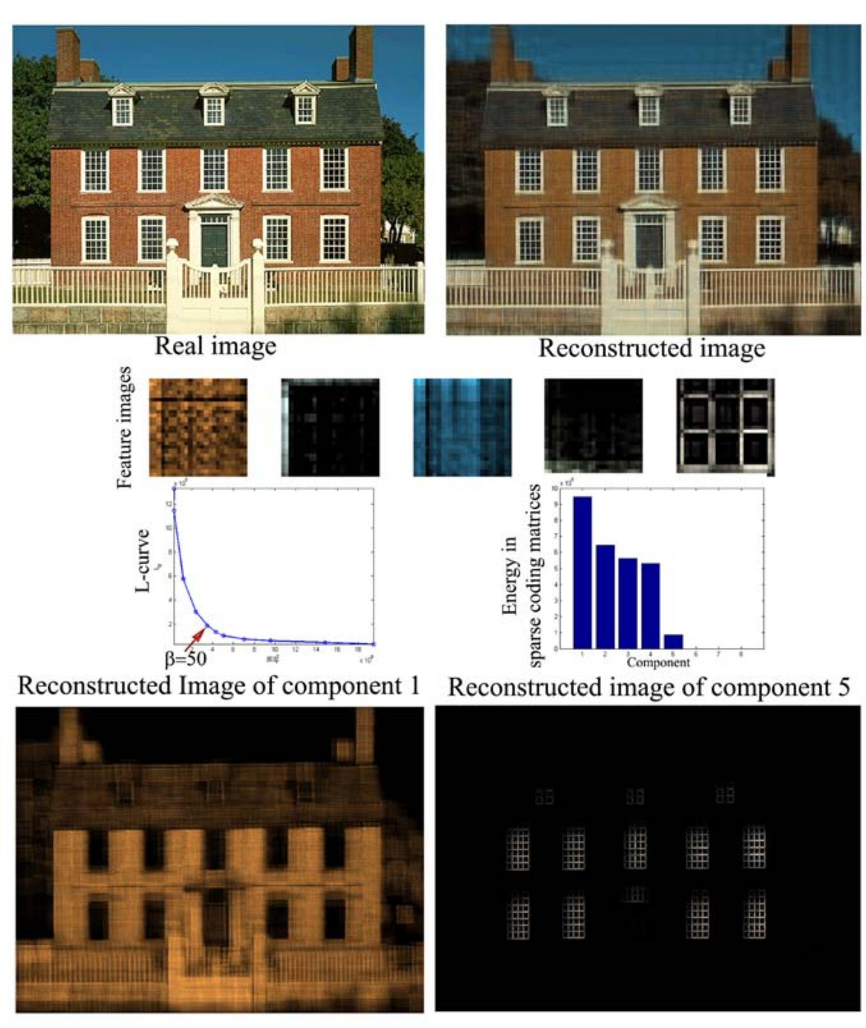


$$I_c(u,v)=s_{c,d}{\color{red}\alpha_d}(u,v)\rightarrow$$

$${\color{blue}\Phi_d}(x-u,y-v)\rightarrow$$



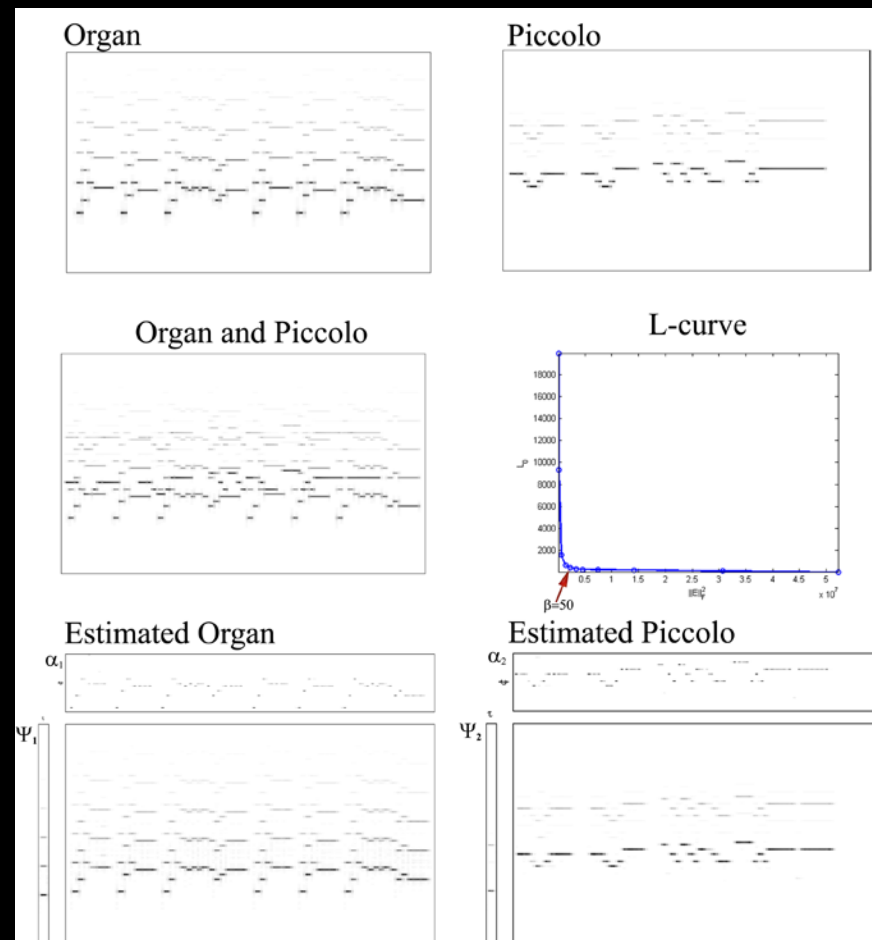

Brodatz D53 image
L-curve
Energy in
sparse coding matrices
β=1
β=250
β=10000
Component
β=1
β=250
β=10000
Reconstructed Image
Feature
Images
Sparse Coding matrix
for first component


Real image
Reconstructed image
Feature images
L-curve
β=50
Energy in
sparse coding matrices
Component
Reconstructed Image of component 1
Reconstructed image of component 5

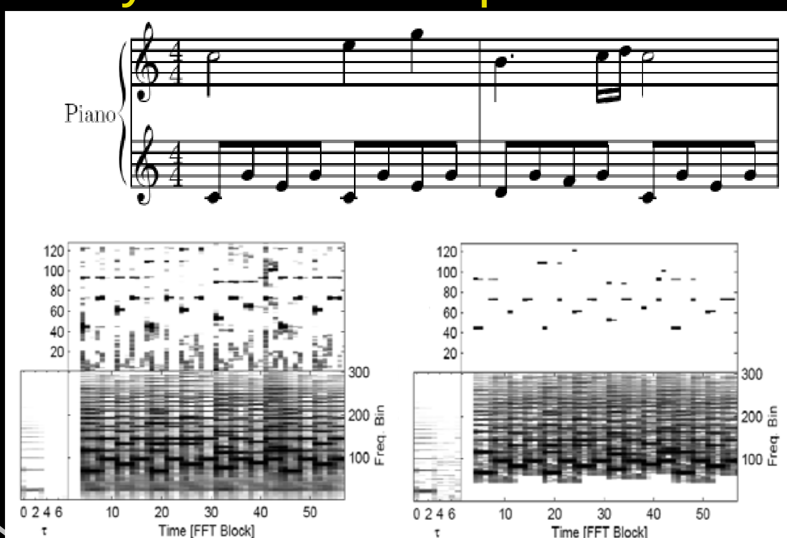
# Analysis of mono signal of mixed Organ and Piccolo

(Music data taken from Y.-G. Zhang, 2005)

$$X(x,y) \approx \sum_{d,u,v} \alpha_d(u,v)\Psi_d(x-u,y-v)$$




## Analysis of mono piano music



## Conclusion:

Modelling data using shift invariance seems highly relevant for a wide range of data types.

Two approaches presented: 1) estimation of specific shift in the frequency domain 2) Estimation of shifts using sparse coding.

Degeneracies encountered in the Candecomp/PARAFAC (CP) model appear to vanish when allowing for shifts. Thus degeneracy in CP often a result of sources of the data being shifted.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