Supplementary Material for Point-Based 3D Reconstruction from Sparse Views under Known Illumination

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A Continuous Adjoint Formulation with Explicit Transmission

This section provides the derivation of the opacity-explicit adjoint gradient used in the main paper. For completeness, we restate the rendering equation as

$$L_o = \tau_\psi \left(L_e + \mathcal{T}_\psi L_o \right), \tag{1}$$

where L_o is outgoing radiance, L_e is emitted radiance, $\mathcal{T}_\psi$ is the light-transport operator, and τ_ψ is the beam-transmittance operator induced by the opacity-bearing scene representation. The image-space objective is written as

$$\mathcal{J}(L_o, \psi) = \left\langle W_c, J(L_o, \psi) \right\rangle, \tag{2}$$

where W_c is the camera response and J is the per-measurement loss.

From Eq. (1), the corresponding state constraint is

$$R(L_o, \psi) = -L_o + \tau_\psi \mathcal{T}_\psi L_o + \tau_\psi L_e. \tag{3}$$

To obtain an unconstrained problem, we introduce a Lagrange multiplier p. The Lagrangian is

$$\begin{aligned} \mathcal{L}(L_o, p, \psi) &= \mathcal{J}(L_o, \psi) + \langle p, R(L_o, \psi) \rangle \\ &= \langle W_c, J(L_o, \psi) \rangle + \langle p, R(L_o, \psi) \rangle, \end{aligned} \tag{4}$$

with

$$\langle f, g \rangle = \int_{A(\psi)} \int_{4\pi} f(x, \omega) g(x, \omega) \, \mathrm{d}\omega \, \mathrm{d}A. \tag{5}$$

Stationarity of the Lagrangian with respect to radiance, $\partial\mathcal{L}/\partial L_o = 0$, together with Eq. (3), gives the adjoint equation

$$\frac{\partial J}{\partial L_o} W_c = -\left(\frac{\partial R}{\partial L_o}\right)^* p = -\left(\tau_\psi \mathcal{T}_\psi - \mathcal{I}\right)^* p = \left(\mathcal{I} - \mathcal{T}_\psi^* \tau_\psi\right) p, \tag{6}$$

where $*$ denotes the adjoint and $\mathcal{I}$ is the identity operator. We treat beam transmittance as self-adjoint, $\tau_\psi^* = \tau_\psi$. A more detailed account of the transport operator and its adjoint is provided by Christensen [1].

When the adjoint equation is satisfied, stationarity with respect to the scene parameters is an optimality condition, and $\partial\mathcal{L}/\partial\psi$ equals the gradient of the cost function [3]. Before reparameterization, this derivative can be written as

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial \mathcal{J}}{\partial \psi} + \left\langle p, \frac{\partial R}{\partial \psi} \right\rangle + \left\langle p, R\left(\nabla_A \cdot \frac{\partial x}{\partial \psi}\right)\right\rangle, \tag{7}$$

where the final term accounts for the parameter dependence of the integration domain $A(\psi)$ [4]. In our implementation, surface integrals over individual surfels are reparameterized over fixed local coordinates. The corresponding parameter dependence is therefore represented through the differentiated integrand and the local surface-area Jacobian, rather than through an explicit moving-domain term.

A.1 Adjoint Light Transport

The rendering equation in Eq. (1) admits the Neumann-series expansion [2]

$$L_o = \left(\mathcal{I} - \tau_\psi \mathcal{T}_\psi\right)^{-1} \tau_\psi L_e = \sum_{n=0}^{\infty} \left(\tau_\psi \mathcal{T}_\psi\right)^n \tau_\psi L_e, \tag{8}$$

where each application of the transport operator corresponds to one light-transport bounce. As observed by Stam [5], the adjoint equation has the same structure but uses the adjoint transport operator $\mathcal{T}_\psi^*$. The adjoint operator propagates an incident quantity, such as radiance or importance, rather than an outgoing quantity [1]. The adjoint variable is therefore

$$p = \sum_{n=0}^{\infty} \left(\mathcal{T}_\psi^* \tau_\psi\right)^n \frac{\partial J}{\partial L_o} W_c = \sum_{n=0}^{\infty} \left(\mathcal{T}_\psi^* \tau_\psi\right)^n \left(L_o - L_{\mathrm{ref}}\right) W_c. \tag{9}$$

Importantly, tracing p depends neither on the number of optimized parameters nor on their individual derivatives.

Because eye-path tracing is the natural implementation strategy in our differentiable renderer, we express the gradient using derivatives of $\mathcal{T}_\psi^*$ rather than derivatives of $\mathcal{T}_\psi$. Substituting Eq. (3) into Eq. (7), taking the partial derivative with respect to ψ while holding the radiance state L_o fixed, and applying the defining property of adjoint operators gives

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial \mathcal{J}}{\partial \psi} + \left\langle p, \frac{\partial}{\partial \psi}\left(-L_o + \tau_\psi \mathcal{T}_\psi L_o + \tau_\psi L_e\right)\right\rangle. \tag{10}$$

This simplifies to

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial \mathcal{J}}{\partial \psi} + \left\langle \frac{\partial \left(\mathcal{T}_\psi^* \tau_\psi\right)}{\partial \psi} p, L_o \right\rangle + \left\langle p, \frac{\partial \tau_\psi}{\partial \psi} L_e \right\rangle, \tag{11}$$

where the light sources are assumed not to be parameterized. The first inner-product term differentiates transported adjoint importance through scattering and transmission, while the second differentiates the transmission of emitted radiance. The surface-position integral in the source term for p collapses through the Dirac delta contained in W_c [Eq. (9)], allowing the gradient to be evaluated by differentiable path tracing from the camera.

References

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