Algorithms and Methods for Fast Model Predictive Control

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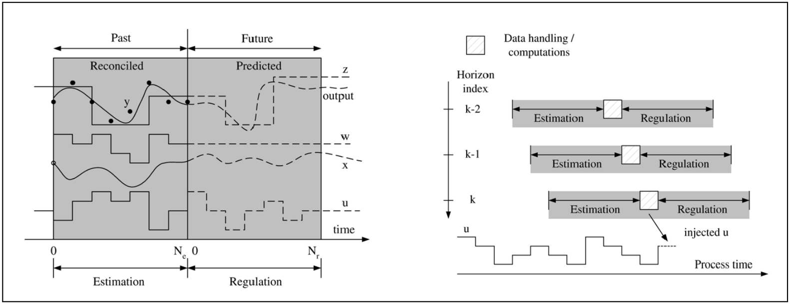
Model Predictive Control (and Moving Horizon Estimation)



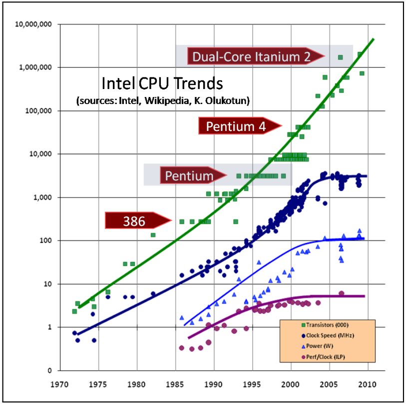
Background: Model Predictive Control

Model Predictive Control

+ optimal control signal
+ easy incorporation of forecasts
+ predictive adaptation to setpoint changes
+ natural handling of constraints and MIMO
+ generalization to non-linear systems
- need for a model
- an optimization problem at each sampling instant

Background: "The Free Lunch is Over" (2005)

- For decades, CPU frequency increase boosted CPU performance
- Around 2002 CPU frequency stalled, transistor count kept doubling every 2 years (Moore's law)
- Use additional transistors to increase CPU performance-per-clock: vectorization (SIMD), parallelization (multi-core)



Background: "The Free Lunch is Over" (2005)

Consequences:

- "If your program is too slow, just wait for the next computer generation" is not true any more
- Vectorization and parallelization require extra programming effort (compilers can't do proper auto-vec. and auto-par.)
- In real-time critical applications, more performance requires more (hardware-exploiting) software optimization

Thesis approach

Algorithms and Methods for Fast Model Predictive Control

- Methods: dense linear algebra implementation methods for embedded optimization (Part I)
- Algorithms: structure-exploiting algorithms for MPC (Parts II and III)
- Both algorithms and their implementation are equally important in the development of fast solvers
- Bottom-up approach: speed-up performance-critical routines to speed-up the overall solvers

Part I

Dense Linear Algebra Routines for Embedded Optimization

Framework: embedded optimization

Assumptions about embedded optimization:

- Computational speed is a key factor: solve optimization problems in real-time on resources-constrained hardware.
- Data matrices are reused several times (e.g. at each optimization algorithm iteration): look for a good data structure.
- Structure-exploiting algorithms can exploit the high-level sparsity pattern: data matrices assumed dense.
- Size of matrices is relatively small (tens or few hundreds): generally fitting in cache.

BLAS

- Basic Linear Algebra Subprograms
- The de-facto standard interface for linear algebra
- Implementations optimized for many computer architectures
  - but optimized for large-scale matrices
  - often poor small-scale performance (large overhead)
- Divided into 3 levels:
  - level 1: vector-vector operations: $\mathcal{O}(n)$ storage, $\mathcal{O}(n)$ flops
  - level 2: matrix-vector operations: $\mathcal{O}(n^2)$ storage, $\mathcal{O}(n^2)$ flops
  - level 3: matrix-matrix operations: $\mathcal{O}(n^2)$ storage, $\mathcal{O}(n^3)$ flops
- an access to memory (memop) is much slower than a flop
  - in level 3 BLAS there is a lot of space for optimization

LAPACK

- Linear Algebra PACKage
- Standard software library for numerical linear algebra
- E.g. Cholesky factorization, matrix inversion
- Built on top of BLAS
  - unblocked routines using level 1 & 2 BLAS (small matrices)
  - blocked routines using level 3 BLAS (large matrices)
- Bad multi-thread scalability (not explicit parallelism)
  - PLASMA project
- Bad small-scale performance (level 1 & 2 BLAS)
  - examples later in the talk

(D)GEMM

- (DP) general matrix-matrix multiplication
- Key sub-operation in all level 3 BLAS & LAPACK
- Often used to benchmark BLAS implementations
- In optimized BLAS, high-performance by employing:
  - blocking for registers
  - machine-specific instructions (e.g. SIMD)
  - special internal matrix format
  - blocking for cache

Computational Performance

- measured in Gflops = (# of flops) / ($10^9\cdot$ solution time in s)
- e.g. `dsyrk + dpotrf` costs $n^3 + \frac{1}{3}n^3 = \frac{4}{3}n^3$ flops
- compared with theoretical peak performance
- measure of CPU utilization
- useful to identify performance bottlenecks
- room for improvement?

Implementation of dsyrk + dpotrf on Intel Ivy-Bridge

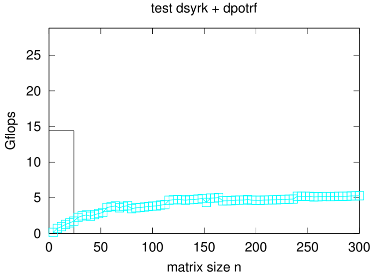
Test operation:

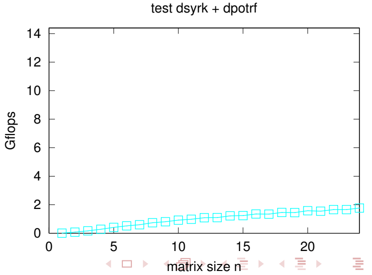
$$\mathcal{L} = \left(\mathcal{Q} + \mathcal{A} \cdot \mathcal{A}^T\right)^{1/2}$$

NetlibBLAS

- Reference BLAS & LAPACK
- triple-loop linear algebra
- machine independent code

[all code is single-threaded]
[all code compiled with gcc]





Triple-loop implementation

- less memops if inner loop over k: each element is computed as

$$c_{ij} = c_{ij} + \sum_{k=0}^{n-1} a_{ik} \cdot b_{kj}, \quad i = 0, \dots, n-1, \quad j = 0, \dots, n-1$$

- issue #1: dependent operations, can not hide latency (since FP instructions are pipelined, latency > throughput)

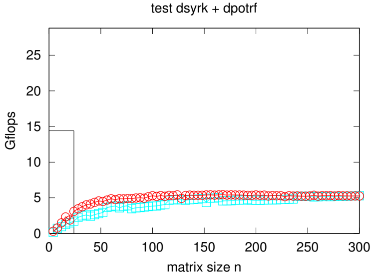
$$\begin{aligned} c_{ij} &= c_{ij} + a_{i0} \cdot b_{0j} \\ c_{ij} &= c_{ij} + a_{i1} \cdot b_{1j} \\ c_{ij} &= c_{ij} + a_{i2} \cdot b_{2j} \\ c_{ij} &= c_{ij} + a_{i3} \cdot b_{3j} \end{aligned}$$

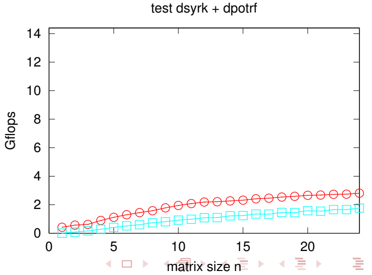
- issue #2: ratio flops/memops=2/2=1

Implementation of `dsyrk` + `dpotrf` on Intel Ivy-Bridge

Code Generation

- e.g. fix the size of the loops: compiler can unroll loops and avoid branches
- need to generate the code for each problem size

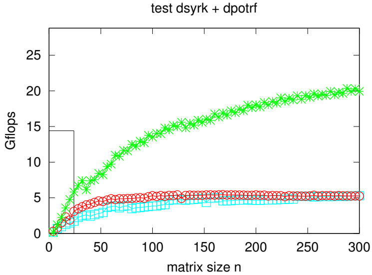


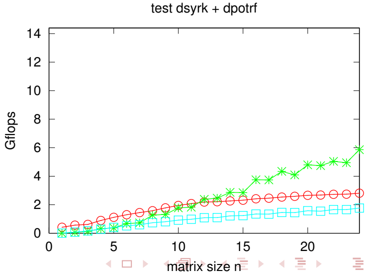


Implementation of dsyrk + dpotrf on Intel Ivy-Bridge

OpenBLAS

- high-performance for large matrices

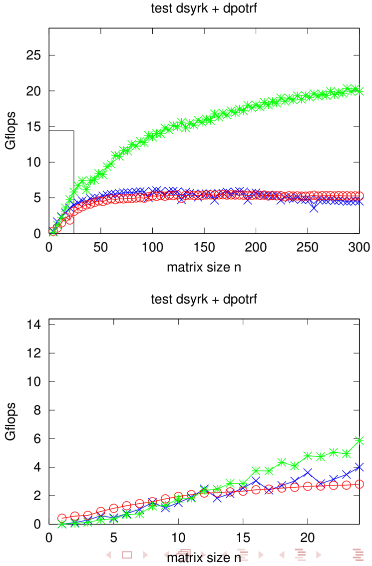




Implementation of dsyrk + dpotrf on Intel Ivy-Bridge

HPMPC - blocking for registers

- HPMPC: library for High-Performance implementation of solvers for Model Predictive Control
- hide latency of instructions
- reuse of matrix elements once in registers



Blocking for registers

- idea: use registers to hold a sub-matrix of C
- e.g. 2×2 sub-matrix in registers

	$b_{k,j+0}$	$b_{k,j+1}$
$a_{i+0,k}$	$c_{i+0,j+0} + a_{i+0,k} \cdot b_{k,j+0}$	$c_{i+0,j+1} + a_{i+0,k} \cdot b_{k,j+1}$
$a_{i+1,k}$	$c_{i+1,j+0} + a_{i+1,k} \cdot b_{k,j+0}$	$c_{i+1,j+1} + a_{i+1,k} \cdot b_{k,j+1}$

- solution #1: independent operations, can hide latency

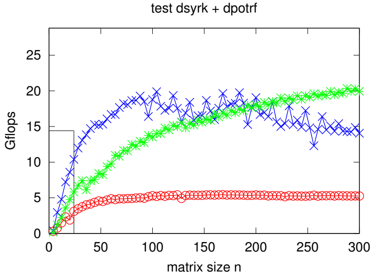
$$
\begin{aligned}
c_{i+0,j+0} &= c_{i+0,j+0} + a_{i+0,0} \cdot b_{0,j+0} \\
c_{i+1,j+0} &= c_{i+1,j+0} + a_{i+1,0} \cdot b_{0,j+0} \\
c_{i+0,j+1} &= c_{i+0,j+1} + a_{i+0,0} \cdot b_{0,j+1} \\
c_{i+1,j+1} &= c_{i+1,j+1} + a_{i+1,0} \cdot b_{0,j+1} \\
c_{i+0,j+0} &= c_{i+0,j+0} + a_{i+0,1} \cdot b_{1,j+0} \\
\ldots &= \ldots
\end{aligned}
$$

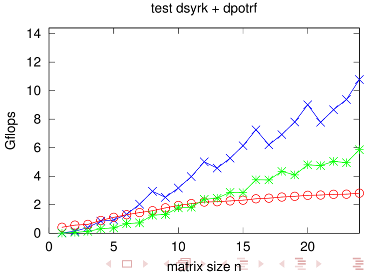
- solution #2: ratio flops/memops=8/4=2

Implementation of `dsyrk` + `dpotrf` on Intel Ivy-Bridge

HPMPC - SIMD instructions

- use SIMD (Single-Instruction Multiple-Data)
- AVX: 4 doubles per vector
- performance drop for *n* multiple of 32 - cache associativity





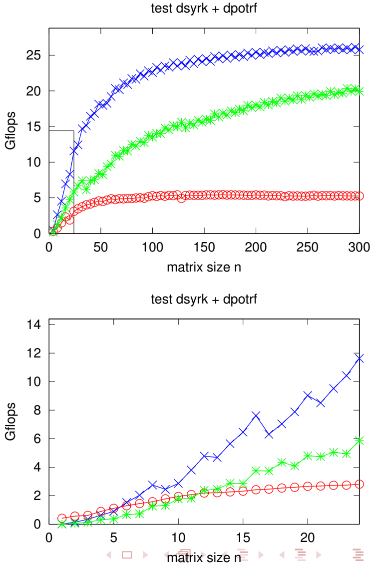
Use of SIMD

- idea: perform the same instructions on small vectors of data, element-wise
- e.g. 2-wide registers, $4x2$ sub-matrix

$$
\begin{array}{c|cc}
 & b_0 & b_1 \\
\hline
\begin{array}{c} a_0 \\ a_1 \end{array} & \begin{bmatrix} c_{00} \\ c_{10} \end{bmatrix} + \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_0 \end{bmatrix} & \begin{bmatrix} c_{01} \\ c_{11} \end{bmatrix} + \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_1 \end{bmatrix} \\
\begin{array}{c} a_2 \\ a_3 \end{array} & \begin{bmatrix} c_{20} \\ c_{30} \end{bmatrix} + \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_0 \end{bmatrix} & \begin{bmatrix} c_{21} \\ c_{31} \end{bmatrix} + \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_1 \end{bmatrix}
\end{array}
$$

- 2-wide SIMD gives up to 2x speed-up

Implementation of dsyrk + dpotrf on Intel Ivy-Bridge



HPMPC - panel-major matrix format

- panel-major matrix format: arrange matrix elements in memory as accessed by the dgemm routine
- smooth performance

Access pattern in optimized BLAS

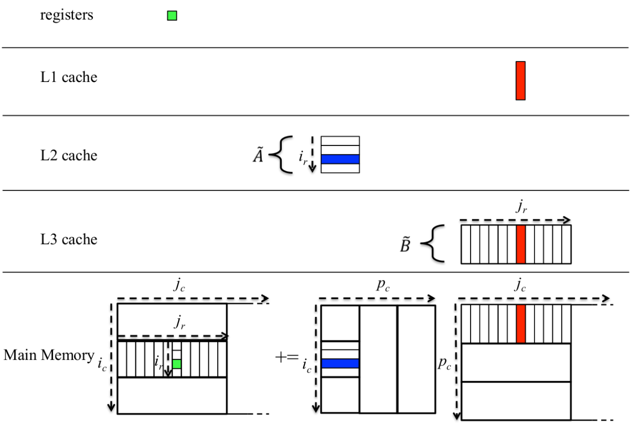
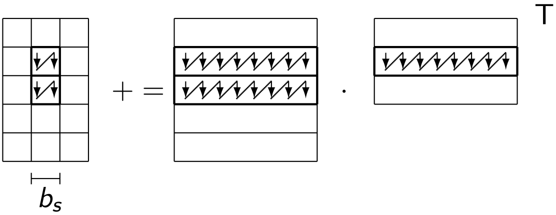


Figure : Access pattern of data in different cache levels for the `dgemm` routine in GotoBLAS/OpenBLAS/BLIS. Data is packed (on-line) into buffers following the access pattern.

Panel-major matrix format



- matrix elements are stored in the same order such as the `gemm` kernel accesses them
- optimal 'NT' variant (namely, A not-transposed, B transposed)
- panels width b_s is the same for the left and the right matrix operand, as well as for the result matrix

Optimized BLAS vs HPMPC software stack

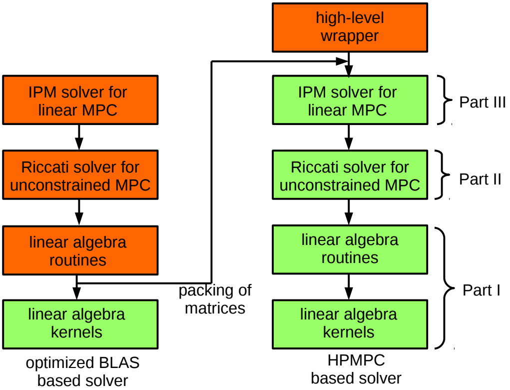
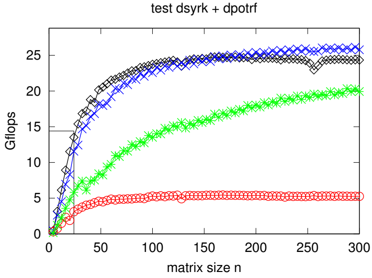


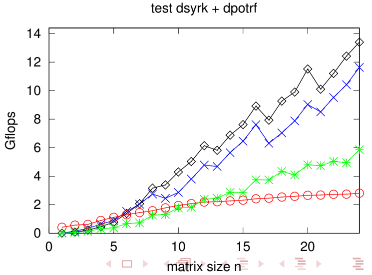
Figure : Structure of a Riccati-based IPM for linear MPC problems when implemented using linear algebra in either optimized BLAS or HPMPC. Routines in the orange boxes use matrices in column-major format, routines in the green boxes use matrices in panel-major format.

Implementation of dsyrk + dpotrf on Intel Ivy-Bridge

HPMPC - merging of linear algebra routines

- specialized kernels for complex operations
- improves small-scale performance
- worse large-scale performance





Merging of linear algebra routines - `dsyrk + dpotrf`

$$\mathcal{L} = \left(\mathcal{Q} + \mathcal{A} \cdot \mathcal{A}^T\right)^{1/2} =$$

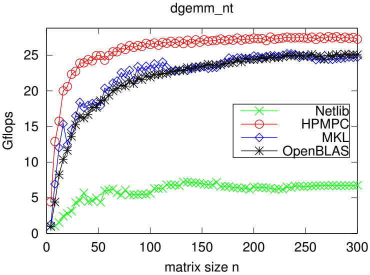
$$\begin{bmatrix} \mathcal{L}_{00} & * & * \\ \mathcal{L}_{10} & \mathcal{L}_{11} & * \\ \mathcal{L}_{20} & \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} = \left(\begin{bmatrix} \mathcal{Q}_{00} & * & * \\ \mathcal{Q}_{10} & \mathcal{Q}_{11} & * \\ \mathcal{Q}_{20} & \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} + \begin{bmatrix} \mathcal{A}_0 \\ \mathcal{A}_1 \\ \mathcal{A}_2 \end{bmatrix} \cdot \begin{bmatrix} \mathcal{A}_0^T & \mathcal{A}_1^T & \mathcal{A}_2^T \end{bmatrix} \right)^{1/2} =$$

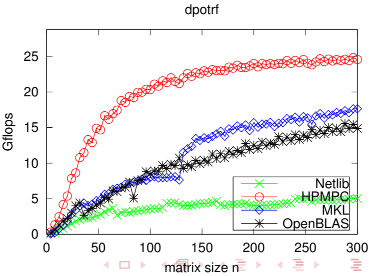
$$\begin{bmatrix} (\mathcal{Q}_{00} + \mathcal{A}_0 \cdot \mathcal{A}_0^T)^{1/2} & * & * \\ (\mathcal{Q}_{10} + \mathcal{A}_1 \cdot \mathcal{A}_0^T)\mathcal{L}_{00}^{-T} & (\mathcal{Q}_{11} + \mathcal{A}_1 \cdot \mathcal{A}_1^T - \mathcal{L}_{10} \cdot \mathcal{L}_{10}^T)^{1/2} & * \\ (\mathcal{Q}_{20} + \mathcal{A}_2 \cdot \mathcal{A}_0^T)\mathcal{L}_{00}^{-T} & (\mathcal{Q}_{21} + \mathcal{A}_2 \cdot \mathcal{A}_1^T - \mathcal{L}_{20} \cdot \mathcal{L}_{10}^T)\mathcal{L}_{11}^{-T} & (\mathcal{Q}_{22} + \mathcal{A}_2 \cdot \mathcal{A}_2^T - \mathcal{L}_{20} \cdot \mathcal{L}_{20}^T - \mathcal{L}_{21} \cdot \mathcal{L}_{21}^T)^{1/2} \end{bmatrix}$$

- each sub-matrix computed using a single specialized routine
  - reduce number of function calls
  - reduce number of load and store of the same data

High-performance LAPACK for small matrices

- Implemented as level 3 BLAS routines
- Blocking at registers level
- Specialized kernels merging `gemm` kernel with unblocked LAPACK routines





Part II

Algorithms for Unconstrained MPC and MHE Problems

Linear Time-Variant Optimal Control Problem

$$\min_{u,x} \quad \sum_{n=0}^{N-1} \frac{1}{2} \begin{bmatrix} u_n \\ x_n \\ 1 \end{bmatrix}^T \begin{bmatrix} R_n & S_n & r_n \\ S_n^T & Q_n & q_n \\ r_n^T & q_n^T & \rho_n \end{bmatrix} \begin{bmatrix} u_n \\ x_n \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_N \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_N & q_N \\ q_N^T & \rho_N \end{bmatrix} \begin{bmatrix} x_N \\ 1 \end{bmatrix}$$

$$s.t. \quad x_{n+1} = A_n x_n + B_n u_n + b_n, \qquad n = 0, \dots, N-1$$

$$x_0 = \hat{x}_0$$

$$0 = D_N x_N + d_N$$

- MPC vs MHE
- equality constraints at last stage

Karush-Kuhn-Tucker optimality conditions

KKT system (for $N = 2$)

$$
\left[\begin{array}{ccc|ccc|cc}
Q_0 & S_0^T & A_0^T & & & & & \\
S_0 & R_0 & B_0^T & & & & & \\
A_0 & B_0 & & -I & & & & \\
\hline
 & & -I & Q_1 & S_1^T & A_1^T & & \\
 & & & S_1 & R_1 & B_1 & & \\
 & & & A_1 & B_1 & & -I & \\
\hline
 & & & & & -I & Q_2 & D_2^T \\
 & & & & & & D_2 &
\end{array}\right]
\left[\begin{array}{c}
x_0 \\ u_0 \\ \lambda_0 \\ \hline x_1 \\ u_1 \\ \lambda_1 \\ \hline x_2 \\ \lambda_2
\end{array}\right]
=
\left[\begin{array}{c}
-q_0 \\ -r_0 \\ -b_0 \\ \hline -q_1 \\ -r_1 \\ -b_1 \\ \hline -q_2 \\ -d_2
\end{array}\right]
$$

- Large, structured system of linear equations
- Sub-matrices are assumed dense or diagonal

Backward Riccati recursion

$$P_n = Q_n + A_n^T P_{n+1} A_n - \\ - (S_n^T + A_n^T P_{n+1} B_n)(R_n + B_n^T P_{n+1} B_n)^{-1}(S + B_n^T P_{n+1} A_n)$$

- structure-exploiting factorization of the KKT matrix
- begins factorization at the last stage
- does not require invertible Hessian
- can not handle additional equality constraints at the last stage
- naturally handles MPC problems
- $\mathcal{O}(N(n_x + n_u)^3)$ flops

Backward Riccati recursion

Main loop

```
...
for n ← N − 1, ..., 0 do
    A_n^T L_{n+1} ← [B_n^T; A_n^T] · L_{n+1,22}                 ▷ trmm
    M_n ← Q_n + (A_n^T L_{n+1}) · (A_n^T L_{n+1})^T              ▷ syrk
    [L_{n,11} ; L_{n,21} L_{n,22}] ← M_n^{1/2}                   ▷ potrf
end for
...
```

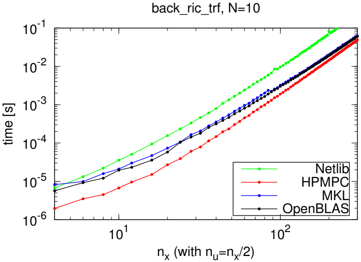
$$\mathcal{A}_n^T \mathcal{L}_{n+1} \leftarrow \begin{bmatrix} B_n^T \\ A_n^T \end{bmatrix} \cdot L_{n+1,22} \qquad \triangleright \texttt{trmm}$$

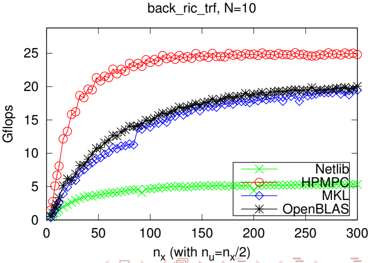
$$\mathcal{M}_n \leftarrow \mathcal{Q}_n + (\mathcal{A}_n^T \mathcal{L}_{n+1}) \cdot (\mathcal{A}_n^T \mathcal{L}_{n+1})^T \qquad \triangleright \texttt{syrk}$$

$$\begin{bmatrix} L_{n,11} & \\ L_{n,21} & L_{n,22} \end{bmatrix} \leftarrow \mathcal{M}_n^{1/2} \qquad \triangleright \texttt{potrf}$$

Backward Riccati recursion

- HPMPC much better for small problems
- performance plot similar to linear algebra ones





Forward Schur-complement recursion

$$\Sigma_{n+1} = Q_n + \left(\begin{bmatrix} A_n & B_n \end{bmatrix} \begin{bmatrix} \Sigma_n & S_n^T \\ S_n & R_n \end{bmatrix}^{-1} \begin{bmatrix} A_n \\ B_n \end{bmatrix} \right)^{-1}$$

- structure-exploiting factorization of the KKT matrix
- begins factorization at the first stage
- requires invertible Hessian (or regularization)
- handles additional equality constraints at the last stage
- naturally handles MHE problems
- $\mathcal{O}(N(n_x + n_u)^3)$ flops

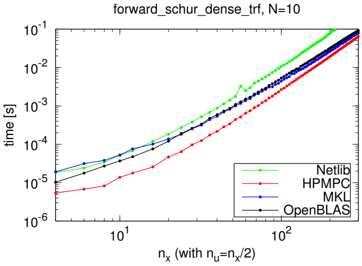
Forward Schur-complement recursion

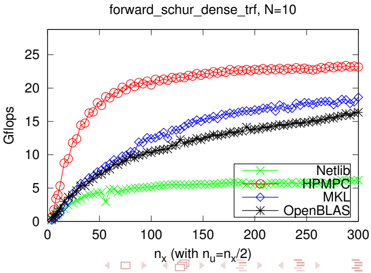
Main loop

$\ldots$
for $n \leftarrow 1, \ldots, N-1$ **do**
  $\Sigma \leftarrow Q_n + U_n \cdot U_n^T$ ▷ `lauum`
  $\mathcal{Q} \leftarrow \begin{bmatrix} \Sigma & 0 \\ S_n & R_n \end{bmatrix}$
  $\mathcal{A} \leftarrow \begin{bmatrix} A_n & B_n \end{bmatrix}$
  $\mathcal{L}_n \leftarrow \mathcal{Q}^{1/2}$ ▷ `potrf`
  $\mathcal{AL}_n \leftarrow \mathcal{A} \cdot \mathcal{L}^{-T}$ ▷ `trsm`
  $P_{inv} \leftarrow \mathcal{AL}_n \cdot \mathcal{AL}_n^T$ ▷ `syrk`
  $L \leftarrow P_{inv}^{1/2}$ ▷ `potrf`
  $U_{n+1} \leftarrow L^{-T}$ ▷ `trtri`
`end for`
$\ldots$

Forward Schur-complement recursion

- similar considerations to backward Riccati recursion
- but slightly worse performance due to more LAPACK routines





Hessian condensing - idea

- Idea: use state-space equation to eliminate states variables from the optimization problem
- Smaller but dense Hessian

$$\begin{bmatrix} B_0^T Q_1 B_0 + B_0^T A_1^T Q_2 A_1 B_0 + B_0^T A_1^T A_2^T Q_3 A_2 A_1 B_0 & * & * \\ B_1^T Q_2 A_1 B_0 + B_1^T A_2^T Q_3 A_2 A_1 B_0 & B_1^T Q_2 B_1 + B_1^T A_2^T Q_3 A_2 B_1 & * \\ B_2^T Q_3 A_2 A_1 B_0 & B_2^T Q_3 A_2 B_1 & B_2^T Q_3 B_2 \end{bmatrix}$$

Hessian condensing - MPC case

Initial state and state space equations

$$x_0 = \hat{x}_0, \quad x_{n+1} = A_n x_n + B_n u_n + b_n$$

rewritten as

$$\bar{A}\bar{x} = \bar{B}\bar{u} + \bar{b} \quad \Rightarrow \quad \bar{x} = \bar{A}^{-1}\bar{B}\bar{u} + \bar{A}^{-1}\bar{b} \doteq \Gamma_u \bar{u} + \Gamma_{x,b}$$

where $(N = 3)$

$$\bar{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} \hat{x}_0 \\ b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} I & & & \\ -A_0 & I & & \\ & -A_1 & I & \\ & & -A_2 & I \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_0 & & \\ & B_1 & \\ & & B_2 \end{bmatrix}$$

Hessian condensing - MPC case

Key idea to have $\mathcal{O}(N^2)$ Hessian condensing algorithms

$$\bar{A}^{-1} = \begin{bmatrix} I & & & \\ -A_0 & I & & \\ & -A_1 & I & \\ & & -A_2 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & & & \\ A_0 & I & & \\ A_1 A_0 & A_1 & I & \\ A_2 A_1 A_0 & A_2 A_1 & A_2 & I \end{bmatrix}$$

- $\bar{A}$ is sparse ($\mathcal{O}(N)$ n.z.) but $\bar{A}^{-1}$ is dense ($\mathcal{O}(N^2)$ n.z.)

$$\Gamma_u = \begin{bmatrix} I & & & \\ -A_0 & I & & \\ & -A_1 & I & \\ & & -A_2 & I \end{bmatrix}^{-1} \begin{bmatrix} & & \\ B_0 & & \\ & B_1 & \\ & & B_2 \end{bmatrix} = \begin{bmatrix} & & \\ B_0 & & \\ A_1 B_0 & B_1 & \\ A_2 A_1 B_0 & A_2 B_1 & B_2 \end{bmatrix}$$

- backsolve vs matrix multiplication: n_x vs $N n_u$ trade-off

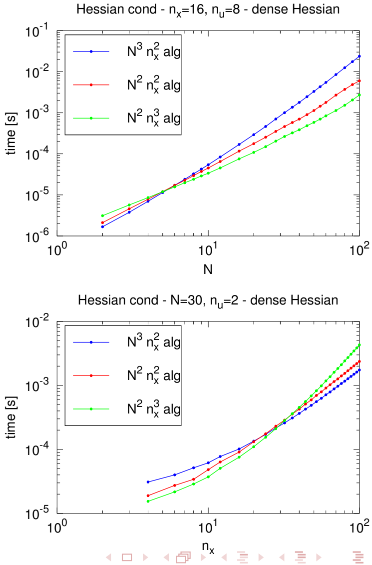
Hessian condensing - MPC case

If $S_n = 0$, condensed Hessian

$$H = \bar{R} + \Gamma_u^T \bar{Q} \Gamma_u$$
$$= \bar{R} + \bar{B}^T \bar{A}^{-T} \bar{Q} \bar{A}^{-1} \bar{B}$$

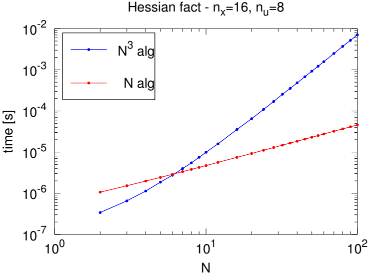
Three algorithms depending on the order of operations

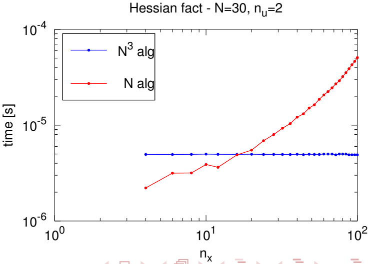
- $\mathcal{O}(N^3)$ and $\mathcal{O}(n_x^2)$
- $\mathcal{O}(N^2)$ and $\mathcal{O}(n_x^2)$
- $\mathcal{O}(N^2)$ and $\mathcal{O}(n_x^3)$



Hessian factorization - MPC case

- $\mathcal{O}(N^3)$ classical Cholesky factorization of condensed Hessian
- $\mathcal{O}(N)$ structure-exploiting Cholesky factorization of permuted condensed Hessian
  - starts form last stage
  - directly builds the factorized Hessian
  - combined with $(\mathcal{O}(N^2)$ and $\mathcal{O}(n_x^2))$ or $(\mathcal{O}(N^2)$ and $\mathcal{O}(n_x^3))$ Hessian condensing algorithms

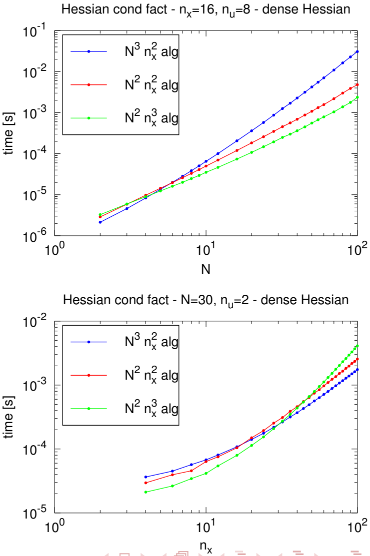




Hessian condensing & factorization - MPC case

Still three algorithms

- $\mathcal{O}(N^3)$ and $\mathcal{O}(n_x^2)$
- $\mathcal{O}(N^2)$ and $\mathcal{O}(n_x^2)$
- $\mathcal{O}(N^2)$ and $\mathcal{O}(n_x^3)$



Hessian condensing - MHE case

State space equations (no initial state constraint)

$$x_{n+1} = A_n x_n + B_n u_n + b_n$$

rewritten as

$$\bar{A}\bar{x} = \bar{B}\bar{u} + \bar{b}$$

where ($N = 3$)

$$\bar{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} \color{red}{0} \\ b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} \color{red}{0} & & & \\ -A_0 & I & & \\ & -A_1 & I & \\ & & -A_2 & I \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} & & \\ B_0 & & \\ & B_1 & \\ & & B_2 \end{bmatrix}$$

Recover invertibility of $\bar{A}$

$$\bar{A}\bar{x} = \begin{bmatrix} I & & & \\ -A_0 & I & & \\ & -A_1 & I & \\ & & -A_2 & I \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} x_0 = \mathcal{\bar{A}}\bar{x} - \mathcal{E}_0 x_0$$

gives

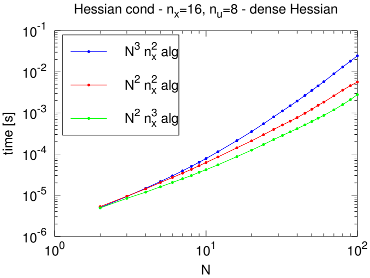
$$\begin{aligned} \bar{x} &= \bar{\mathcal{A}}^{-1}\mathcal{E}_0 x_0 + \bar{\mathcal{A}}^{-1}\bar{B}\bar{u} + \bar{\mathcal{A}}^{-1}\bar{b} \\ &= \bar{\mathcal{A}}^{-1}\bar{\mathcal{B}}\bar{v} + \bar{\mathcal{A}}^{-1}\bar{b} \end{aligned}$$

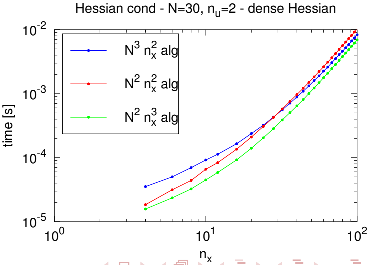
where

$$\bar{\mathcal{B}} = \left[\begin{array}{c|ccc} I & & & \\ & B_0 & & \\ & & B_1 & \\ & & & B_2 \end{array}\right], \quad \bar{v} = \left[\begin{array}{c} x_0 \\ \hline u_0 \\ u_1 \\ u_2 \end{array}\right]$$

Hessian condensing - MHE case

- x_0 as additional input (of size n_x) at stage -1
- all algorithms for MPC can be employed
- $\mathcal{O}(n_x^3)$ can not be avoided
- one algorithm is always better
- same applies for condensed Hessian factorization





Part III

Algorithms for Constrained and Nonlinear MPC Problems

Linear MPC problem

$$
\begin{aligned}
\min_{u,x} \quad & \sum_{n=0}^{N-1} \frac{1}{2} \begin{bmatrix} u_n \\ x_n \\ 1 \end{bmatrix}^T \begin{bmatrix} R_n & S_n & r_n \\ S_n^T & Q_n & q_n \\ r_n^T & q_n^T & \rho_n \end{bmatrix} \begin{bmatrix} u_n \\ x_n \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_N \\ 1 \end{bmatrix}^T \begin{bmatrix} Q_N & q_N \\ q_N^T & \rho_N \end{bmatrix} \begin{bmatrix} x_N \\ 1 \end{bmatrix} \\
s.t. \quad & x_{n+1} = A_n x_n + B_n u_n + b_n, \quad n = 0, \dots, N-1 \\
& x_0 = \hat{x}_0 \\
& u_n^{\mathrm{l}} \leq u_n \leq u_n^{\mathrm{u}}, \quad n = 0, \dots, N-1 \\
& x_n^{\mathrm{l}} \leq x_n \leq x_n^{\mathrm{u}}, \quad n = 1, \dots, N
\end{aligned}
$$

- only box constraints considered here

Interior Point Methods (IPMs) - general idea

- General QP program & KKT system

$$
\begin{aligned}
\min_{x,u} \quad & \frac{1}{2}x^T Hx + g^T x \\
s.t. \quad & Ax = b \\
& Cx \geq d
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
Hx + g - A^T\pi - C^T\lambda &= 0 \\
Ax - b &= 0 \\
Cx - d - t &= 0 \\
\lambda^T t &= 0 \\
(\lambda, t) &\geq 0
\end{aligned}
$$

- Newton method (2nd order method) for the KKT system

$$
\begin{bmatrix} H & -A^T & -C^T & 0 \\ A & 0 & 0 & 0 \\ C & 0 & 0 & -I \\ 0 & 0 & T_k & \Lambda_k \end{bmatrix}
\begin{bmatrix} \Delta x \\ \Delta \pi \\ \Delta \lambda \\ \Delta t \end{bmatrix}
= -
\begin{bmatrix} Hx_k - A^T\pi_k - C^T\lambda_k + g \\ A\pi_k - b \\ Cx_k - t_k - d \\ \Lambda_k T_k e + \sigma\mu_k e \end{bmatrix}
$$

Interior Point Methods (IPMs) - general idea

- structured system, can be rewritten as (augmented system)

$$\begin{bmatrix} H + C^T(T_k^{-1}\Lambda_k)C & -A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \pi_k \end{bmatrix} =$$
$$= -\begin{bmatrix} g - C^T(\Lambda_k e + T_k^{-1}\Lambda_k d + T_k^{-1}\sigma\mu_k e) \\ b \end{bmatrix}$$

- KKT system of an equality constrained QP

Riccati-based IPM for the linear MPC problem

- In the linear MPC problem, KKT system of a LTV-OCP
- Most expensive operation: compute prediction-correction search directions (factorization of KKT system uses level 3 BLAS & LAPACK)
- Backward Riccati recursion (cubic & quadratic number of flops in stage variables number)
- All other operations in IPMs: linear number of flops in stage variables number

Riccati-based IPM in HPMPC

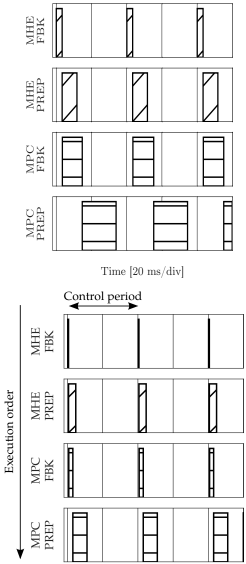
Table : Comparison of solvers for the box-constrained linear MPC problem: low- and high-level interfaces for the IPM in HPMPC, FORCES IPM and FORCES_Pro IPM. Run times are presented in seconds. For each problem size and solver, the number of IPM iterations is fixed to 10.

n_x	n_u	n_b	N	HPMPC low-level	HPMCP high-level	FORCES	FORCES Pro
4	1	5	10	$5.39 \cdot 10^{-5}$	$6.31 \cdot 10^{-5}$	$1.1 \cdot 10^{-4}$	$1.0 \cdot 10^{-4}$
8	3	11	10	$9.05 \cdot 10^{-5}$	$1.04 \cdot 10^{-4}$	$3.4 \cdot 10^{-4}$	$3.1 \cdot 10^{-4}$
12	5	17	30	$5.07 \cdot 10^{-4}$	$5.74 \cdot 10^{-4}$	$2.11 \cdot 10^{-3}$	$1.84 \cdot 10^{-3}$
22	10	32	10	$3.94 \cdot 10^{-4}$	$4.60 \cdot 10^{-4}$	$3.96 \cdot 10^{-3}$	$3.29 \cdot 10^{-3}$
30	14	44	10	$7.03 \cdot 10^{-4}$	$8.17 \cdot 10^{-4}$	$9.47 \cdot 10^{-3}$	$7.49 \cdot 10^{-3}$
60	29	89	30	$1.10 \cdot 10^{-2}$	$1.26 \cdot 10^{-2}$	$1.67 \cdot 10^{-1}$	$1.25 \cdot 10^{-1}$

Conclusion

Arrival point of the PhD work:

- High-performance QP solvers for linear MPC
- Riccati-based IPM for MPC and Schur-complement recursion for MHE interfaced with ACADO
- NMPC of a rotational start-up of a airbone wind energy system



Possible future directions - library

- split the library
  - BLASFEO (?): linear algebra routines for embedded optimization
  - HPMPC: algorithms for MPC built on top of it
- expand the library
  - add LU factorization for e.g. implicit integrators
  - add LDL factorization
  - embed partial condensing into Riccati-based IPM
- improve the library
  - agree on (and fix) interfaces
  - kernels in assembly to reduce code size
  - (re-)add single-precision support
  - add support for embedded hardware (e.g. Cortex M)
  - multi CPU cores

Possible future directions - direct sparse solvers

Direct sparse solvers (e.g. MA57 in IPOPT)

- built on top of level 3 BLAS (e.g. dgemm)
- analyzes the sparsity pattern of the problem, and gathers the non-zero elements into dense sub-matrices
- trade-off between sparsity exploitation (small sub-matrices) and BLAS performance (large sub-matrices): small-scale linear algebra performance is the key
- may lack the right routine in standard BLAS (e.g. in MA57, `dsyrk` with different factor matrices)

Re-implement MA57 on top of BLASFEO?

Thanks for your attention

Questions and comments?

Trend in (Intel) computing architectures

Table : Intel computer architectures: from 2-years cycle to 3 years-cycle

year	arch.	proc.	ISA	DP flops/cycle
2006/07	Merom	65 nm	SSSE3	4
2007/08	Penryn	45 nm	SSE4.1	4
2008/09	Nehalem	45 nm	SSE4.2	4
2010	Westmere	32 nm	SSE4.2	4
2011	Sandy-Bridge	32 nm	AVX	8
2012	Ivy-Bridge	22 nm	AVX	8
2013	Haswell	22 nm	AVX2/FMA3	16
2014	Haswell-refresh	22 nm	AVX2/FMA3	16
2014/15	Broadwell	14 nm	AVX2/FMA3	16
2015/16	Skylake	14 nm	AVX2/FMA3	16
2016/17	Kaby Lake	14 nm	AVX2/FMA3?	16?
2017/18?	Cannonlake	10 nm	AVX512?	32?

Code stack in HPMPC

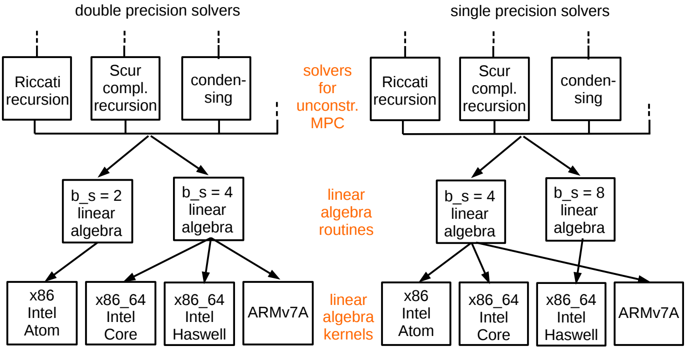
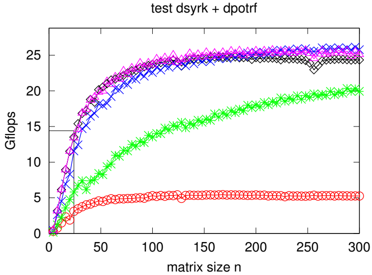


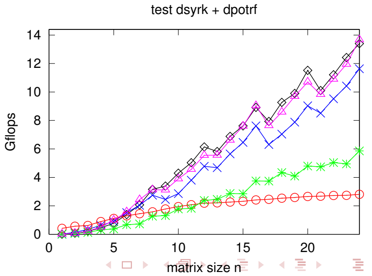
Figure : Structure of the linear algebra routines in HPMPC. The linear algebra kernels are tailored to each computer architecture. The linear algebra routines depend only on the panel height b_s (that may be different for single and double precision). The routines at higher levels in the routines hierarchy are completely architecture-independent.

Implementation of dsyrk + dpotrf on Intel Ivy-Bridge

HPMPC - swapping the order of outer loops

- has to be considered in case of not-squared kernels
- improves the L1 cache reuse
- machine-dependent code





Backward Riccati recursion

- Main operations per stage:
  - update

$$Q + A \cdot P \cdot A^T = Q + A \cdot (L \cdot L^T) \cdot A^T = Q + (A \cdot L) \cdot (A \cdot L)^T$$

    $\frac{7}{3}n_x^3 + 3n_x^2 n_u + n_x n_u^2$ flops
  - factorization-solution-downgrade

$$\mathcal{L} \leftarrow R^{-1}$$
$$L \leftarrow M \cdot \mathcal{L}^{-T}$$
$$P \leftarrow P - L \cdot L^T$$

    $n_x^2 n_u + n_x n_u^2 + \frac{1}{3}n_u^3$ flops
- Total flops: $N(\frac{7}{3}n_x^3 + 4n_x^2 n_u + 2n_x n_u^2 + \frac{1}{3}n_u^3)$

Froward Schur-complement recursion

- Main operations per stage:
  - computation of Schur complement

$$Q+A\cdot P^{-1}\cdot A^T = Q+A\cdot(L\cdot L^T)^{-1}\cdot A^T = Q+(A\cdot L^{-T})\cdot(A\cdot L^{-T})^T$$

    $\frac{7}{3}n_x^3 + 4n_x^2 n_u + 2n_x n_u^2 + \frac{1}{3}n_u^3$ flops
  - inversion of positive definite matrix

$$Q^{-1} = (L\cdot L^T)^{-1} = L^{-T}\cdot L^{-1}$$

    n_x^3 flops
- Total flops:
  - dense Hessian $N(\frac{10}{3}n_x^3 + 4n_x^2 n_u + 2n_x n_u^2 + \frac{1}{3}n_u^3)$
  - diagonal Hessian $N(\frac{10}{3}n_x^3 + n_x^2 n_u)$