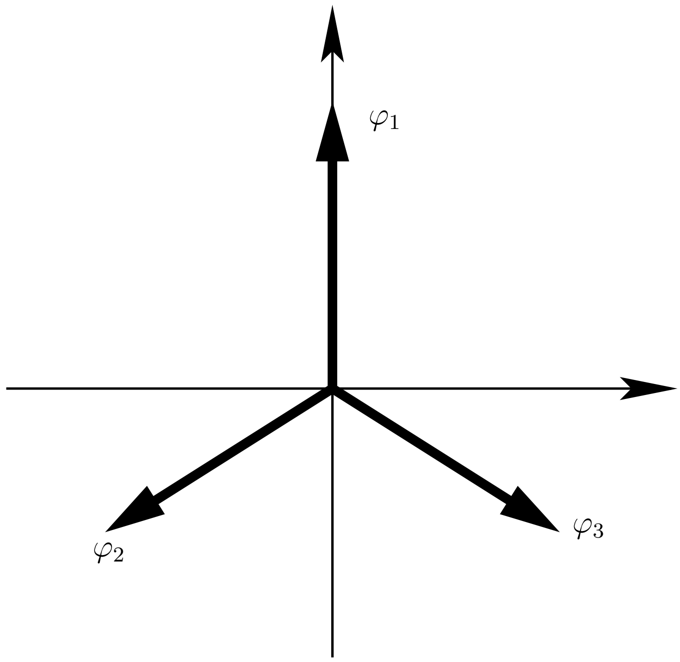
Frames and Their Application on Robust Signal Transmission



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Abstract

In this report we will demonstrate the use of frames (over-complete "bases") in robust signal transmission. We begin by developing the necessary mathematical theory for frames in finite dimensional vector spaces. More specifically we show how to perform a frame expansion using the dual frame, we investigate the properties of the eigenvalues of the frame operator and we introduce the harmonic tight frames, that are particularly well-suited for our application.

We then show how frames can be useful in lossy signal transmission, where the signal is subject to errors modelled as the loss of some coefficients, referred to as erasures. We find that uniform tight frames minimize the error from quantization and minimize the average and worst-case MSE caused by a single erasure, as well as giving nice results for the multiple erasure case.

Finally we demonstrate numerically the advantages of using frames to achieve robust signal transmission.

Preface

This report is the result of a combined Bachelor project (for the first author) and a special course (for the second author) at the Department of Mathematics, MAT, at the Technical University of Denmark, DTU. The subject is finite dimensional frames and their application on robust signal transmission. The work is carried out in the spring of 2007 and the extent of the project is 15 ECTS points.

We would like to take this opportunity to thank our supervisor Ole Christensen for a very delightful collaboration and many inspiring mathematical conversations.

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1 Preliminaries

Definition 1.1 *Throughout this report we will denote matrices by upper-case and vectors by lower-case letters, e.g.*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Definition 1.2 *For a complex matrix $A = [a_{ij}] \in \mathbb{C}^{n \times m}$ we denote by $A^* = [a^*_{ij}]$ the* Hermitian transpose *of A, defined by*

$$a^*_{ij} = \overline{a_{ji}} \quad \forall i, j.$$

That is, A^* is obtained by transposing A and complex conjugating each entry.

Definition 1.3 *In $\mathbb{C}^N$ we will use the inner product $\langle \cdot, \cdot \rangle : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$*

$$\langle x, y \rangle = \sum_{i=1}^{N} x_i \overline{y_i} = y^* x, \quad \text{for} \quad x, y \in \mathbb{C}^N.$$

Definition 1.4 *For two matrices $A, B \in \mathbb{C}^{N \times N}$ we define the partial ordering "$\leq$" by*

$$\begin{aligned} & A \leq B \\ \Leftrightarrow \quad & x^* A x \leq x^* B x \quad \forall x \in \mathbb{C}^N, \end{aligned}$$

whenever the numbers x^*Ax and x^*Bx are real.

Definition 1.5 Kronecker's delta function δ_{ij} *is defined by*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Definition 1.6 *In $\mathbb{R}^N$ we denote the set of all unit vectors by S^{N-1}:*

$$S^{N-1} = \{x \in \mathbb{R}^N \mid \|x\| = 1\}.$$

The set is called the unit ball *in $\mathbb{R}^N$.*

2 Introduction

In this report we will demonstrate how over-complete bases, known as *frames*, can be useful in signal transmission. We will show that the over-completeness of the frame gives a robustness towards losses in the transmission.

The basis for this report has been the paper *Quantized Frame Expansions with Erasures* by Goyal, Kovacevic and Kelner, [5]. All major results in the report are from this paper. Our assignment has been to read and understand the paper and to fill out the gaps in the argumentation where necessary. This has not been an easy task, due to the lack of details in the paper. We have constructed and proved several lemmas and propositions to facilitate the more complex proofs.

In the first section we will introduce the concept of a frame and go through a number of their properties. When we have established the basic theory, we will move on to introduce the application in signal transmission and describe how frames can be useful. In the following section we continue by considering what happens when erasures are introduced, and we arrive at a family of frames that are particularly efficient for robust signal transmission. Finally we will demonstrate some of the thoretical results numerically.

Remark: Since the report is mainly derived from [5], we will not cite the paper every time a result from it is presented. In addition it should be noted that in order to maintain a flow through the report, we have stated several helpful propositions in Appendix A, B and C.

3 Theory of frames

3.1 From bases to frames

Orthonormal bases (ONB) have a wide range of applications in applied mathematics. The basic idea is to express a complicated (in some sense) function as a linear combination of a family of simple functions. If $\{e_i\}_{i\in I}$, is an orthonormal basis of a finite-dimensional vector space V, then for any $f \in V$ we can write

$$f = \sum_{i\in I} \langle f, e_i\rangle e_i, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is the inner product of V. The scalars $c_i = \langle f, e_i\rangle$ are called *the transform coefficients*. In some cases, however, an ONB is not desirable. For instance having an ONB imposes some heavy restrictions on the basis functions. They must all be orthogonal to each other and of unit length. It might be possible to have an expression similar to (3.1) where the "basis" functions can be selected with greater flexibility. These considerations give rise to the theory of *frames* - a generalization of bases.

3.2 Introduction to frames

In this section frames in finite dimensional spaces will be introduced and a number of their properties will be examined. We begin by defining a frame by the so-called *frame-condition*.

Definition 3.1 *Let $\Phi = \{\varphi_k\}_{k=1}^M$ be a set of vectors in $\mathbb{R}^N$ (or $\mathbb{C}^N$). Φ is called a* frame *if there exist $0 < A \le B < \infty$ such that*

$$A\,\|x\|^2 \le \sum_{k=1}^{M} |\langle x, \varphi_k\rangle|^2 \le B\,\|x\|^2 \qquad \forall x \in \mathbb{R}^N. \tag{3.2}$$

If we can choose $A = B$ the frame is called tight*. The number $r = M/N$ is called the* redundancy *of the frame.*

The scalars A and B are called *frame bounds*. The maximal value of A and minimal value of B are called the *optimal* frame bounds. The lower frame bound establishes a fundamental property of the set of frame vectors as explained in the following proposition.

Proposition 3.2 *Given a set of vectors $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$, the following holds:*

$$\exists A > 0 : \; A\,\|x\|^2 \le \sum_{k=1}^{M} |\langle x, \varphi_k\rangle|^2 \quad \forall x \in \mathbb{R}^N \qquad \Leftrightarrow \qquad \operatorname{span}\{\Phi\} = \mathbb{R}^N.$$

Proof. "$\Rightarrow$": The proof is by contradiction. For a set of vectors Φ we always know that $\text{span}\{\Phi\} \subseteq \mathbb{R}^N$. In order to reach a contradiction we assume $\text{span}\{\Phi\} \subset \mathbb{R}^N$. From the assumption we get $\text{span}\{\Phi\}^\perp \neq \{0\}$. For an $x \in \text{span}\{\Phi\}^\perp \backslash \{0\}$ it is true that $\forall k = 1, \ldots, M: \ \langle x, \varphi_k \rangle = 0$. This implies that $\sum_{k=1}^M |\langle x, \varphi_k \rangle|^2 = 0$ and since $x \neq 0$, we get $\|x\| \neq 0$, which forces $A = 0$.

"$\Leftarrow$": Since $\text{span}(\Phi) = \mathbb{R}^N$ we cannot have $\langle x, \varphi_k \rangle = 0$ for all k unless $x = 0$. In other words,

$$\forall x \neq 0: \qquad \sum_{k=1}^{M} |\langle x, \varphi_k \rangle|^2 > 0. \tag{3.3}$$

Define the function Λ by

$$\Lambda : S^{N-1} \to \mathbb{R}, \quad \Lambda x = \sum_{k=1}^{M} |\langle x, \varphi_k \rangle|^2 \quad \forall x \in \mathbb{R}^N.$$

The domain of definition is the unit-ball in $\mathbb{R}^N$, so only x with $\|x\| = 1$ are considered. This is sufficient since all non-zero vectors can be scaled to unit-length. Λ is continuous and due to its domain of definition it has compact support. Due to Proposition C.1 this means that the infimum of the function occurs for some x, that we will denote x_0. We can let $A = \sum_{k=1}^M |\langle x_0, \varphi_k \rangle|^2$ and then we have

$$\exists x_0: \qquad A = \sum_{k=1}^{M} |\langle x_0, \varphi_k \rangle|^2 = \inf_{\|x\|=1} \sum_{k=1}^{M} |\langle x, \varphi_k \rangle| > 0,$$

where the last inequality follows by (3.3). The infimum is smaller than or equal to any other function value which means that

$$\forall x, \|x\| = 1: \qquad \sum_{k=1}^{M} |\langle x_0, \varphi_k \rangle|^2 \leq \sum_{k=1}^{M} |\langle x, \varphi_k \rangle|^2,$$

or equivalently

$$\forall x, \|x\| = 1: \qquad A\,\|x\|^2 \leq \sum_{k=1}^{M} |\langle x, \varphi_k \rangle|^2.$$

This completes the proof when $\|x\| = 1$ and the general result follows by scaling as mentioned earlier.

$\square$

Assume that $\Phi = \{\varphi_k\}_{k=1}^M$ is a frame for $\mathbb{R}^N$. Since the lower bound implies that the frame vectors span $\mathbb{R}^N$, we must always have $M \geq N$. If we have a finite number M of vectors that span $\mathbb{R}^N$, the Cauchy-Schwartz inequality yields

$$\sum_{k=1}^{M} |\langle x, \varphi_k \rangle|^2 \leq \sum_{k=1}^{M} \|x\|^2 \|\varphi_k\|^2 = \|x\|^2 \sum_{k=1}^{M} \|\varphi_k\|^2, \tag{3.4}$$

Name	Abbr.	Description	Alternate name
Orthonormal basis	ONB	$\langle \varphi_i, \varphi_j \rangle = \delta_{ij}, \forall i, j$	
Uniform frame	UNF	$\|\|\varphi_i\|\| = 1, \forall\, i$	Unit-norm Frame
Tight frame	TF	$A = B$	
Parseval tight frame	PTF	$A = B = 1$	
Uniform tight frame	UNTF	$A = B, \|\|\varphi_i\|\| = 1, \forall\, i$	Unit-norm tight frame

Table 1: Short description of various frames, inspired by [7]. It is assumed that the considered set of vectors form a frame. The abbreviations UNF and UNTF are related to their alternate names.

which shows that we can choose $B = \sum_{k=1}^{M} \|\varphi_k\|^2$. In other words, *any finite number of vectors that span* $\mathbb{R}^N$ *will constitute a frame*.

There exist different classes of frames as seen in Table 1. We have already mentioned tight frames (TF), that satisfy $A = B$. Furthermore if $A = B = 1$ the frame is called a *Parseval tight frame* (PTF). A frame where all the vectors have the same norm is called *equal-norm frame* and if the norm is 1 it is called *unit-norm* or *uniform* (UNF). Finally if a frame is both tight and uniform it is called a *uniform tight frame* (UNTF). For reasons to follow our primary concern will be uniform tight frames.

At this point we give a simple example to make the concept of a frame a little more concrete.

Example 3.3 *Consider in* $\mathbb{R}^2$ *the set of vectors* $\Phi = \{\varphi_k\}_{k=1}^{3}$, *where*

$$\varphi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \qquad \varphi_3 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

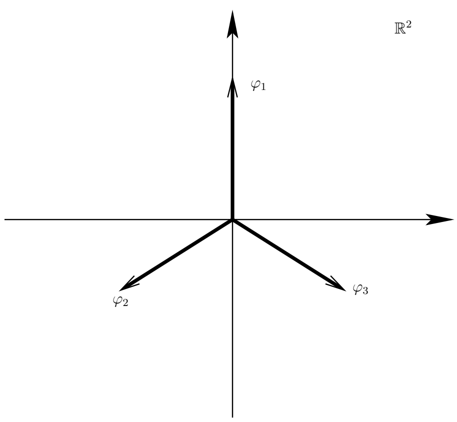

Figure 1: The Mercedes Benz frame.

For any $x \in \mathbb{R}^2$ *we have by an elementary calculation*

$$\sum_{k=1}^{3} | \langle x, \varphi_k \rangle |^2 = \frac{3}{2} \|x\|^2,$$

so the frame is tight with $A = B = 3/2$*. This frame in particular is called the* Mercedes Benz frame*. The frame is also uniform since*

$$\forall k = 1, 2, 3 : \ \|\varphi_k\| = 1.$$

The frame is therefore a uniform tight frame.

We are interested in obtaining an expression similar to (3.1) for frames. If this is possible, we will have a way to express any vector from the considered space in terms of the frame vectors. In order to do so, it will be useful to have an operator to express the transform from the vector space to the transform coefficients.

Definition 3.4 *For a frame* $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{R}^N$ *the corresponding* analysis operator F *is defined by*

$$F : \mathbb{R}^N \to \mathbb{R}^M : \quad (Fx)_k = \langle x, \varphi_k \rangle \quad \forall k \in \{1, \dots, M\}, \tag{3.5}$$

for an $x \in \mathbb{R}^N$*.*

Since the operator for each frame vector computes the inner product with x, the operator can be conveniently expressed using a matrix containing all the frame vectors conjugate transposed in the rows:

$$F = \begin{pmatrix} — & \varphi_1^* & — \\ — & \varphi_2^* & — \\ & \vdots & \\ — & \varphi_M^* & — \end{pmatrix}. \tag{3.6}$$

We will let F refer both to the analysis operator and the matrix representation of it. The frame condition (3.2) can be expressed in terms of the analysis operator:

Proposition 3.5 *For a frame the frame condition* (3.2) *can be equivalently expressed by use of the partial ordering "*$\leq$*" from Definition 1.4 as*

$$AI_N \leq F^*F \leq BI_N, \tag{3.7}$$

where F *is the corresponding analysis operator. The composed operator* F^*F *is called the* frame operator*. In particular for a tight frame* $F^*F = AI_N$*.*

Proof. The left-hand side of (3.2) is rewritten

$$A\|x\|^2 = A \langle x, x \rangle = Ax^*x = x^*AI_Nx.$$

Similarly we get $B\|x\|^2 = x^* B I_N x$. For F^*F we get

$$\begin{aligned} x^* F^* F x = (Fx)^*(Fx) &= \begin{bmatrix} \cdots & \overline{\langle x, \varphi_k \rangle} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \langle x, \varphi_k \rangle \\ \vdots \end{bmatrix} \\ &= \sum_{k=1}^{M} \langle x, \varphi_k \rangle \, \overline{\langle x, \varphi_k \rangle} = \sum_{k=1}^{M} | \langle x, \varphi_k \rangle |^2. \end{aligned}$$

Substituting the three expressions into (3.2) we obtain

$$x^* A I_N x \leq x^* F^* F x \leq x^* B I_N x,$$

which is equivalent to (3.7) by Definition 1.4.

For a tight frame we have $A = B$, and $F^*F = AI_N$ follows directly.

□

We have now defined the frame operator F^*F, but what happens when it is applied to an $x \in \mathbb{R}^N$? It turns out that it can help us get an expression similar to (3.1) using frames. First of all, we know by (3.5) that

$$Fx = \begin{pmatrix} \langle x, \varphi_1 \rangle \\ \langle x, \varphi_2 \rangle \\ \vdots \\ \langle x, \varphi_M \rangle \end{pmatrix}.$$

For F^*Fx we get

$$\begin{aligned} F^* F x &= \begin{pmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_M \\ | & | & & | \end{pmatrix} \begin{pmatrix} \langle x, \varphi_1 \rangle \\ \langle x, \varphi_2 \rangle \\ \vdots \\ \langle x, \varphi_M \rangle \end{pmatrix} \\ &= \varphi_1 \langle x, \varphi_1 \rangle + \varphi_2 \langle x, \varphi_2 \rangle + \cdots + \varphi_M \langle x, \varphi_M \rangle \\ &= \sum_{k=1}^{M} \langle x, \varphi_k \rangle \, \varphi_k. \end{aligned} \tag{3.8}$$

In order to proceed we need the following lemma about the operator F^*F.

Lemma 3.6 *The frame operator F^*F is self-adjoint and invertible.*

Proof. The two postulates are considered seperately:

- Self-adjoint: $(F^*F)^* = F^* (F^*)^* = F^*F$.

- Invertible: From Proposition 3.11 that we will prove later, we know that all the eigenvalues of F^*F are nonzero. This implies that F^*F is invertible.

□

By the lemma we know that $(F^*F)^{-1}$ exists. According to (3.8), we have for $x \in \mathbb{R}^N$

$$x = (F^*F)(F^*F)^{-1} x = \sum_{k=1}^{M} \left\langle (F^*F)^{-1} x, \varphi_k \right\rangle \varphi_k.$$

We also know that F^*F is self-adjoint, so $(F^*F)^{-1}$ is self-adjoint and we can continue with

$$x = \sum_{k=1}^{M} \left\langle x, (F^*F)^{-1} \varphi_k \right\rangle \varphi_k. \tag{3.9}$$

The vectors $(F^*F)^{-1}\varphi_k$ are very important and have their own name, as explained by the following theorem.

Theorem 3.7 *For a frame $\Phi = \{\varphi_k\}_{k=1}^M$ with bounds A and B and analysis operator F, the set $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k=1}^M$, where*

$$\widetilde{\varphi}_k = (F^*F)^{-1}\varphi_k \qquad \forall k = 1, \ldots, M \tag{3.10}$$

is a frame with bounds B^{-1} and A^{-1}. The frame is called the canonical dual frame *of Φ and has the analysis operator*

$$\widetilde{F} = F(F^*F)^{-1} \tag{3.11}$$

and frame operator

$$\widetilde{F}^*\widetilde{F} = (F^*F)^{-1}. \tag{3.12}$$

Proof. We begin by finding the expressions for $\widetilde{F}$ and $\widetilde{F}^*\widetilde{F}$. By hermitian transposition of (3.10) we get for the k'th dual frame vector (using Lemma 3.6)

$$\begin{aligned}\widetilde{\varphi}_k^* &= \left((F^*F)^{-1}\varphi_k\right)^* = \varphi_k^* \left((F^*F)^{-1}\right)^* \\ &= \varphi_k^* \left((F^*F)^*\right)^{-1} = \varphi_k^*(F^*F)^{-1}.\end{aligned}$$

To form $\widetilde{F}$ all the $\widetilde{\varphi}_k^*$ are stacked:

$$\widetilde{F} = \begin{pmatrix} — & \widetilde{\varphi}_1^* & — \\ — & \widetilde{\varphi}_2^* & — \\ & \vdots & \\ — & \widetilde{\varphi}_M^* & — \end{pmatrix} = \begin{pmatrix} — & \varphi_1^* & — \\ — & \varphi_2^* & — \\ & \vdots & \\ — & \varphi_M^* & — \end{pmatrix} (F^*F)^{-1} = F(F^*F)^{-1},$$

which proves (3.11). $\widetilde{F}^*\widetilde{F}$ can then be found to be

$$\begin{aligned}\widetilde{F}^*\widetilde{F} &= \left(F(F^*F)^{-1}\right)^* \left(F(F^*F)^{-1}\right) = \left((F^*F)^{-1}\right)^* F^*F(F^*F)^{-1}\\ &= ((F^*F)^*)^{-1} = (F^*F)^{-1},\end{aligned}$$

which proves (3.12).

Now to show that $\widetilde{\Phi}$ is a frame, we want to apply Proposition B.4. From equation (3.7) we know that $AI_N \leq F^*F \leq BI_N$. By Lemma 3.6 F^*F is self-adjoint and hence so is $(F^*F)^{-1}$. Also, AI_N and BI_N are trivially self-adjoint. Furthermore it is easy to see that $(F^*F)^{-1}$ commutes with AI_N, F^*F and BI_N, so by Proposition B.4

$$(F^*F)^{-1} AI_N \leq (F^*F)^{-1} (F^*F) \leq (F^*F)^{-1} BI_N,$$

which reduces to

$$\begin{aligned}(F^*F)^{-1} &\leq \frac{1}{A} I_N\\ (F^*F)^{-1} &\geq \frac{1}{B} I_N.\end{aligned}$$

Since we have proved $(F^*F)^{-1}$ is the frame operator for the dual frame, (3.7) gives that $\widetilde{\Phi}$ is a frame with bounds B^{-1} and A^{-1}.

$\square$

Although there frequently exist other frames that are dual to Φ, the term canonical will often be omitted. Using the (canonical) dual frame we can rewrite (3.9) and we end up with the expression

$$x = \sum_{k=1}^{M} \langle x, \widetilde{\varphi}_k \rangle \, \varphi_k \qquad \forall x \in \mathbb{R}^N. \tag{3.13}$$

In general it is not an easy task to compute the dual frame[1], but if the frame is tight there is a simple result. For a tight frame we know from Proposition 3.5 that $F^*F = AI_N$. This means that $(F^*F)^{-1} = 1/A \cdot I_N$ and the definition of the dual frame gives us $\widetilde{\varphi}_k = 1/A \cdot \varphi_k$, $\forall k = 1, \ldots, M$. Insertion of this result into (3.9) yields

$$x = \sum_{k=1}^{M} \left\langle x, \frac{1}{A} I_N \varphi_k \right\rangle \varphi_k \qquad \forall x \in \mathbb{R}^N,$$

or simpler

$$x = \frac{1}{A} \sum_{k=1}^{M} \langle x, \varphi_k \rangle \, \varphi_k \qquad \forall x \in \mathbb{R}^N. \tag{3.14}$$

Except for the scaling factor $1/A$ this is exactly the same expression as (3.1) for an orthonormal basis! This is a very powerful result.

[1]This is discussed on page 365 in [1].

Example 3.8 *In Example 3.3 we found that the Mercedes Benz frame is tight with $A = 3/2$. Now (3.14) gives us the following representation of any vector from $\mathbb{R}^2$:*

$$x = \frac{2}{3}\sum_{k=1}^{3} \langle x, \varphi_k\rangle\ \varphi_k \qquad \forall x \in \mathbb{R}^2.$$

If for instance we look at $x_0 = (5,1)^T$, we find the following inner products:

$$\begin{aligned}\langle x_0, \varphi_1\rangle &= 1,\\ \langle x_0, \varphi_2\rangle &= -\frac{5\sqrt{3}+1}{2},\\ \langle x_0, \varphi_3\rangle &= \frac{5\sqrt{3}-1}{2}.\end{aligned}$$

We form the sum:

$$\begin{aligned}x_0 &= \frac{2}{3}\sum_{k=1}^{3} \langle x_0, \varphi_k\rangle\ \varphi_k\\ &= \frac{2}{3}\left(1 \cdot \begin{pmatrix} 0\\ 1\end{pmatrix} - \frac{5\sqrt{3}+1}{2} \cdot \begin{pmatrix} -\frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{pmatrix} + \frac{5\sqrt{3}-1}{2} \cdot \begin{pmatrix} \frac{\sqrt{3}}{2}\\ -\frac{1}{2}\end{pmatrix}\right),\end{aligned}$$

and we have hereby obtained an explicit expression for x_0 in terms of the frame vectors.

Let us look a bit more at the relationship between a frame and its dual. It turns out that the dual frame of the dual is the original frame itself, as expressed by the following theorem.

Theorem 3.9 *Given a frame Φ and its dual $\widetilde{\Phi}$. The dual of $\widetilde{\Phi}$, denoted $\widetilde{\widetilde{\Phi}}$, is equal to the original frame, that is $\widetilde{\widetilde{\Phi}} = \Phi$.*

Proof. Using the definition of the dual frame from Theorem 3.7 twice we get

$$\begin{aligned}\widetilde{\varphi}_k &= (F^*F)^{-1}\varphi_k \qquad \forall k = 1, \dots, M,\\ \widetilde{\widetilde{\varphi}}_k &= (\widetilde{F}^*\widetilde{F})^{-1}\widetilde{\varphi}_k \qquad \forall k = 1, \dots, M.\end{aligned}$$

By the expression (3.12) and insertion of the first equation into the second we find

$$\begin{aligned}\widetilde{\widetilde{\varphi}}_k &= \left((F^*F)^{-1}\right)^{-1}\widetilde{\varphi}_k = (F^*F)\widetilde{\varphi}_k\\ &= (F^*F)(F^*F)^{-1}\varphi_k = \varphi_k \qquad \forall k = 1, \dots, M.\end{aligned}$$

□

So there is in fact a duality between a frame and its dual. Using this relationship we can see that the roles of the frame and its dual can be interchanged in (3.13), which is expressed be the following theorem.

Theorem 3.10 *Given a frame* $\Phi = \{\varphi_k\}_{k=1}^M$ *and its dual* $\widetilde{\Phi} = \{\widetilde{\varphi}_k\}_{k=1}^M$ *for* $\mathbb{R}^N$*. For any vector* $x \in \mathbb{R}^N$ *it holds that*

$$x = \sum_{k=1}^{M} \langle x, \widetilde{\varphi}_k \rangle \, \varphi_k \qquad \textit{and} \qquad x = \sum_{k=1}^{M} \langle x, \varphi_k \rangle \, \widetilde{\varphi}_k \qquad \forall x \in \mathbb{R}^N. \tag{3.15}$$

This is a useful fact. The latter is the frame expansion that we will use later on in the application on robust signal transmission.

The coefficients of the frame expansion are the elements of Fx as expressed by (3.6). Given the coefficients, the vector x can be reconstructed using the left-inverse $F^\dagger$, given by $F^\dagger = (F^*F)^{-1}F^*$. It is easy to see that $F^\dagger$ acts as an inverse, since

$$F^\dagger F x = (F^*F)^{-1}F^*Fx = x.$$

Using (3.11) the left-inverse can also be written in terms of the dual analysis operator:

$$F^\dagger = (F^*F)^{-1}F^* = \left(F(F^*F)^{-1}\right)^* = \widetilde{F}^*, \tag{3.16}$$

where it has been used that $(F^*F)^{-1}$ is self-adjoint. The left-inverse will be needed later for reconstructing signals from their frame coefficients.

3.3 Properties of frames

In the following a number of properties of finite dimensional frames are stated and proved. The main part of the properties are concerned with the eigenvalues of the frame operator. The results will be needed in more advanced proofs later in the report.

Proposition 3.11 *Given a frame with frame bounds* A *and* B*, all the eigenvalues of* F^*F *belong to the interval* $[A, B]$*. In particular, for a tight frame all the eigenvalues are equal to* A*.*

Proof. Take any eigenpair (λ, v) of F^*F, where v is non-zero, then $F^*Fv = \lambda v$. Multiplying by v^* yields

$$v^*F^*Fv = v^*\lambda v = v^*\lambda I_N v.$$

By applying (3.7) to v we get

$$v^*AI_Nv \leq v^*\lambda I_N v \leq v^*BI_Nv,$$

which reduces to $A \leq \lambda \leq B$. Since for a tight frame $B = A$, all the eigenvalues of a tight frame are equal to A.

□

Proposition 3.12 *For a frame $\{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ with analysis operator F the sum of the eigenvalues of F^*F equals the sum of the squared lengths of the frame vectors:*

$$\sum_{i=1}^{N} \lambda_i = \sum_{i=k}^{M} \|\varphi_k\|^2.$$

If the frame is uniform as well, the sum of the eigenvalues equals M.

Proof. The eigenvalues of F^*F are called $\{\lambda_i\}_{i=1}^N$. By Proposition A.7 and Proposition A.4 we have

$$\sum_{i=1}^{N} \lambda_i = \mathrm{tr}(F^*F) = \mathrm{tr}(FF^*).$$

Using (3.6) we see that the k'th diagonal element of

$$FF^* = \begin{pmatrix} - & \varphi_1^* & - \\ - & \varphi_2^* & - \\ & \vdots & \\ - & \varphi_M^* & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_M \\ | & | & & | \end{pmatrix}$$

can be written as $\varphi_k^*\varphi_k = \|\varphi_k\|^2$. Using this we obtain

$$\mathrm{tr}(FF^*) = \sum_{k=1}^{M} \varphi_k^*\varphi_k = \sum_{k=1}^{M} \|\varphi_k\|^2,$$

or

$$\sum_{i=1}^{N} \lambda_i = \sum_{k=1}^{M} \|\varphi_k\|^2.$$

When the frame is uniform, all the lengths are 1 and this simplifies to

$$\sum_{i=1}^{N} \lambda_i = \sum_{k=1}^{M} 1 = M.$$

□

Proposition 3.13 *Given a tight frame in $\mathbb{R}^N$ with analysis operator F, the matrix F^*F has eigenvalue A with multiplicity N. If the frame is uniform, then additionally $A = M/N = r$ holds.*

Proof. By (3.7) we have $F^*F = AI_N$ due to tightness of the frame. By Proposition 3.11 we know that all the eigenvalues must be equal to A. Since an $N \times N$

matrix has exactly N eigenvalues, the first part is proved. If the frame is uniform, we have by Proposition 3.12

$$\sum_{i=1}^{N} \lambda_i = M.$$

Since all eigenvalues are equal to A this simplifies to $NA = M$, or

$$A = \frac{M}{N} = r.$$

□

Let us continue with a result for tight frames. We know that the φ_k^* constitute the rows of F, but consider for now the columns of F instead. Denote by f_i the i'th column of F. For a tight frame we know that

$$F^*F = \begin{pmatrix} — & f_1^* & — \\ — & f_2^* & — \\ & \vdots & \\ — & f_N^* & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ f_1 & f_2 & \cdots & f_N \\ | & | & & | \end{pmatrix} = AI_N$$

from which it is seen that

$$\forall i, j = 1, \ldots, N : \quad f_i^* f_j = A\delta_{ij}. \tag{3.17}$$

In other words, the N columns of F are orthogonal. Since $N \leq M$ we can extend F by adding $M - N$ orthogonal columns to obtain a basis for $\mathbb{R}^M$. This is done using the Gram-Schmidt method. If in addition the frame is uniform, we have the following result.

Proposition 3.14 *Given a uniform tight frame Φ with analysis operator F of size $M \times N$. Then there exists an orthogonal $M \times M$ matrix U such that F is the first N columns of $\sqrt{M/N} \cdot U$.*

Proof. By Proposition 3.13 we have $F^*F = M/N \cdot I_N$. From the expression (3.17) we get that

$$\forall i = 1, \ldots, N : \qquad \|f_i\| = (f_i^* f_i)^{1/2} = \sqrt{M/N}.$$

This means that all of the N columns in the $M \times N$ matrix F has Euclidian norm $\sqrt{M/N}$. Since Φ is a frame, the rows and hence also the columns span an N-dimensional subspace of the M-dimensional space. We now append $M - N$ columns to F, such that the all the columns are linearly independent. Now the columns span all M dimensions.

Apply the Gram-Schmidt procedure on the columns and scale each vector to length $\sqrt{M/N}$ to obtain an orthogonal basis. This does not affect the original columns of F, because (3.17) holds. Denote the obtained matrix by V. By scaling all the columns by the factor $\sqrt{N/M}$ we get an orthonormal basis, which is equivalent to saying that the matrix $U = \sqrt{N/M} \cdot V$ is orthogonal.

Now by taking the first N columns of $V = \sqrt{M/N} \cdot U$ we get exactly F, and the proof is complete.

□

To clarify the procedure described in the proof, we give the following example.

Example 3.15 *Consider the Mercedes-Benz frame again. To the frame operator F we append the vector $(0, 0, 1)^T$, so we have the following matrix*

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 \end{array}\right).$$

We now apply the Gram-Schmidt procedure to obtain another vector a that is orthogonal to the other two:

$$\begin{aligned} a &= f_3 - \sum_{k=1}^{2} \left\langle \frac{f_k}{\|f_k\|}, f_3 \right\rangle \frac{f_k}{\|f_k\|} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{1}{\sqrt{6}} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}. \end{aligned}$$

This vector is scaled to the length $\sqrt{M/N} = \sqrt{3/2}$ and the obtained vector is

$$\sqrt{\frac{3}{2}} \frac{a}{\|a\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

This vector is appended to F instead of $(0, 0, 1)^T$ to get the matrix V. Now the matrix

$$U = \sqrt{\frac{2}{3}} V = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

is orthogonal, since it is easily checked that

$$U^* U = U U^* = I_3.$$

As the proposition states, we have now found an orthogonal matrix U such that F can be taken as the first 2 columns of $V = \sqrt{3/2} \cdot U$.

3.4 Equivalence between frames

Consider the Mercedes Benz frame from Example 3.3. Rotate the three vectors some fixed angle around the origo and look at the obtained system. Although we have a new set of vectors, the relative position of the vectors have not changed. They still form the Mercedes Benz star and they still form a frame, see Figure 2. In some sense the frame has not changed under this operation. We want to formalize this notion of having frames that in some sense are the same. To do this we introduce an equivalence relation between frames. It will be natural to include not only rotations but also more general operations, e.g. reflection of the entire frame in a hyperplane[2] and change of sign of some of the frame vectors.

Definition 3.16 *In the vector space $\mathbb{R}^N$ two frames $\Phi = \{\varphi_k\}_{k=1}^M$ and $\Psi = \{\psi_k\}_{k=1}^M$, with analysis operators F_Φ and F_Ψ respectively, are said to be* equivalent *if we can write*

$$\psi_k = \sigma_k U \varphi_k \qquad \forall k = 1, \ldots, M, \tag{3.18}$$

where σ_k is either 1 or -1 and U is some unitary matrix. If Φ and Ψ are equivalent we write

$$\Phi \simeq \Psi.$$

Notice that $\sigma_k = -1$ corresponds to negating the k'th vector and that U captures the unitary operations. In $\mathbb{R}^2$ this corresponds to rotations and reflections, while in higher dimensional spaces the interpretation is more complex. We have $\psi_k^* = \sigma_k \varphi_k^* U^*$. The ψ_k^* are stacked to form the analysis operator F_Ψ from F_Φ:

$$\begin{pmatrix} & \vdots & \\ - & \psi_k^* & - \\ & \vdots & \end{pmatrix} = \begin{pmatrix} \ddots & & \\ & \sigma_k & \\ & & \ddots \end{pmatrix} \begin{pmatrix} & \vdots & \\ - & \varphi_k^* & - \\ & \vdots & \end{pmatrix} U^*.$$

By letting $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_M)$ and using the analysis operators this can be written

$$F_\Psi = \Sigma F_\Phi U^*. \tag{3.19}$$

(3.19) is a useful way to express (3.18) using the analysis operators.

Theorem 3.17 *The relation "$\simeq$" from Definition 3.16 is an equivalence relation.*

Proof. To prove that "$\simeq$" is an equivalence relation we have to prove that it is reflexive, symmetric and transitive. By $\mathcal{F}(\mathbb{R}^N)$ we denote the set of frames for $\mathbb{R}^N$.

[2] A hyperplane is a generalization of the concept of a line in the 2-dimensional space or a plane in the 3-dimensional space. A reflection in a hyperplane in the 2-dimensional space therefore corresponds to the reflection of all frame vectors in a line and similarly in higher dimensions.

- Reflexive: $\forall \Phi \in \mathcal{F}(\mathbb{R}^N) : \ \Phi \simeq \Phi$. Let $U = I_N$ and $\forall k = 1, \ldots, M$: $\sigma_k = 1$. Then U is trivially unitary and

$$\varphi_k = \sigma_k U \varphi_k \qquad \forall k = 1, \ldots, M.$$

- Symmetric: $\forall \Phi, \Psi \in \mathcal{F}(\mathbb{R}^N) : \ \Phi \simeq \Psi \quad \Rightarrow \quad \Psi \simeq \Phi$. Assume that $\Phi \simeq \Psi$, then

$$\psi_k = \sigma_k U \varphi_k \qquad \forall k = 1, \ldots, M.$$

Since U is unitary, multiplication by $\sigma_k U^*$ yields

$$\sigma_k U^* \psi_k = \varphi_k \qquad \forall k = 1, \ldots, M,$$

since $\sigma_k \cdot \sigma_k = 1 \ \forall k = 1, \ldots, M$. U^* is unitary since U is unitary, so the relation is symmetric.

- Transitive: $\forall \Phi, \Psi, \Omega \in \mathcal{F}(\mathbb{R}^N) : \ (\Phi \simeq \Psi) \wedge (\Psi \simeq \Omega) \quad \Rightarrow \quad \Phi \simeq \Omega$. We have

$$\begin{aligned} \psi_k &= \sigma_k U_1 \varphi_k & \forall k = 1, \ldots, M, \\ \omega_k &= \widetilde{\sigma}_k U_2 \psi_k & \forall k = 1, \ldots, M \end{aligned}$$

for some sequences $\{\sigma_k\}_{k=1}^M$, $\{\widetilde{\sigma}_k\}_{k=1}^M$ and unitary matrices U_1, U_2. By substituting the first expression into the second we obtain

$$\omega_k = \widetilde{\sigma}_k \sigma_k U_2 U_1 \varphi_k \qquad \forall k = 1, \ldots, M.$$

Let $\widehat{\sigma}_k = \widetilde{\sigma}_k \sigma_k$, $\forall k = 1, \ldots, M$ and the $\widehat{\sigma}_k$ will all be 1 or -1. Let $U = U_2 U_1$ and by Proposition B.3 U is unitary. We have

$$\omega_k = \widehat{\sigma}_k U \varphi_k \qquad \forall k = 1, \ldots, M.$$

□

Example 3.18 *To illustrate the concept of equivalence between frames we give the following example. In $\mathbb{R}^2$ take the Mercedes-Benz frame and rotate the three vectors some fixed angle, see Figure 2. Another example is seen in Figure 3 where two of the vectors are changed to go in the opposite directions. These two frames are both equivalent to the Mercedes Benz frame from Figure 1.*

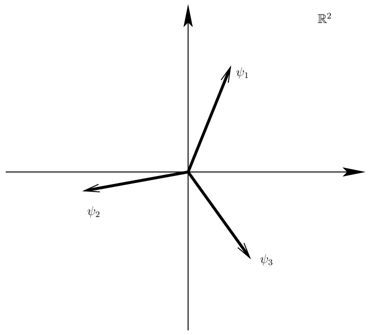

Figure 2: Rotated Mercedes-Benz frame

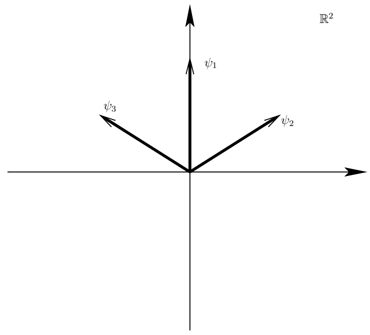

Figure 3: Flipped Mercedes-Benz frame

Corollary 3.19 *Tightness of a frame Φ implies tightness of any frame equivalent to Φ. The same is true for uniformity.*

Proof. Consider now two equivalent frames Φ and Ψ in $\mathbb{R}^N$, with $F_\Psi = \Sigma F_\Phi U^*$ as in (3.19). If Φ is tight, it follows that Ψ is tight as well, since by Proposition 3.5 $F_\Phi^* F_\Phi = AI_N$ and

$$\begin{aligned} F_\Psi^* F_\Psi &= (\Sigma F_\Phi U^*)^* \Sigma F_\Phi U^* = U F_\Phi^* \Sigma^* \Sigma F_\Phi U^* \\ &= U F_\Phi^* F_\Phi U^* = U A I_N U^* = A I_N, \end{aligned}$$

where it is used that $\Sigma^* \Sigma = I_M$.

If Φ is uniform we have $\forall k = 1, \dots, M : \ \|\varphi_k\| = 1$ and by (3.18)

$$\|\psi_k\| = \|\sigma_k U \varphi_k\| = |\sigma_k| \cdot \|U \varphi_k\| = \|\varphi_k\| = 1 \qquad \forall k = 1, \dots, M,$$

since a unitary matrix preserves the norm. Hence, Ψ is uniform as well.

□

As it is known from basic algebra an equivalence relation splits the considered space into equivalence classes. This is useful when trying to get an overview of the frames for a specific vector space. In one case there is a particularly simple result: It turns out that there is only one equivalence class of uniform tight frames if $M = N + 1$. To prove this we need the following lemma.

Lemma 3.20 *Given two orthonormal bases $\{f_k\}_{k=1}^N$ and $\{g_k\}_{k=1}^N$ for a vector space $\mathbb{R}^N$. Then the bases are equivalent with respect to "$\simeq$".*

Proof. Let $\{f_k\}_{k=1}^N$ and $\{g_k\}_{k=1}^N$ be orthonormal bases of $\mathbb{R}^N$ and $\{e_k\}_{k=1}^N$ be the canonical basis of $\mathbb{R}^N$, that is $e_k = (0, 0, \dots, 0, 1, 0, \dots, 0)$, $\forall k = 1, \dots, N$ with a 1 in the k'th position. If we can prove that $\{f_k\}_{k=1}^N$ is equivalent to $\{e_k\}_{k=1}^N$, then the same holds for $\{g_k\}_{k=1}^N$ and transitivity of the equivalence relation will imply that $\{f_k\}_{k=1}^N$ is equivalent to $\{g_k\}_{k=1}^N$.

It is easily seen that the following holds:

$$\forall k = 1, \dots, N \qquad f_k = \begin{pmatrix} | & | & & | \\ f_1 & f_2 & \cdots & f_N \\ | & | & & | \end{pmatrix} e_k.$$

Since $\{f_k\}_{k=1}^N$ is an orthonormal basis the matrix is unitary. By Definition 3.16 we get that $\{f_k\}_{k=1}^N \simeq \{e_k\}_{k=1}^N$. The argument applies to $\{g_k\}_{k=1}^N$ as well, and transitivity then yields $\{f_k\}_{k=1}^N \simeq \{g_k\}_{k=1}^N$ as mentioned.

$\square$

Theorem 3.21 *Consider the vector space $\mathbb{R}^N$. All uniform tight frames $\Phi = \{\varphi_k\}_{k=1}^M$ with $M = N+1$ belong to the same equivalence class.*

Proof. Let Φ be a uniform tight frame for $\mathbb{R}^N$ with analysis operator F and $M = N+1$. By Proposition 3.14 there exists an orthogonal matrix U such that F is the first N columns of $\overline{F} = \sqrt{M/N} \cdot U$. In an orthogonal matrix each row and column has norm 1. That means that each row of $\overline{F}$ has norm $\sqrt{M/N}$, or since $M = N+1$

$$\sum_{j=1}^{N+1} \overline{F}_{ij}^2 = \frac{N+1}{N} \qquad \text{for} \qquad i = 1, 2, \dots, N+1.$$

The uniformity of the frame implies that $\|\varphi_k^*\| = \|\varphi_k\| = 1, \forall k = 1, 2, \dots, N+1$ or

$$\sum_{j=1}^{N} \overline{F}_{ij}^2 = 1 \qquad \text{for} \qquad i = 1, 2, \dots, N+1,$$

since the φ_k^* constitute the rows of F. Subtraction of this from the previous expression leaves only the last term, so

$$\overline{F}_{i,N+1}^2 = \frac{1}{N} \qquad \text{for} \qquad i = 1, 2, \dots, N+1.$$

This means that the last column of $\overline{F}$ contains the vector

$$\frac{1}{\sqrt{N}} \begin{pmatrix} \pm 1 \\ \pm 1 \\ \vdots \\ \pm 1 \end{pmatrix}$$

for some choice of signs. Since $\overline{F}$ is a scaled orthogonal matrix, the columns are orthogonal to each other, and in particular the N first columns are orthogonal to the last. In other words the span of the N first columns is equal to the orthogonal complement to $(\pm 1, \pm 1, \ldots, \pm 1)$. That is, if we denote the j'th column of $\overline{F}$ by $\overline{f}_j$ we have

$$\text{span}\left\{\overline{f}_j\right\}_{j=1}^{N} = \text{span}\left\{\overline{f}_{N+1}\right\}^{\perp}.$$

For a given $\overline{f}_{N+1}$ we have a fixed N-dimensional subspace with the orthonormal basis $\left\{\sqrt{N/M} \cdot \overline{f}_k\right\}_{k=1}^{N}$, where the scaling factor $\sqrt{N/M}$ comes from $\overline{F} = \sqrt{M/N} \cdot U$. By Lemma 3.20 we know that all orthonormal bases of the subspace are equivalent, so all frames with this choice of $\overline{f}_{N+1}$ are equivalent. For another choice of $\overline{f}_{N+1}$ only the signs change. This can be handled by changing the signs of the corresponding σ_k from Definition 3.16. So no matter which $\overline{f}_{N+1}$ is chosen, the frames are equivalent.

□

It is not possible to generalize this result to when M exceeds $N+1$. However, in $\mathbb{R}^2$ the uniform tight frames are completely characterized, as expressed by the following theorem.

Theorem 3.22 *Consider the vector space $\mathbb{R}^2$ and a sequence of real scalars $\{\alpha_k\}_{k=1}^{M}$. The following statements are equivalent:*

1. $\Phi = \{\varphi_k = (\cos\alpha_k, \sin\alpha_k)\}_{k=1}^{M}$ *is a uniform tight frame,*

2. $\sum_{k=1}^{M} z_k = 0$, *where* $z_k = e^{2i\alpha_k}$ *for* $k = 1, \ldots, M$.

Proof. "1. ⇒ 2.": The analysis operator of Φ is

$$F = \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 \\ \cos\alpha_2 & \sin\alpha_2 \\ \vdots & \vdots \\ \cos\alpha_M & \sin\alpha_M \end{pmatrix}.$$

Since the frame is uniform and tight, Proposition 3.13 gives

$$\begin{aligned} F^*F &= \begin{pmatrix} \cos\alpha_1 & \cos\alpha_2 & \cdots & \cos\alpha_M \\ \sin\alpha_1 & \sin\alpha_2 & \cdots & \sin\alpha_M \end{pmatrix} \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 \\ \cos\alpha_2 & \sin\alpha_2 \\ \vdots & \vdots \\ \cos\alpha_M & \sin\alpha_M \end{pmatrix} \\ &= \begin{pmatrix} \frac{M}{2} & 0 \\ 0 & \frac{M}{2} \end{pmatrix}. \end{aligned} \tag{3.20}$$

This yields the following:

$$\sum_{k=1}^{M} \cos^2 \alpha_k = \frac{M}{2}, \tag{3.21}$$

$$\sum_{k=1}^{M} \sin^2 \alpha_k = \frac{M}{2}, \tag{3.22}$$

$$\sum_{k=1}^{M} \cos \alpha_k \sin \alpha_k = 0. \tag{3.23}$$

By subtracting (3.22) from (3.21) we get

$$\sum_{k=1}^{M} \cos^2 \alpha_k - \sum_{k=1}^{M} \sin^2 \alpha_k = \frac{M}{2} - \frac{M}{2},$$

and using the identity $\cos 2x = \cos^2 x - \sin^2 x$ we find

$$\sum_{k=1}^{M} \cos 2\alpha_k = 0. \tag{3.24}$$

By multiplying (3.23) with 2 a similar result is found for the sine-function

$$2 \sum_{k=1}^{M} \cos \alpha_k \sin \alpha_k = 2 \cdot 0,$$

or equivalently

$$\sum_{k=1}^{M} \sin 2\alpha_k = 0. \tag{3.25}$$

where the identity $\sin 2x = 2 \sin x \cos x$ has been used. Taking the sum of (3.24) and i times (3.25) equals 0:

$$\sum_{k=1}^{M} \cos 2\alpha_k + i \cdot \sum_{k=1}^{M} \sin 2\alpha_k = 0,$$

or equivalently

$$\sum_{k=1}^{M} e^{2i\alpha_k} = 0.$$

Let $z_k = e^{2i\alpha_k}$ and we are done.

"2. $\Rightarrow$ 1.": Given that $\sum_{k=1}^{M} e^{2i\alpha_k} = 0$ we want to prove tightness and uniformity of $\Phi = \{\varphi_k = (\cos \alpha_k, \sin \alpha_k)\}_{k=1}^{M}$.

- Tightness: We immediately get that

$$\sum_{k=1}^{M} \cos 2\alpha_k = \sum_{k=1}^{M} \text{Re } e^{2i\alpha_k} = 0,$$
$$\sum_{k=1}^{M} \sin 2\alpha_k = \sum_{k=1}^{M} \text{Im } e^{2i\alpha_k} = 0,$$

since a complex number is zero if and only if both its real and imaginary parts are zero. As in (3.24) and (3.25) we obtain

$$\sum_{k=1}^{M} \cos^2 \alpha_k = \sum_{k=1}^{M} \sin^2 \alpha_k,$$
$$\sum_{k=1}^{M} \cos \alpha_k \sin \alpha_k = 0.$$

As in (3.20) we obtain for the frame operator

$$F^*F = \begin{pmatrix} \sum_{k=1}^{M} \cos^2 \alpha_k & 0 \\ 0 & \sum_{k=1}^{M} \sin^2 \alpha_k \end{pmatrix}.$$

Since we have just shown that the diagonal elements are equal, the frame is tight.

- Uniformity: For each frame element $\varphi_k = (\cos \alpha_k, \sin \alpha_k)$ we have

$$\|\varphi_k\| = \sqrt{\cos^2 \alpha_k + \sin^2 \alpha_k} = 1,$$

so the frame is uniform.

□

3.5 Construction of uniform tight frames

After having discussed a number of properties of finite dimensional frames, it will be interesting to know how a frame can be constructed for any given M and N. This will be useful when considering the application of frames on signal transmission later on. The family of frames we will introduce are called *harmonic frames* and they are uniform and tight as explained by the following theorem.

Theorem 3.23 *Consider the complex vectorspace $\mathbb{C}^N$. Define the set of vectors $\Phi = \{\varphi_{k+1}\}_{k=0}^{M-1}$ by*

$$(\varphi_{k+1})_{j+1} = \frac{1}{\sqrt{N}} W_M^{kj}, \quad j = 0, \ldots, N-1, \ k = 0, \ldots, M-1 \qquad (3.26)$$

where

$$W_M = e^{2\pi i/M}$$

is the M'th complex root of unity. Then Φ is a uniform tight frame. It is called the complex harmonic tight frame.

Proof. First we prove that Φ is a tight frame and then uniformity.

- Tight frame: Let F denote the analysis operator. According to Proposition 3.5 we can show that Φ is a tight frame by showing that $F^*F = AI_N$ for some constant A. We know that the frame vectors constitute the columns of F^*, so we know the $(j+1,k+1)$'th element of F^*

$$F^*_{j+1,k+1} = \frac{1}{\sqrt{N}} W_M^{kj},$$

and therefore

$$F_{k+1,j+1} = \left(\frac{1}{\sqrt{N}} W_M^{kj}\right)^*.$$

Now we will show that $F^*F = M/NI_N$ by showing that the (a,b)'th element $(F^*F)_{ab} = M/N\delta_{ab}$:

$$\begin{aligned}
(F^*F)_{ab} &= \sum_{p=0}^{M-1} F^*_{a,p+1} F_{p+1,b} \\
&= \sum_{p=0}^{M-1} \frac{1}{\sqrt{N}} W_M^{p(a-1)} \left(\frac{1}{\sqrt{N}} W_M^{p(b-1)}\right)^* \\
&= \frac{1}{N} \sum_{p=0}^{M-1} e^{2\pi i p(a-1)/M} e^{-2\pi i p(b-1)/M} \\
&= \frac{1}{N} \sum_{p=0}^{M-1} e^{2\pi i p(a-b)/M}.
\end{aligned}$$

When $a = b$, that is along the diagonal, the terms all reduce to 1, and the sum is therefore M/N. When $a \neq b$ we rewrite to

$$(F^*F)_{ab} = \frac{1}{N} \sum_{p=0}^{M-1} \left(e^{2\pi i(a-b)/M}\right)^p,$$

which is recognized as a geometric series. By Proposition C.2 we get

$$\begin{aligned}(F^*F)_{ab} &= \frac{1}{N}\frac{1-\left(e^{2\pi i(a-b)/M}\right)^M}{1-e^{2\pi i(a-b)/M}}\\ &= \frac{1}{N}\frac{1-\left(e^{2\pi i}\right)^{(a-b)}}{1-e^{2\pi i(a-b)/M}}\\ &= \frac{1}{N}\frac{1-1}{1-e^{2\pi i(a-b)/M}} = 0.\end{aligned}$$

This means that $F^*F = M/NI_N$, so Φ is a tight frame.

- Uniformity: From the definition it is seen that

$$\varphi_{k+1} = \frac{1}{\sqrt{N}}\begin{pmatrix} W_M^{k\cdot 0}\\ W_M^{k\cdot 1}\\ \vdots \\ W_M^{k\cdot(N-1)}\end{pmatrix}, \qquad k = 0,\dots,M-1.$$

Since each entry is a power of a root of unity the absolute value is 1, and

$$\begin{aligned}\|\varphi_{k+1}\| &= \left|\frac{1}{\sqrt{N}}\right|\sqrt{\left|W_M^{k\cdot 0}\right|^2+\left|W_M^{k\cdot 1}\right|^2+\cdots+\left|W_M^{k\cdot(N-1)}\right|^2}\\ &= \frac{1}{\sqrt{N}}\sqrt{1+1+\cdots+1}\\ &= \frac{\sqrt{N}}{\sqrt{N}} = 1 \quad \text{for} \quad k = 0,\dots,M-1.\end{aligned}$$

$\square$

There exists an analogous version in the real case, which is stated below. It is a bit more cumbersome and the proof will be omitted.

Theorem 3.24 *Consider the real vectorspace $\mathbb{R}^N$. Define the set of vectors $\Phi = \{\varphi_{k+1}\}_{i=0}^{M-1}$ by*

$$\varphi_{k+1} = \sqrt{\frac{2}{N}}\left[\cos\frac{k\pi}{M}, \cos\frac{3k\pi}{M}, \dots, \cos\frac{(N-1)k\pi}{M},\right.$$
$$\left.\sin\frac{k\pi}{M}, \sin\frac{3k\pi}{M}, \dots, \sin\frac{(N-1)k\pi}{M}\right]^T$$

for $k = 0, 1, \dots, M-1$ if N is even and by

$$\varphi_{k+1} = \sqrt{\frac{2}{N}}\left[\frac{1}{\sqrt{2}}, \cos\frac{2k\pi}{M}, \cos\frac{4k\pi}{M}, \dots, \cos\frac{(N-1)k\pi}{M},\right.$$
$$\left.\sin\frac{2k\pi}{M}, \sin\frac{4k\pi}{M}, \dots, \sin\frac{(N-1)k\pi}{M}\right]^T$$

for $k = 0, 1, \ldots, M-1$ *if* N *is odd. Then* Φ *is a uniform tight frame. It is called the* real harmonic tight frame.

Example 3.25 *In the following example we illustrate harmonic frames in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$*. In* $\mathbb{R}^2$ *our examples of frames have respectively 4 and 9 frame vectors.*

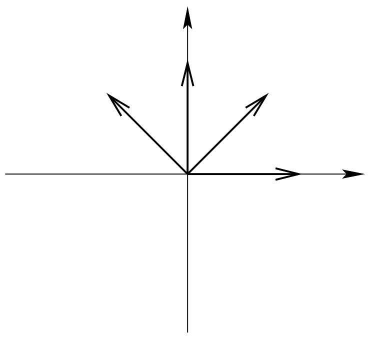

Figure 4: The harmonic frame for $\mathbb{R}^2$ with 4 frame vectors.

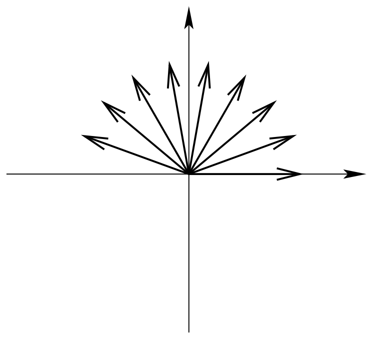

Figure 5: The harmonic frame for $\mathbb{R}^2$ with 9 frame vectors.

In $\mathbb{R}^3$ *the chosen frame has 20 frame vectors.*

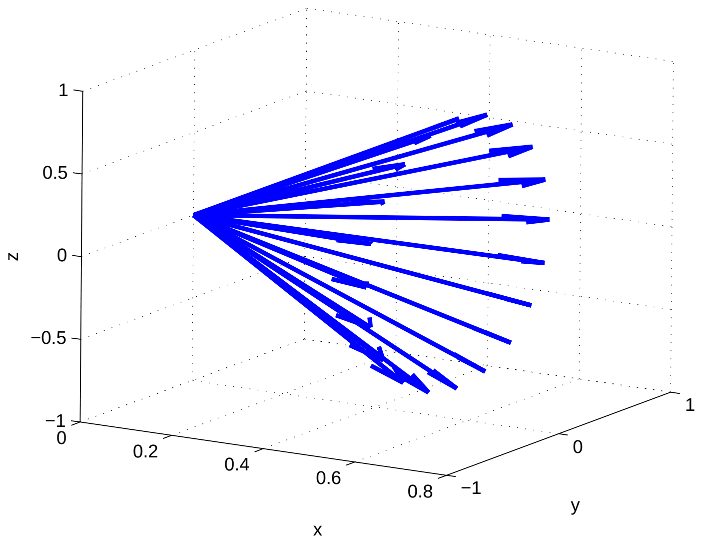

Figure 6: The harmonic frame for $\mathbb{R}^3$ with 20 frame vectors. The vectors all have the same x-coordinate and form a cone.

Notice how the frame vectors are "spread out" evenly. This is a nice property of the harmonic tight frames.

The existence of the harmonic frames for any given M and N combined with Theorem 3.21 immediately yields the following nice result.

Corollary 3.26 *Given the vector space $\mathbb{R}^N$ and any uniform tight frame $\Phi = \{\varphi_i\}_{i=1}^{M}$ with $M = N + 1$. Then Φ is equivalent to a harmonic tight frame.*

This completes the introduction to the theory of frames. We have seen a number of properties of frames, how to make a frame expansion, introduced an equivalence relation and shown how to construct a (harmonic) frame for any given M and N. We will now proceed to consider how frames can be useful in signal transmission. More specifically we will investigate why it is advantageous to use frames instead of orthonormal bases and which frames give the best robustness towards losses of frame coefficients during transmission.

4 Signal transmission

Digital signal transmission is a method of communication in which a *sender* transmits digitally stored information to a *recipient*. Various methods and requirements exist, and the most common will be addressed in the following. There are basically two different types of communication lines, referred to as *one-way* and *two-way* transmission lines respectively. Which one to choose depends on the application. A classical example of a two-way transmission is the telephone, where the recipient can inform the sender that a part of the message was not heard, and ask the sender to repeat the message. A one-way transmission could be a public speech, where the recipient cannot inform the speaker that a certain part of the message was not heard. This is analoguos to the case of a digital transmission line. A two-way digital transmission line can inform the sender that a part of the information was not received, in order for the sender to repeat this part until the entire information stream has been received. It is however crucial that the receiver can identify which pieces of information that did not arrive correctly. This is for instance solved by partitioning the information into *packages* and equipping each package with a unique checksum for identification.

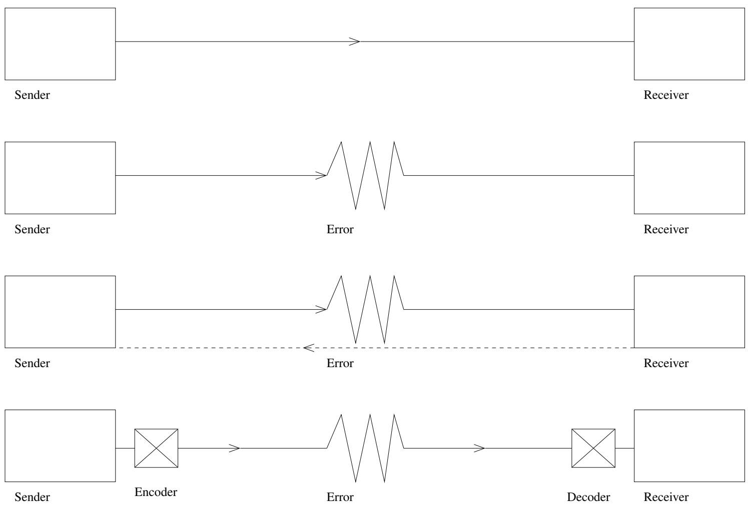

Figure 7: An overview of different types of signal transmission. First a simple one-way transmission line where errors are absent. Second, in real life errors *do* occur, in our case an error corresponds to the loss of a package. Third, by using a two-way transmission line the receiver can inform the sender about missing packages. Fourth, instead of using a two-way system a robust one-way system can be used. This demands an intelligent way of sending the information with some redundancy.

In some applications, for instance streaming of video over the internet or digital radio, two-way communication is not possible, since it can take a variable time to recover the lost parts. Instead one-way communication is used. In a one-way transmission it is desired that the information is received without any losses. Unfortunately it is usually not possible to send data without encountering missing packages. Often the received information will not suffice, and in order to compensate, the sender must send the information in an intelligent way such that the receiver can reconstruct all (or almost all) the information based only on the received data. This is often done by use of so-called "Error-correcting codes" that are based on algebra. Here we will use another approach where the application of frames will prove useful. As we will show in the following sections, frames have very good ability to withstand these losses. However, first we must consider a process known as quantization that is necessary in digital signal transmission.

4.1 Quantization

When operating with continuous signals in a digital communication system, it is necessary to *quantize* the signal in order to achieve a discrete representation of the signal. First the continuous signal is sampled in order to get a discrete signal, the amplitude of each signal element is then rounded to the nearest value in a given set of discrete values. This set of values can for instance be a set of integers, or a set of numbers with a fixed number of decimals, depending on the required accuracy.

Example 4.1 *This example will demonstrate the necessity of quantization. A piece of equipment measures a continuous signal in the interval* $[0; 10[$*, and samples this signal to a precision of the fifth decimal. Each element of the signal has a range of one milion values, since it can assume all possible values in the interval* $[0.00000; 9.99999]$*. In a digital communication system decimal numbers are converted to a binary representation in order to be transmitted using bits. This implies that at least a 20 bit representation (*$2^{20} = 1048576$*) is needed in order to represent each element of the signal.*

The discrete signal is now to be sent through a digital transmission line, but the transmission line is limited to transmit only seven bits per milli-second. In order to send the signal in real-time, the values must be limited to seven bits, thus only permitting $2^7 = 128$ *different values. For decimal numbers this means that the signal can only assume values to the precision of one decimal, thus yielding the range* $[0.0; 9.9]$*. This is due to the fact that for a one-decimal number in this interval, a total of one hundred possible values can occur, but when a second decimal is introduced, the total number of different values exceed 128, precisely a total of 1000 different values.*

Quantization of a signal y, can be interpreted by adding an error vector η to the signal. The quantized signal $\widehat{y}$ is given by

$$\widehat{y} = y + \eta. \tag{4.1}$$

An appropriate stepsize Δ is chosen. In the previous example Δ would be 0.1, because the signal was rounded to precision of one decimal. The stepsize Δ is chosen depending on the application's requirements and limitations. In *uniform* *quantization* each element η_i is a number within the interval $\left[-\frac{\Delta}{2};\frac{\Delta}{2}\right[$, with the asssumption that the elements in η are uniformly distributed. The assumption that η is uniformly distributed is based on the observation that each element can assume a random value within the interval. In addition the elements in η are assumed to be pairwise independent.

From basic statistics, (9.6.8) in [10], we know that if a stochastic variable X is uniformly distributed on the interval $[a, b]$, the mean is given by $E[X] = \frac{a+b}{2}$, which yields

$$E[\eta_i] = \frac{-\Delta/2 + \Delta/2}{2} = 0. \tag{4.2}$$

The variance of X is $\sigma^2 = \frac{(b-a)^2}{12}$, which for η_i gives $\sigma^2 = \frac{\Delta^2}{12}$. Since η_i and η_j are assumed independent for $i \neq j$,

$$E[\eta_i\eta_j] = E[\eta_i]E[\eta_j] = 0, \text{ for } i \neq j.$$

The variance of the η_i can be expressed by

$$\sigma^2 = E[\eta_i^2] - (E[\eta_i])^2 = E[\eta_i^2] - 0^2 = E[\eta_i\eta_i].$$

The two results can be summarized to

$$E[\eta_i] = 0 \quad \text{and} \quad E[\eta_i\eta_j] = \delta_{ij}\sigma^2. \tag{4.3}$$

This quantization noise model makes it possible to evaluate the error introduced when working with a frame expansion.

Example 4.2 *The following example demonstrates the quantization error η. We wish to quantize a randomly picked signal $y = \{y_i\}_{i=1}^4$ with the stepsize $\Delta = 0.2$, thus yielding the quantized signal $\widehat{y}$.*

y_i		η_i	$\widehat{y}_i$
2.5468		0.0532	2.6
7.5846	$\rightarrow$	0.0154	7.6
1.2548		−0.0548	1.2
0.9518		0.0482	1.0

We will now study the use of frames in one-way signal transmission. Using (3.15) we can perform a frame expansion of a signal x:

$$x = \sum_{k=1}^{M} \langle x, \varphi_k \rangle \, \widetilde{\varphi}_k.$$

Instead of sending just the signal itself, the frame coefficients

$$y = \begin{pmatrix} \langle x, \varphi_1 \rangle \\ \langle x, \varphi_2 \rangle \\ \vdots \\ \langle x, \varphi_M \rangle \end{pmatrix}$$

are sent. This process is referred to as *encoding*, whereas the reconstruction from the received coefficients is called *decoding*. If the analysis operator of the frame is called F, the frame coefficients are computed by $y = Fx$. The new signal y is then quantized by adding an error η to y, thus yielding the quantized frame expansion

$$\widehat{y} = Fx + \eta, \tag{4.4}$$

where η is a vector containing all the error-elements. For the receiver, who only has access to $\widehat{y}$, it is impossible to reconstruct the original signal x perfectly, due to the noise. The quantization error introduced in the reconstruction is the subject of the next section.

4.2 Reconstruction Error

The goal of this section is to determine which frames that minimize the error between the original signal and a signal reconstructed from the quantized frame coefficients. To measure the error of an appoximation to a signal we will use the so-called *MSE*.

Definition 4.3 *Given a discrete signal x of length N, and an approximation $\widehat{x}$ to x, the mean-squared-error* MSE *is defined by*

$$\mathrm{MSE} = \frac{1}{N} E\left[||x - \widehat{x}||^2\right]. \tag{4.5}$$

If quantization was not present, we know that we can achieve a perfect reconstruction using the left-inverse $F^\dagger$ to the analysis operator F as described in (3.15) and (3.16):

$$x = F^\dagger F x = \sum_{k=1}^{M} \langle x, \varphi_k \rangle \ \tilde{\varphi}_k,$$

where $F^\dagger = \tilde{F}^*$ and $\tilde{F}$ is the dual analysis operator. If quantization is applied to the signal, we send $\widehat{y} = Fx + \eta$ instead of $y = Fx$. The reconstructed signal $\widehat{x}$ is then

$$\widehat{x} = \tilde{F}^*(Fx + \eta) = \sum_{k=1}^{M} (\langle x, \varphi_k \rangle + \eta_k)\tilde{\varphi}_k. \tag{4.6}$$

As mentioned our task is to minimize the MSE, so we begin by considering $||x - \widehat{x}||$

$$\begin{aligned} ||x - \widehat{x}|| &= \left|\left| \sum_{k=1}^{M} \langle x, \varphi_k \rangle \tilde{\varphi}_k - \sum_{k=1}^{M} (\langle x, \varphi_k \rangle + \eta_k)\tilde{\varphi}_k \right|\right| \\ &= \left|\left| - \sum_{k=1}^{M} \eta_k \tilde{\varphi}_k \right|\right|. \end{aligned} \tag{4.7}$$

The mean-squared error (MSE) is then

$$\begin{aligned}\text{MSE} &= \frac{1}{N}E\left[||x-\widehat{x}||^2\right] \\ &= \frac{1}{N}E\left[\left|\left|\sum_{k=1}^{M}\eta_k\tilde{\varphi}_k\right|\right|^2\right],\end{aligned}$$

which can be rewritten using linearity of the inner product,

$$\begin{aligned}\text{MSE} &= \frac{1}{N}E\left[\left\langle\sum_{k=1}^{M}\eta_k\tilde{\varphi}_k,\sum_{i=1}^{M}\eta_i\tilde{\varphi}_i\right\rangle\right] \\ &= \frac{1}{N}E\left[\sum_{k=1}^{M}\sum_{i=1}^{M}\langle\eta_k\tilde{\varphi}_k,\eta_i\tilde{\varphi}_i\rangle\right] \\ &= \frac{1}{N}E\left[\sum_{i=1}^{M}\sum_{k=1}^{M}\eta_i\eta_k\tilde{\varphi}_i^*\tilde{\varphi}_k\right].\end{aligned}$$

Linearity of the mean and the use of (4.3) allows us to rewrite the expression as

$$\begin{aligned}\text{MSE} &= \frac{1}{N}\sum_{i=1}^{M}\sum_{k=1}^{M}E\left[\eta_i\eta_k\right]\tilde{\varphi}_i^*\tilde{\varphi}_k \\ &= \frac{1}{N}\sum_{i=1}^{M}\sum_{k=1}^{M}\delta_{ik}\sigma^2\tilde{\varphi}_i^*\tilde{\varphi}_k.\end{aligned}$$

The only non-zero terms are for $i=k$, so this reduces to

$$\text{MSE} = \frac{1}{N}\sigma^2\sum_{k=1}^{M}||\tilde{\varphi}_k||^2.$$

Using Proposition 3.12, Proposition A.7 and (3.12), it can be futher reduced to

$$\begin{aligned}\text{MSE} &= N^{-1}\sigma^2\text{tr}(\tilde{F}^*\tilde{F}) \\ &= N^{-1}\sigma^2\text{tr}((F^*F)^{-1}).\end{aligned}$$

Since the frame operator F^*F is self-adjoint, there exists some unitary matrix V such that

$$F^*F = V\Lambda V^*$$

where $\Lambda = \text{diag}\{\lambda_i\}_{i=1}^N$. This implies by the use of Proposition B.2

$$(F^*F)^{-1} = (V\Lambda V^*)^{-1} = (V^*)^{-1}\Lambda^{-1}V^{-1} = V\Lambda^{-1}V^*.$$

By Proposition A.5, and that V is unitary, the mean squared error can be rewritten as

$$\begin{aligned}\mathrm{MSE} &= N^{-1}\sigma^2 \mathrm{tr}(V\Lambda^{-1}V^*)\\ &= N^{-1}\sigma^2 \mathrm{tr}(V^*V\Lambda^{-1})\\ &= N^{-1}\sigma^2 \mathrm{tr}(\Lambda^{-1}).\end{aligned}$$

Since $\mathrm{tr}(\Lambda^{-1})$ is the same as the sum of all the inverted eigenvalues of F^*F, we arrive at the simple expression for the MSE

$$\mathrm{MSE} = \frac{1}{N}\sigma^2 \sum_{i=1}^{N} \frac{1}{\lambda_i}. \tag{4.8}$$

This expression can help determine which frames minimize the MSE. For convenience we will only consider uniform frames. The result is stated in the following theorem. A lemma is however needed in order to complete the proof. The lemma is a well-known result, see for instance (39.9) in [11], and the proof will be omitted.

Lemma 4.4 *Let $\{a_i\}_{i=1}^N$ be a sequence of real numbers. Denote by μ_H and μ the harmonic and the arithmetic mean respectively,*

$$\mu_H = \frac{N}{\sum_{i=1}^N \frac{1}{a_i}} \qquad\qquad \mu = \frac{1}{N}\sum_{i=1}^N a_i.$$

The following inequality holds with equality if and only if all the a_i are equal:

$$\mu_H \le \mu.$$

Theorem 4.5 *When encoding with a uniform frame and decoding with the left-inverse (4.6) under the noise model (4.3), the MSE is minimum if and only if the frame is tight.*

Proof. Due to Proposition 3.12 the sum of the eigenvalues of F^*F is equal to M. To minimize the MSE is therefore equivalent to minimizing the sum $\sum_{i=1}^N \lambda_i^{-1}$ subject to the constraint that $\sum_{i=1}^N \lambda_i = M$ is constant. To do that we notice that the first and the second sums are closely related to the harmonic mean μ_H and the usual arithmetic mean μ of the λ_i respectively.

By Lemma 4.4 we know that $\mu_H \le \mu$. This is equivalent to $\mu_H^{-1} \ge \mu^{-1}$:

$$\frac{1}{N}\sum_{i=1}^N \frac{1}{\lambda_i} = \mu_H^{-1} \ge \mu^{-1} = \frac{N}{\sum_{i=1}^N \lambda_i}$$

which is equivalent to

$$\sum_{i=1}^N \frac{1}{\lambda_i} \ge \frac{N^2}{M}$$

where N^2/M is constant. Clearly the sum is minimized if and only if equality holds which is true if and only if all the λ_i are equal according to Lemma 4.4. This is equivalent to having a tight frame.

□

We emphasize that the theorem states that uniform tight frames should be used in order to achieve an MSE as small as possible. Depending of the type of frame, different bounds of the MSE exist.

Theorem 4.6 *Consider recontruction* (4.6) *with noise* η *satisfying* (4.3) *and the mean-squared error* (MSE) *from Definition 4.3. For any frame, the* MSE *is given by (4.8) and satisfies*

$$B^{-1}\sigma^2 \leq \mathrm{MSE} \leq A^{-1}\sigma^2. \tag{4.9}$$

For a uniform frame,

$$\frac{N}{M}\sigma^2 \leq \mathrm{MSE} \leq A^{-1}\sigma^2. \tag{4.10}$$

For a uniform tight frame,

$$\mathrm{MSE} = \frac{N}{M}\sigma^2 = r^{-1}\sigma^2. \tag{4.11}$$

Proof. The bounds in (4.9) follow from Proposition 3.11, where the eigenvalues $\lambda_i \in [A, B]\ \forall i$, so by inversion $1/\lambda_i \in [1/B, 1/A]\ \forall i$, and

$$\sum_{i=1}^{N} B^{-1} \leq \sum_{i=1}^{N} \frac{1}{\lambda_i} \leq \sum_{i=1}^{N} A^{-1}. \tag{4.12}$$

By multiplication of $\frac{1}{N}\sigma^2$

$$\frac{1}{N}\sigma^2 \sum_{i=1}^{N} B^{-1} \leq \frac{1}{N}\sigma^2 \sum_{i=1}^{N} \frac{1}{\lambda_i} \leq \frac{1}{N}\sigma^2 \sum_{i=1}^{N} A^{-1},$$

which in terms of the MSE is simplified to

$$\frac{1}{N}\sigma^2\frac{N}{B} \leq \mathrm{MSE} \leq \frac{1}{N}\sigma^2\frac{N}{A}.$$

This directly yields the bounds in (4.9). For a uniform tight frame the bounds are equal, since $A = B = r = M/N$, and (4.11) follows. If the frame is only uniform, we know by Theorem 4.5 that the MSE cannot be lower than for a tight frame, which gives the lower bound in (4.10).

□

Notice the particularly simple expression for the MSE for a uniform tight frame, which only depends on the redundancy of the frame.

This completes the considerations concerning the error from quantization. In the following section we will investigate what happens if some of the information is lost during the transmission.

5 Erasures

5.1 Introducing erasures

When dealing with one-way signal transmission, it is an essential property to be able to tolerate a certain amount of losses during the transmission. This is due to the fact, that the recipient does not have any means of communicating to the sender that a certain part of a signal has not been received. If this option was possible, the sender could just send the missing part again, and therefore complete the signal transmission. Instead the reciever must obtain enough information from the remaining signal, in order to compensate for the missing parts. It might not be possible to obtain a perfect recontruction, but typically some norm-difference in comparison to the original signal is allowed. The following example demonstrates the importance of the ability to withstand erasures.

Example 5.1 *Given a signal made of a linear combination of cosine-functions that is to be sent through a one-way transmission line, we will here demonstrate the consequences of using a non-robust series expansion. The signal is here shown in Figure 8. We construct a series expansion of the signal using an ONB (the Discrete Fourier Transform), and send it to the receiver. During the transmission a single element of the series expansion is lost, and the missing element is in this case replaced with a zero. The signal is then recontructed with the remaining information. As seen in Figure 9, the signal does not give a good reconstruction.*

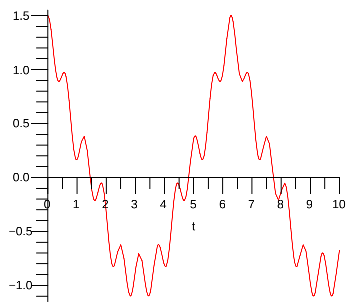

Figure 8: Original signal to be sent using a one-way transmission line, using a ONB series expansion.

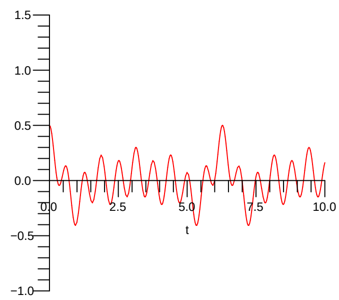

Figure 9: Recontructed signal with one erasure using an ONB series expansion. It is far from close to the original.

Fortunately some frames have very good abilities to withstand losses due to their over-completeness. In general these losses are referred to as erasures, and the index set corresponding to the erasures is denoted E. It should be emphasized that we assume that the recipient knows this index set. If a signal $\widehat{y}$ loses $|E|$ elements, $\{\widehat{y}_k\}_{k\in E}$, during a transmission, the goal is to reconstruct the signal as well as possible using the remaining information. To do so, the recipient uses the frame consisting of the frame vectors corresponding to the coefficients that were received. More precisely, $\Phi_E = \{\varphi_k\}_{k \notin E}$ is used, where it is assumed that Φ_E is still a frame. This assumption is used throughout the rest of this report, since it gives rise to a number of nice results. Furthermore this does not pose a problem, since we will show that the assumption holds for the harmonic tight frames. The resulting $(M - |E|) \times N$ analysis operator F_E is then the matrix with φ_k^* as the k'th row, where $k \notin E$.

It will be important that the frames we use are as stable towards erasures as possible. The frame is not suited for the application if the remaining vectors after only a few erasures is no longer a frame. We introduce the concept of *maximal robustness* in the following definition.

Definition 5.2 *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame in $\mathbb{R}^N$. The frame Φ is said to have maximal robustness if $\operatorname{span}\{\varphi_i\}_{i \in P} = \mathbb{R}^N$ for any set $P \subseteq \{1 \dots M\}$ where $|P| = N$.*

As the definition says, a frame is called maximally robust if it can withstand up to $M - N$ erasures and still span the considered space. The next example demonstrates that not all frames, even tight frames, are maximally robust.

Example 5.3 *This example shows that even though we have a frame, it is not necessarily maximally robust. Figure 10 illustrates the uniform tight frame $\Phi = \{\varphi_i\}_{i=1}^4 \in \mathbb{R}^2$ given by*

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \varphi_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

In figure 11 the two vectors φ_2 and φ_4 are removed, corresponding to losses of the second and fourth coefficient. It is easy to see that the two remaining vectors do not span $\mathbb{R}^2$, thus not being maximally robust. There exist similar examples, where even one erasure leads to a incomplete set.

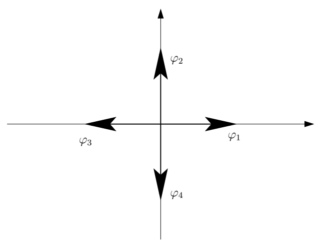

Figure 10: Example of a non-robust frame in $\mathbb{R}^2$ before erasures occur.

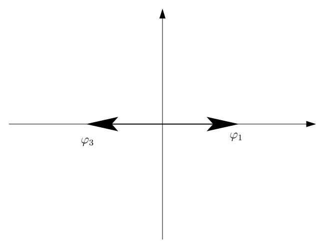

Figure 11: The remaining vectors after two erasures. They do not span $\mathbb{R}^2$.

If a number of frame elements are removed from the frame, it is not always possible to maintain the frame bounds. It is clear that if $|E|$ is greater than $M - N$, then the span of the remaining vectors does not cover $\mathbb{R}^N$. The first subject of attention is to which extent erasures can occur, and still maintain a frame.

Theorem 5.4 *Let $e \in \mathbb{N}$ and $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ be a uniform tight frame, with $M/N > e$. For any set E, where $|E| = e$, then $\Phi_E = \{\varphi_k\}_{k=1,\, k\notin E}^M$ is a uniform frame, and has the lower frame bound $A_E = M/N - e$ and upper frame bound $B_E = M/N$.*

Proof. The frame bound of a tight frame is

$$\sum_{k=1}^{M} |\langle x, \varphi_k \rangle|^2 = A \quad \forall x \in S^{N-1}. \tag{5.1}$$

If a set E, with $|E| = e$, of frame elements, φ_i, $i \in E$, is deleted from the frame, this corresponds to removing all the terms $|\langle x, \varphi_i \rangle|^2$ $\forall i \in E$ from the sum in (5.1). Since Φ is a uniform tight frame, $||\varphi_i|| = 1\,\forall i$, then by the Cauchy-Schwartz inequality we have that

$$0 \leq |\langle x, \varphi_i \rangle| \leq ||x|| \cdot ||\varphi_i|| = 1, \quad \forall x \in S^{N-1}. \tag{5.2}$$

This implies that

$$0 \leq \sum_{i \in E} |\langle x, \varphi_i \rangle|^2 \leq e, \tag{5.3}$$

yielding

$$A - e \leq \sum_{k=1,\, k\notin E}^{M} |\langle x, \varphi_k \rangle|^2 \leq A \quad \forall x \in S^{N-1}. \tag{5.4}$$

For a tight frame $A = M/N$, and due to the assumption that $M/N > e$, the lower frame bound is still positive, and the upper frame bound is still maintained, thus Φ_E is still a frame. The frame is of course still uniform since the remaining frame vectors have not been changed.

□

The theorem says that any uniform tight frame is stable towards any number of erasures lower than M/N. It is an extended version of Theorem 4.1 in [5]. For an arbitrary, but fixed, number of erasures e, we are now able construct frames that can withstand e erasures by selecting a sufficiently large M. This is not satisfactory, however. This is due to the fact that if one for instance wishes to send a signal using a one-way communication-line, it will be necessary to send a signal of double length, in order to maintain a frame, with just two erasures. For more erasures the signal length must be even larger.

Fortunately is it possible to contruct frames that have more robustness to erasures than this. An important example of such frames is the class of harmonic frames. This is the content of the following theorem.

Theorem 5.5 *Let $\Phi = \{\varphi_k\}_{k=1}^M$ be a complex harmonic frame in $\mathbb{C}^N$ given by Theorem 3.23 or a real harmonic frame in $\mathbb{R}^N$ given by Theorem 3.24. Then any subset of N or more vectors from Φ form a frame. In other words, a harmonic frame is maximally robust.*

The proof of Theorem 5.5 is a bit messy and not very enlightening and will not be given here. It can be found in Appendix A.6 in [5].

The harmonic frames are maximally robust and from Section 3.5 we know how to construct them for any given N-dimensional space and with any number M of frame vectors. Hence they form a nice family of frames for our application.

We are now ready to investigate how the MSE is affected by erasures.

5.2 MSE after occurences of erasures

In section 4.2 the MSE was calculated for the error introduced by quantization. In the following the effect of erasures is taken into consideration, and the MSE is calculated. In Section 5.3, the MSE is calculated for a frame with only a single erasure. This result is in Section 5.4 expanded to the general case with e erasures.

Recall that for a given frame Φ we assume to have a new frame Φ_E after the erasures have occured. This gives rise a new analysis operator F_E. The new frame operator is then given by the expression $F_E^* F_E$, with eigenvalues $\{\lambda_{E,i}\}_{i=1}^N$. Then $\{1/\lambda_{E,i}\}_{i=1}^N$ are the eigenvalues for $(F_E^* F_E)^{-1}$. Since the new frame is used for the reconstruction, the MSE is by (4.8)

$$\mathrm{MSE}_E = \frac{\sigma^2}{N} \sum_{i=1}^{N} \frac{1}{\lambda_{E,i}}. \tag{5.5}$$

Expressed using trace we have

$$\mathrm{MSE}_E = N^{-1} \sigma^2 \mathrm{tr}\left((F_E^* F_E)^{-1} \right). \tag{5.6}$$

This is the general expression of the MSE after e erasures that we will use in the following sections to reach some relatively nice expressions both with a single erasure and more.

5.3 MSE with one erasure

At first we wish to study the case of just one erasure from a uniform tight frame Φ. The goal of the section is to prove Theorem 5.14. The theorem states that among all uniform frames the tight frames minimize the average and the maximum MSE over all possible single erasures. We begin by finding the MSE for a single erasure when the frame is tight and then we proceed to the theorem.

To compute the MSE for one erasure from a uniform tight frame we proceed as follows. Since the frame vectors are named arbitrarily, we can for convenience turn our attention to the case where $\langle \varphi_1, x \rangle$ is the erased component. The remaining frame is then denoted $\Phi_{\{1\}} = \Phi \backslash \{\varphi_1\}$ with analysis operator F_1. Since the frame operators in question can be written as $F^* F = \sum_{k=1}^M \varphi_k \varphi_k^* = \frac{M}{N} I$ and $F_1^* F_1 = \sum_{k=2}^M \varphi_k \varphi_k^*$, we have

$$F_1^* F_1 = \frac{M}{N} I - \varphi_1 \varphi_1^*. \tag{5.7}$$

To calculate the MSE we know by (5.6) that we need $\operatorname{tr}(F_1^*F_1)^{-1}$. To invert $F_1^*F_1$ we use Proposition C.6 which yields [3]

$$(F_1^*F_1)^{-1} = \left(\frac{M}{N}I - \varphi_1\varphi_1^*\right)^{-1} = \frac{N}{M}I + \frac{N^2}{M(M-N)}\varphi_1\varphi_1^*.$$

Since the MSE can be calculated by use of the trace, we find the trace

$$\operatorname{tr}(F_1^*F_1)^{-1} = \operatorname{tr}\left(\frac{N}{M}I + \frac{N^2}{M(M-N)}\varphi_1\varphi_1^*\right),$$

and by use of linearity of trace, Proposition A.4 and that φ_i are of unit-norm, we then get

$$\begin{aligned}\operatorname{tr}(F_1^*F_1)^{-1} &= \frac{N}{M}\operatorname{tr}(I_N) + \frac{N^2}{M(M-N)}\operatorname{tr}(\varphi_1\varphi_1^*)\\ &= \frac{N^2}{M} + \frac{N^2}{M(M-N)}\operatorname{tr}(\varphi_1^*\varphi_1)\\ &= \frac{N^2}{M}\left(1 + \frac{1}{M-N}\right).\end{aligned}$$

Combined with (5.6) we then obtain

$$\begin{aligned}\mathrm{MSE}_1 &= \frac{\sigma^2}{N}\frac{N^2}{M}\left(1 + \frac{1}{M-N}\right)\\ &= \left(1 + \frac{1}{M-N}\right)\mathrm{MSE}_0, \end{aligned} \tag{5.8}$$

where MSE_0 is the mean square error from quantization for a uniform tight frame given by (4.11). Notice that the MSE does not depend on which particular erasure has occured but only on $M - N$. This is a surprisingly simple result and we will state it in a theorem.

Theorem 5.6 *Consider encoding with a uniform tight frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ and decoding with the left-inverse (4.6) under the noise model (4.3). The mean squared error resulting from one single erasure is given by*

$$\mathrm{MSE}_1 = \left(1 + \frac{1}{M-N}\right)\mathrm{MSE}_0,$$

where MSE_0 is given by (4.11).

After having determined the MSE for a single erasure we will proceed to prove Theorem 5.14. In order to do so, we need a number of lemmas. Most of them will be proved but some will only be stated. The following pages have been challenging for our part, due to the lack of details in this part of [5]. It is possible that the reader will also find that it takes some effort to read.

[3] Substitute $A = M/N \cdot I_N$, $B = \varphi_1$, $C = 1$ and $D = \varphi_1^*$.

Lemma 5.7 *Let $w \in \mathbb{C}^N$ with $||w|| = 1$. The eigenvalues of $I_N - ww^*$ are 0 and 1 with multiplicity 1 and $N-1$ respectively. Futhermore w is an eigenvector corresponding to the eigenvalue 0.*

Proof. We begin by finding bounds for the eigenvalues $\{\lambda_i\}_{i=1}^N$ for $I_N - ww^*$. Let (λ, v) be an eigenpair of $I_N - ww^*$, then $(I_N - ww^*)v = \lambda v$ holds, and by taking the inner product with v we deduce

$$\langle (I_N - ww^*)v, v\rangle = \langle \lambda v, v\rangle \ ,$$

or

$$||v||^2 - ||w^*v||^2 = \lambda ||v||^2,$$

that can be rewritten to

$$||w^*v||^2 = (1-\lambda)||v||^2. \tag{5.9}$$

Since w^*v is a scalar, then $||w^*v|| = |w^*v|$, and the following holds

$$\begin{aligned} ||w^*v||^2 &= |w^*v|^2 = |\langle v, w\rangle|^2 \\ &\leq ||v||^2||w||^2 = ||v||^2, \end{aligned} \tag{5.10}$$

by use of the Cauchy-Schwartz inequality. From this we get $(1-\lambda)||v||^2 \leq ||v||^2$ which implies that $\lambda \geq 0$. On the other hand, since $||w^*v||^2$ and $||v||^2$ are none-negative, (5.9) implies that $\lambda \leq 1$. So all eigenvalues belong to the interval $[0, 1]$.

w is an eigenvector of $I_N - ww^*$ corresponding to the eigenvalue 0, since

$$\begin{aligned} ((I_N - ww^*) - 0 \cdot I_N)\, w &= I_N w - ww^*w \\ &= w - w||w||^2 = w - w = 0. \end{aligned}$$

We now wish to prove that the remaining eigenvalues are all 1. First note that the matrix $I_N - ww^*$ is self-adjoint since $(I_N - ww^*)^* = I_N^* - (ww^*)^* = I_N - ww^*$. Hence the spectral theorem applies, and we can write $I_N - ww^* = U\Lambda U^*$ where U is unitary and $\Lambda = \text{diag}\{\lambda_i\}_{i=1}^N$. This is then rewritten to

$$I_N - U^*ww^*U = \Lambda.$$

By using the linearity of trace (Proposition A.2) we find that

$$\text{tr}(I_N) - \text{tr}(U^*ww^*U) = \text{tr}(\Lambda).$$

Using the cyclic property of trace (Proposition A.5), the expression can be rewritten as

$$\text{tr}(I_N) - \text{tr}(w^*UU^*w) = \sum_{i=1}^N \lambda_i,$$

and using Proposition A.7, we can deduce that

$$\sum_{i=1}^{N} \lambda_i = N - ||w||^2 = N - 1.$$

We know that $\lambda_i \in [0, 1]$ and that a single eigenvalue is equal to 0. This forces the remaining $N - 1$ eigenvalues to be equal to 1.

□

Lemma 5.8 *Let $M \in \mathbb{C}^{N \times N}$ and let $w \in \mathbb{C}^N$ with $||w|| = 1$. Then the following inequality holds, with equality if and only if w is an eigenvector of M:*

$$w^* M^* M w \geq |w^* M w|^2 .$$

Proof. First we show that the matrix $I_N - ww^*$ is positive semidefinite, that is for any $x \in \mathbb{C}^N$, $x^*(I_N - ww^*)x \geq 0$. The left hand side is rewritten to

$$\begin{aligned} x^*(I_N - ww^*)x &= x^*x - x^*ww^*x \\ &= ||x||^2 - (w^*x)^*(w^*x) \\ &= ||x||^2 - ||w^*x||^2. \end{aligned}$$

The inequality (5.10) implies that $||x||^2 - ||w^*x||^2 \geq 0$ thus proving that $I_N - ww^*$ is positive semidefinite. This means that

$$0 \leq (Mw)^*(I_N - ww^*)(Mw) = w^*M^*Mw - (w^*M^*w)(w^*Mw).$$

By rearranging we then get

$$w^*M^*Mw \geq (w^*Mw)^*(w^*Mw),$$

which is equivalent to

$$w^* M^* M w \geq |w^* M w|^2 . \tag{5.11}$$

After having proved that the inequality holds, we proceed to the equality part. Let $e_N = w$ and choose $\{e_i\}_{i=1}^{N-1}$ to be normalized eigenvectors such that $\{e_i\}_{i=1}^{N}$ forms an orthonormal basis of $\mathbb{C}^N$. Due to Lemma 5.7 we have

$$(I_N - ww^*)e_i = \begin{cases} e_i & \text{for} \quad i = 1, \dots, N-1 \\ 0 & \text{for} \quad i = N. \end{cases} \tag{5.12}$$

Using the orthonormal basis any vector from $\mathbb{C}^N$ can be written as a linear combination of the eigenvectors, for instance there exist constants $\{c_i\}_{i=1}^{N}$ so that

$$Mw = \sum_{i=1}^{N} c_i e_i. \tag{5.13}$$

The equality corresponding to (5.11) is equivalent to

$$\langle (I_N - ww^*)Mw, Mw\rangle = 0. \tag{5.14}$$

(5.12) and (5.13) can be used to rewrite the left hand side of (5.14):

$$\begin{aligned}\langle (I_N - ww^*)Mw, Mw\rangle &= \left\langle (I_N - ww^*)\sum_{i=1}^{N} c_i e_i, \sum_{i=1}^{N} c_i e_i \right\rangle \\ &= \left\langle \sum_{i=1}^{N} c_i (I_N - ww^*) e_i, \sum_{i=1}^{N} c_i e_i \right\rangle \\ &= \left\langle \sum_{i=1}^{N-1} c_i e_i, \sum_{i=1}^{N} c_i e_i \right\rangle .\end{aligned}$$

From this it is seen that

$$\langle (I_N - ww^*)Mw, Mw\rangle = \sum_{i=1}^{N-1} |c_i|^2,$$

which is zero if and only if $\{c_i\}_{i=1}^{N-1}$ are all 0. This has major consequences for (5.13), where all but the last term disappear:

$$Mw = c_N e_N = c_N w. \tag{5.15}$$

But this is equivalent to saying that w is an eigenvector of M, which completes the proof, since all arguments work in both directions.

□

Definition 5.9 *The definition is adopted from [12].*

- *A subset C of $\mathbb{R}^N$ is said to be* convex, *if for any $x, y \in C$ and any $t \in [0,1]$, it holds that*

$$(1-t)x + ty \in C.$$

- *A function $f : C \to \mathbb{R}$ on a convex set C is said to be* convex, *if for any $x, y \in C$ and any $t \in [0,1]$, it holds that*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

  If "$\leq$" can be replaced by "$<$" the function is said to be strictly convex.

For a function f on the interval $I \subset \mathbb{R}$ the convexity can equivalently be expressed by

$$\frac{d^2}{dx^2} f(x) \geq 0 \textit{ for } x \in I,$$

where again the term "strictly" is used if "$\leq$" can be replaced by "$<$".

Lemma 5.10 *Let V be a bounded convex set and $f : V \to \mathbb{R}$ be a strictly convex function. Then f has a* unique *minimizer in V.*

Proof. See Proposition 1.2 in [3].

Lemma 5.11 *Let $f : V \to \mathbb{R}$ be a convex function and let $\{\lambda_i\}_{i=1}^M$ be a sequence of real positive numbers with sum 1. Then for all sequences $\{x_i\}_{i=1}^M \subset V^M$ it holds that*

$$f\left(\sum_{i=1}^{M} \lambda_i x_i\right) \leq \sum_{i=1}^{M} \lambda_i f(x_i). \tag{5.16}$$

Proof. See page 39-40 in [12].

Lemma 5.12 *Let f be a convex function over a bounded set $V \subset \mathbb{R}$ and let $V^M = V \times \cdots \times V$, M times. Consider sequences $x = \{x_i\}_{i=1}^M \in V^M$ with the constraint $\sum_{i=1}^M x_i = N$, and assume $N/M \in V$. Then the function $F : V^M \to \mathbb{R} : F(x) = \sum_{i=1}^M f(x_i)$ has a unique minimizer among all such sequences x and the minimum occurs if and only if $x_i = N/M$, $\forall i$.*

Proof. First notice that F is convex, since it is a sum of convex functions. From Lemma 5.10 we then know that F has a unique global minimizer in V^M. To determine this minimizer we need the result from Lemma 5.11. In the notation of Lemma 5.11 we let $\lambda_i = 1/M$, $\forall i$. This means that

$$\sum_{i=1}^{M} \lambda_i x_i = \sum_{i=1}^{M} \frac{1}{M} x_i = \frac{1}{M} \sum_{i=1}^{M} x_i = \frac{N}{M}.$$

This can be used to rewrite the inequality (5.16) to

$$Mf\left(\frac{N}{M}\right) \leq \sum_{i=1}^{M} f(x_i), \quad \forall x,$$

which in terms of the function F is equivalent to

$$F\left(\frac{N}{M}, \frac{N}{M}, \ldots, \frac{N}{M}\right) \leq F(x), \quad \forall x.$$

In other words, $(N/M, N/M, \ldots, N/M)$ is a minimizer of F, and we already know that the minimizer is uniquely determined. This means that F is minimized over V^M if and only if $x_i = N/M$, $\forall i$.

□

Lemma 5.13 *Let F^*F be the frame operator of the frame $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$, $M > N$, and define the numbers $v_i = \varphi_i^*(F^*F)^{-1}\varphi_i, \forall i = 1, \ldots, M$. Then $v_i \in]0, 1[, \forall i$.*

Proof. Denote the frame bounds of Φ by A and B. Then $1/B$ and $1/A$ are frame bounds for the dual frame with frame operator $(F^*F)^{-1}$. This means that

$$\varphi_i^*(F^*F)^{-1}\varphi_i \geq \frac{1}{B} > 0, \quad \forall i.$$

This establishes the lower bound.

For the upper bound we proceed as follows. Using the dual frame vectors we expand φ_i:

$$\varphi_i = \sum_{k=1}^{M} \left\langle \varphi_i, (F^*F)^{-1}\varphi_k \right\rangle \varphi_k.$$

On the other hand we can make the straightforward expansion

$$\varphi_i = \sum_{k=1}^{M} \delta_{ik}\varphi_k.$$

By Proposition C.3 we get

$$\sum_{k=1}^{M} |\delta_{ik}|^2 = \sum_{k=1}^{M} |\left\langle \varphi_i, (F^*F)^{-1}\varphi_k \right\rangle|^2 + \sum_{k=1}^{M} |\delta_{ik} - \left\langle \varphi_i, (F^*F)^{-1}\varphi_k \right\rangle|^2.$$

Notice that the left hand side is equal to 1 and rewrite the right hand side to get

$$\begin{aligned} 1 = |\left\langle \varphi_i, (F^*F)^{-1}\varphi_i \right\rangle|^2 + \sum_{k=1, k\neq i}^{M} |\left\langle \varphi_i, (F^*F)^{-1}\varphi_k \right\rangle|^2 \\ + \sum_{k=1}^{M} |\delta_{ik} - \left\langle \varphi_i, (F^*F)^{-1}\varphi_k \right\rangle|^2. \end{aligned}$$

All the terms are positive, so they are all (and in particular the first) less than or equal to 1, which means that

$$\varphi_i^*(F^*F)^{-1}\varphi_i \leq 1, \quad \forall i.$$

Now assume $\varphi_i^*(F^*F)^{-1}\varphi_i = 1$. From Proposition C.4 we get that $\Phi\backslash\{\varphi_i\}$, is incomplete, but this contradicts our assumption that the deletion of any frame vector from Φ leaves a frame. This means that $\varphi_i^*(F^*F)^{-1}\varphi_i < 1, \forall i$, and the proof is complete.

□

We have now completed the lemmas and are ready to prove the main result.

Theorem 5.14 *Consider encoding with a uniform tight frame and decoding with the left-inverse* (4.6) *under the noise model* (4.3)*. The* MSE *averaged over all possible erasures of one frame element,*

$$\overline{\text{MSE}}_1 = \frac{1}{M}\sum_{k=1}^{M} \text{MSE}_{\{k\}},$$

is minimum if and only if the original frame is tight. Also, a tight frame minimizes the maximum distortion caused by one erasure

$$\max_{k=1,2,\ldots,M} \text{MSE}_{\{k\}}.$$

Proof. We wish to consider the average MSE over all possible single erasures. Recall that for reconstruction of a signal with the erasure set E we used F_E instead of F, where F is the analysis operator for the *entire* frame. The present reconstruction is with a single erasure, so $|E| = 1$. Let i denote the index of the erased frame element φ_i from F. We can express our frame operator F^*F in the following way

$$F^*F = \begin{pmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_M \\ | & | & & | \end{pmatrix} \begin{pmatrix} - & \varphi_1^* & - \\ - & \varphi_2^* & - \\ & \cdots & \\ - & \varphi_M^* & - \end{pmatrix} = \sum_{k=1}^{M} \varphi_k \varphi_k^*.$$

A single erasure corresponds to removing one term in the expression. Removing the i'th term from the sum, we then define

$$H_i = \sum_{k=1, k\neq i}^{M} \varphi_k \varphi_k^* = F^*F - \varphi_i \varphi_i^*,$$

where H_i corresponds to the frame operator with the i'th element removed. By (5.6), the i'th MSE can be expressed as

$$\text{MSE}_{\{i\}} = N^{-1}\sigma^2 \text{tr}\left(H_i^{-1}\right).$$

By averaging over all possible erasures we get

$$\overline{\text{MSE}}_1 = \frac{1}{M}\sum_{i=1}^{M} \frac{\sigma^2}{N} \text{tr}\left(H_i^{-1}\right). \tag{5.17}$$

Our task is now to show that this expression is minimized if and only if the original frame is tight. By Proposition C.6 we can rewrite $H_i^{-1} = (F^*F - \varphi_i \varphi_i^*)^{-1}$ to [4]

$$H_i^{-1} = (F^*F)^{-1} + (F^*F)^{-1}\varphi_i \left[1 - \varphi_i^*(F^*F)^{-1}\varphi_i\right]^{-1} \varphi_i^*(F^*F)^{-1}.$$

[4] Substitute $A = F^*F$, $B = \varphi_i$, $C = 1$ and $D = \varphi_i^*$.

To find the trace of H_i^{-1}, we will begin by using linearity of the trace:

$$\mathrm{tr}(H_i^{-1}) = \mathrm{tr}\left((F^*F)^{-1}\right) + \left[1 - \varphi_i^*(F^*F)^{-1}\varphi_i\right]^{-1} \mathrm{tr}\left((F^*F)^{-1}\varphi_i\varphi_i^*(F^*F)^{-1}\right),$$

since $\left[1 - \varphi_i^*(F^*F)^{-1}\varphi_i\right]$ is a scalar. By use of the cyclic property of trace (Proposition A.5), the expression can be rewritten as

$$\mathrm{tr}(H_i^{-1}) = \mathrm{tr}\left((F^*F)^{-1}\right) + \left[1 - \varphi_i^*(F^*F)^{-1}\varphi_i\right]^{-1} \mathrm{tr}\left(\varphi_i^*(F^*F)^{-2}\varphi_i\right).$$

Observing that $\varphi_i^*(F^*F)^{-2}\varphi_i$ is a scalar yields

$$\mathrm{tr}(H_i^{-1}) = \mathrm{tr}\left((F^*F)^{-1}\right) + \frac{\varphi_i^*(F^*F)^{-2}\varphi_i}{1 - \varphi_i^*(F^*F)^{-1}\varphi_i}.$$

Substituting in (5.17) gives

$$\begin{aligned}\overline{\mathrm{MSE}}_1 &= \frac{\sigma^2}{MN}\sum_{i=1}^{M}\left(\mathrm{tr}\left((F^*F)^{-1}\right) + \frac{\varphi_i^*(F^*F)^{-2}\varphi_i}{1 - \varphi_i^*(F^*F)^{-1}\varphi_i}\right)\\ &= \frac{\sigma^2}{N}\mathrm{tr}\left((F^*F)^{-1}\right) + \frac{\sigma^2}{MN}\sum_{i=1}^{M}\frac{\varphi_i^*(F^*F)^{-2}\varphi_i}{1 - \varphi_i^*(F^*F)^{-1}\varphi_i}. \end{aligned} \tag{5.18}$$

We recognize the first term from Theorem 4.5, and so we know that this is minimized if and only if the frame is tight. This means that we can turn our attention to the second term.

This part is more tricky and requires some more work. The idea is similar to the one used in the proof of Theorem 4.5: A function of the λ_i was minimized with respect to the constraint that $\sum_{i=1}^{N}\lambda_i$ was constant. Now we will introduce the numbers v_i that will be used in a similar fashion:

$$v_i = \varphi_i^*(F^*F)^{-1}\varphi_i \quad \text{for} \quad i = 1, \ldots, M.$$

$(F^*F)^{-1}$ is the dual frame operator and we know that the bounds for the dual frame are $1/B$ and $1/A$. Due to this, and $||\varphi_i|| = 1$, we see that $v_i \in \left[\frac{1}{B}, \frac{1}{A}\right]$. The sum of the v_i is constant, but this requires a number of steps to be seen. Using the definition of v_i and the fact that it is a scalar we find

$$\begin{aligned}\sum_{i=1}^{M} v_i &= \sum_{i=1}^{M}\varphi_i^*(F^*F)^{-1}\varphi_i\\ &= \sum_{i=1}^{M}\mathrm{tr}\left(\varphi_i^*(F^*F)^{-1}\varphi_i\right).\end{aligned}$$

The cyclic permutation property and linearity of the trace yield

$$\begin{aligned}\sum_{i=1}^{M} v_i &= \sum_{i=1}^{M} \operatorname{tr}\left((F^*F)^{-1}\varphi_i\varphi_i^*\right) \\ &= \operatorname{tr}\left(\sum_{i=1}^{M}(F^*F)^{-1}\varphi_i\varphi_i^*\right) \\ &= \operatorname{tr}\left((F^*F)^{-1}\sum_{i=1}^{M}\varphi_i\varphi_i^*\right).\end{aligned}$$

Recognizing $\sum_{i=1}^{M}\varphi_i\varphi_i^*$ as F^*F, this simplifies to

$$\sum_{i=1}^{M} v_i = \operatorname{tr}\left((F^*F)^{-1}(F^*F)\right) = \operatorname{tr}(I_N) = N. \tag{5.19}$$

Since we are interested in minimizing the second expression in (5.18), we now turn our attention to the numerator. By use of Lemma 5.8 we obtain the inequality

$$\varphi_i^*(F^*F)^{-2}\varphi_i \geq \left|\varphi_i^*(F^*F)^{-1}\varphi_i\right|^2 = |v_i|^2 = v_i^2,$$

since v_i is positive. Using this inequality the second expression from (5.18) can be rewritten as

$$\sum_{i=1}^{M} \frac{\varphi_i^*(F^*F)^{-2}\varphi_i}{1-\varphi_i^*(F^*F)^{-1}\varphi_i} \geq \sum_{i=1}^{M} \frac{v_i^2}{1-v_i}. \tag{5.20}$$

Observe that the function $f(x) = \frac{x^2}{1-x}$ for $x \in]0,1[$ is strictly convex since

$$\frac{d^2}{dx^2}f(x) = \frac{2}{(1-x)^3} > 0 \quad \text{for} \quad x \in]0,1[.$$

Since $N/M \in]0,1[$ and f is convex over $]0,1[$, we know by Lemma 5.12, that the right hand side of (5.20) is minimized if and only if $v_i = N/M, \forall i$.

To prove that this is equivalent to having a tight frame we proceed as follows: If Φ is tight, then $F^*F = M/N \cdot I_N$, so $(F^*F)^{-1} = N/M \cdot I_N$. This implies that

$$v_i = \varphi_i^*(F^*F)^{-1}\varphi_i = \frac{N}{M}||\varphi_i||^2 = \frac{N}{M} \quad \forall i.$$

Equality holds in (5.20), since the denominators on each side are the same and

$$\varphi_i^*(F^*F)^{-2}\varphi_i = \varphi_i^*\left(\frac{N}{M}I_N\right)^2\varphi_i = \left(\frac{N}{M}\right)^2 = v_i^2,$$

which means that for a tight frame the left hand side of (5.20) attains its minimum.

Conversely, if $v_i = N/M, \forall i$, then the right hand side of (5.20) is minimized, and according to Lemma 5.8 equality holds if and only if φ_i is an eigenvector of $(F^*F)^{-1}$. Denote the corresponding eigenvalue by ν_i and we have $(F^*F)^{-1}\varphi_i = \nu_i\varphi_i$, which by multiplication by φ_i^* yields

$$\varphi_i^*(F^*F)^{-1}\varphi_i = \nu_i.$$

But this means that $\nu_i = v_i = N/M$, and thus all eigenvalues of $(F^*F)^{-1}$ are equal. Hence the dual frame of Φ is tight, which in turn implies that Φ itself is tight. This completes the proof of the first part of the theorem.

We now wish to prove that the maximum MSE caused by one erasure is minimized if the frame is tight. Let $\mathcal{F}_{M\times N}$ denote the set of all uniform frames with M vectors for $\mathbb{R}^N$. For a given frame $\Phi \in \mathcal{F}_{M\times N}$ there exist an erasure that gives an MSE larger than the MSE for all other erasures from this frame. Since this is true for all frames in $\mathcal{F}_{M\times N}$, there exists a frame $\widehat{\Phi}$ that has the lowest maximum MSE in comparison to all the other frames. That is, $\widehat{\Phi}$ minimizes the maximum MSE caused by one erasure among all frames in $\mathcal{F}_{M\times N}$.

Consider now $\widehat{\Phi}$. We know that the maximum is always greater than or equal to the average:

$$\max_{k=1,\ldots,M} \mathrm{MSE}_{\{k\}}\left(\widehat{\Phi}\right) \geq \overline{\mathrm{MSE}}_1\left(\widehat{\Phi}\right),$$

so we have a lower bound for the maximum value. In the first part of this proof, we showed that the average MSE is minimized if and only if the frame is tight. When the frame is tight, each erasure contributes equally to the average due to (5.8). This means that the maximum value of $\mathrm{MSE}_{\{k\}}$ can be equal to the average by choosing a tight frame. Since the maximum value cannot be lowered further, this proves that the minimax solution is obtained when the frame is tight.

□

The theorem says that among the uniform frames the tight frames will give the best performance when considering the MSE caused by a single erasure. To elaborate this further, we give the following example.

Example 5.15 *This example will illustrate what Theorem 5.14 states, namely that a tight frame has better robustness towards single erasures both in average and in worst-case.*

Consider in $\mathbb{R}^2$ the Mercedes Benz frame Φ and the frame Ψ with analysis operators

$$F = \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

respectively, see Figures 12 and 13. Now consider the deletion of one frame vector. The MSE can be computed by (4.8). For each erasure of a vector from Φ a non-tight frame is left with eigenvalues $\lambda(F_1^* F_1) = \{\frac{3}{2}, \frac{1}{2}\}$. The MSE is then

$$\text{MSE} = \frac{\sigma^2}{N}\left(\frac{1}{3/2} + \frac{1}{1/2}\right) = \frac{8}{3}\frac{\sigma^2}{N}.$$

Since this is the case for each erasure, the average case and worst-case values will also be $8/3 \cdot \sigma^2/N$.

For Ψ the erasure of ψ_3 results in a tight frame with $\lambda(G_{\{3\}}^* G_{\{3\}}) = \{1, 1\}$ and an MSE of $2\sigma^2/N$. This is better than for Φ, even though the original frame Ψ was not tight. However if instead ψ_1 or ψ_2 is deleted, we get $\lambda(G_{\{1\}}^* G_{\{1\}}) = \lambda(G_{\{2\}}^* G_{\{2\}}) = \{1 + \sqrt{1/2}, 1 - \sqrt{1/2}\}$ and

$$\text{MSE} = \frac{\sigma^2}{N}\left(\frac{1}{1+\sqrt{1/2}} + \frac{1}{1+\sqrt{1/2}}\right) = 4\frac{\sigma^2}{N},$$

which is much poorer than for Φ. In average and worst-case Ψ gives $10/3 \cdot \sigma^2/N$ and $4 \cdot \sigma^2/N$ which is worse than the $8/3 \cdot \sigma^2/N$ for Φ. So the tight frame has better performance in both cases, which was the result of Theorem 5.14.

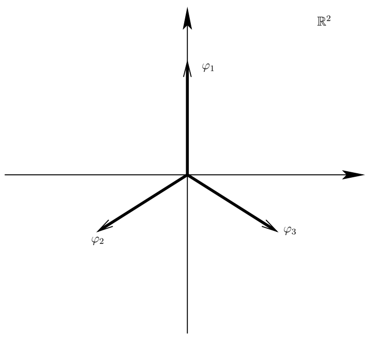

Figure 12: The Mercedes Benz frame Φ.

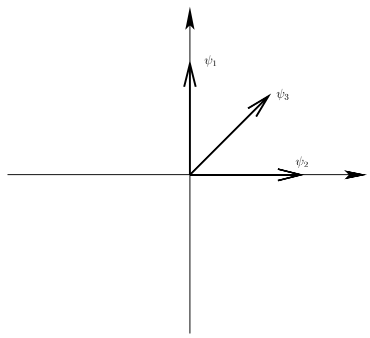

Figure 13: The non-tight frame Ψ.

5.4 MSE with more than one erasure

The natural thing to do next would be to try and extend Theorem 5.14 to cover more than one erasure. Unfortunately a similar result is not true for an arbitrary number of erasures, in fact already in the case of two erasures it is not true that all tight frames minimize the MSE, which is seen from the following example.

Example 5.16 *This example will illustrate that not all tight frames minimize the MSE in the case of two erasures.*

Consider in $\mathbb{R}^2$ *the uniform frames* Φ *and* Ψ *with analysis operators*

$$F = \begin{pmatrix} 0 & 1 \\ \frac{9}{10} & \sqrt{1-\left(\frac{9}{10}\right)^2} \\ 0 & 1 \\ -\sqrt{1-\left(\frac{9}{10}\right)^2} & \frac{9}{10} \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 & 1 \\ \frac{3}{5} & \frac{4}{5} \\ 0 & 1 \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

respectively, see Figures 14 and 15.

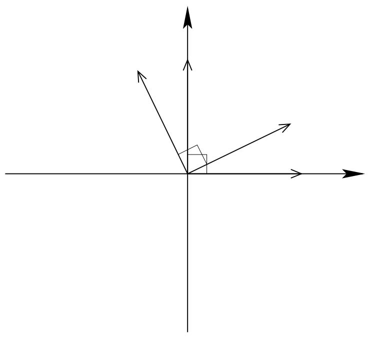

Figure 14: The tight frame Φ.

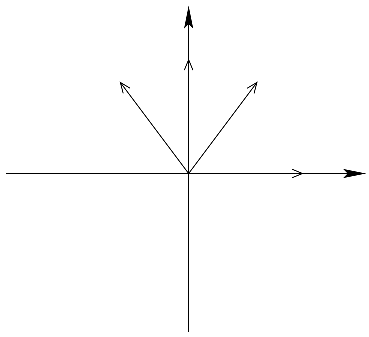

Figure 15: The non-tight frame Ψ.

Φ *is tight and* Ψ *is non-tight, since it is easy to show that*

$$F^*F = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad G^*G = \begin{pmatrix} \frac{43}{25} & 0 \\ 0 & \frac{57}{25} \end{pmatrix}.$$

However, using the same procedure as in Example 5.15 we find (numerically) the average and worst-case MSE*'s summarized by Table 2. It is clear that the tight frame has poorer perfomance in this case.*

Frame	$\overline{\text{MSE}}_2$	$\max \text{MSE}$
Φ	4.9985	10.5263
Ψ	3.5885	5.5556

Table 2: Comparison of MSE's. The factor σ^2/N has been omitted for simplicity. Even though Φ is tight, the MSE is larger in both cases than for Ψ that is non-tight.

However, inspired by the nice properties in the case of zero and one erasure, and the overall simplicity of tight frames compared to general frames, we will consider the MSE for a tight frame in the case where an arbitrary number of erasures can occur.

Let Φ be a uniform tight frame and assume that e erasures have occured at positions given by the index set E. Assume furthermore that Φ_E is a frame, and let F_E denote the corresponding analysis operator. By (5.6) we know that the MSE depends on the trace of $(F_E^*F_E)^{-1}$. Since Φ is tight, $F_E^*F_E$ can be written as

$$F_E^*F_E = \frac{M}{N}I_N - QQ^*,$$

where Q is an $N \times e$ matrix with columns $\{\varphi_k\}_{k\in E}$. To arrive at a nice expression for the MSE, we perform a number of steps. First we use Proposition C.6 [5]:

$$\begin{aligned}(F_E^*F_E)^{-1} &= \frac{N}{M}I_N + \frac{N}{M}I_N Q\left(I_e - Q^*\frac{N}{M}I_N Q\right)^{-1}Q^*\frac{N}{M}I_N \\ &= \frac{N}{M}I_N + \frac{N^2}{M^2}Q\left(I_e - \frac{N}{M}Q^*Q\right)^{-1}Q^*. \end{aligned} \tag{5.21}$$

We compute $\operatorname{tr}(F_E^*F_E)^{-1}$ from (5.21):

$$\begin{aligned}\operatorname{tr}(F_E^*F_E)^{-1} &= \operatorname{tr}\left[\frac{N}{M}I_N + \frac{N^2}{M^2}Q\left(I_e - \frac{N}{M}Q^*Q\right)^{-1}Q^*\right] \\ &= \frac{N^2}{M} + \frac{N^2}{M^2}\operatorname{tr}\left[Q\left(I_e - \frac{N}{M}Q^*Q\right)^{-1}Q^*\right].\end{aligned}$$

To simplify this, we can make a series expansion of the matrix inversion using Neumann's theorem (Theorem A.5.3 in [1])

$$\left(I_e - \frac{N}{M}Q^*Q\right)^{-1} = \sum_{k=0}^{\infty}\left(\frac{N}{M}Q^*Q\right)^k,$$

which together with the cyclic property of trace gives the following

$$\begin{aligned}\operatorname{tr}(F_E^*F_E)^{-1} &= \frac{N^2}{M} + \frac{N^2}{M^2}\operatorname{tr}\left[\left(I_e - \frac{N}{M}Q^*Q\right)^{-1}Q^*Q\right] \\ &= \frac{N^2}{M} + \frac{N^2}{M^2}\operatorname{tr}\left[\sum_{k=0}^{\infty}\left(\frac{N}{M}Q^*Q\right)^k Q^*Q\right] \\ &= \frac{N^2}{M} + \frac{N^2}{M^2}\sum_{k=0}^{\infty}\left(\frac{N}{M}\right)^k \operatorname{tr}\left((Q^*Q)^{k+1}\right).\end{aligned}$$

This is inserted into (5.6):

$$\begin{aligned}\mathrm{MSE}_E &= \frac{\sigma^2}{N}\left(\frac{N^2}{M} + \frac{N^2}{M^2}\sum_{k=0}^{\infty}\left(\frac{N}{M}\right)^k \operatorname{tr}\left((Q^*Q)^{k+1}\right)\right) \\ &= \frac{N\sigma^2}{M}\left(1 + \frac{1}{M}\sum_{k=0}^{\infty}\left(\frac{N}{M}\right)^k \operatorname{tr}\left((Q^*Q)^{k+1}\right)\right) \\ &= \left(1 + \frac{1}{M}\sum_{k=0}^{\infty}\left(\frac{N}{M}\right)^k \operatorname{tr}\left((Q^*Q)^{k+1}\right)\right)\mathrm{MSE}_0,\end{aligned}$$

[5] Let $A = M/N \cdot I_N, B = Q, C = I_e, D = Q^*$

where MSE_0 is the MSE for a uniform tight frame with zero erasures as in (4.11). To simplify this expression further, we proceed as follows. Let $\{\mu_i\}_{i=1}^{e}$ denote the eigenvalues of Q^*Q. By Proposition A.8, we can rewrite the series

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{N}{M}\right)^k \operatorname{tr}\left((Q^*Q)^{k+1}\right) &= \sum_{k=0}^{\infty}\left(\frac{N}{M}\right)^k \sum_{i=1}^{e}\mu_i^{k+1}\\ &= \sum_{i=1}^{e}\mu_i \sum_{k=0}^{\infty}\left(\frac{N}{M}\mu_i\right)^k \\ &= \sum_{i=1}^{e}\frac{\mu_i}{1-(N/M)\mu_i},\end{aligned} \tag{5.22}$$

where the convergence of $\sum_{k=0}^{\infty}\left(\frac{N}{M}\mu_i\right)^k$ will be addressed shortly. The expression for the MSE_E simplifies to the following theorem.

Theorem 5.17 *Consider encoding with a maximally robust uniform tight frame $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{R}^N$ with analysis operator F and decoding with the left-inverse (4.6) under the noise model (4.3). Let E with $e = |E|$ and $1 < e \leq M - N$ be the index set of erasures. The MSE is then*

$$\mathrm{MSE}_E = \left(1 + \sum_{i=1}^{e}\frac{\mu_i}{1-(N/M)\mu_i}\right)\mathrm{MSE}_0 \tag{5.23}$$

where Q is an $N \times e$ matrix with $\{\varphi_k\}_{k\in E}$ in the columns and $\{\mu_i\}_{i=1}^{e}$ are the eigenvalues of Q^*Q.

The series in (5.22) is geometric and is convergent if and only if $|(N/M)\mu_i| < 1$. We will show that $|(N/M)\mu_i| \leq 1$, where equality holds if and only if Φ_E is not a frame. To do that we need the following lemma.

Lemma 5.18 *Let $A \in \mathbb{C}^{m\times n}$. Then the non-zero eigenvalues of A^*A and AA^* are equal.*

Proof. Without loss of generality we can assume $m > n$. Make the singular value decomposition (SVD) of A:

$$A = U\Sigma V^*,$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are unitary and $\Sigma \in \mathbb{R}^{m\times n}$ has the singular values $\{\sigma_i\}_{i=1}^{n}$ in the diagonal and zeros everywhere else. Notice that by hermitian transposition we get the SVD of A^*:

$$A^* = V\Sigma^* U^*.$$

Now use the SVDs and the fact that U and V are unitary to find expressions for A^*A and AA^*:

$$\begin{aligned} A^*A &= V\Sigma^*U^*U\Sigma V^* = V\Sigma^*\Sigma V^* \\ &= V\begin{pmatrix} |\sigma_1|^2 & & & \\ & |\sigma_2|^2 & & \\ & & \ddots & \\ & & & |\sigma_n|^2 \end{pmatrix} V^* \end{aligned}$$

and

$$\begin{aligned} AA^* &= U\Sigma V^*V\Sigma^*U^* = U\Sigma\Sigma^*U^* \\ &= U\left(\begin{array}{cccc|c} |\sigma_1|^2 & & & & \\ & |\sigma_2|^2 & & & \mathbf{0} \\ & & \ddots & & \\ & & & |\sigma_n|^2 & \\ \hline & & \mathbf{0} & & \mathbf{0} \end{array}\right) U^*. \end{aligned}$$

From this it can be seen that $\{|\sigma_i|^2\}_{i=1}^n$ are the eigenvalues of A^*A and $\{|\sigma_i|^2\}_{i=1}^n \cup \{0\}_{i=1}^{m-n}$ are the eigenvalues of AA^*. Except for the zeros the eigenvalues are the same, and the proof is complete.

□

Denote by $\{\mu_i\}_{i=1}^e$ the eigenvalues for Q^*Q. By Lemma 5.18 we know that these eigenvalues are equal to the non-zero eigenvalues of QQ^*. F^*F and QQ^* can be expressed as

$$F^*F = \sum_{k=1}^{M} \varphi_k\varphi_k^* \quad \text{and} \quad QQ^* = \sum_{k=1}^{e} \varphi_k\varphi_k^*.$$

Now let v be a normalized eigenvector for QQ^* corresponding to the eigenvalue μ_i. We then have

$$\begin{aligned} v^*F^*Fv &= v^*\left(\sum_{k=1}^{M} \varphi_k\varphi_k^*\right)v \\ &= v^*QQ^*v + v^*\left(\sum_{k=e+1}^{M} \varphi_k\varphi_k^*\right)v \\ &= \mu_i + \sum_{k=e+1}^{M} |v^*\varphi_k|^2 \geq \mu_i \end{aligned} \tag{5.24}$$

Since Φ is tight, $F^*F = (M/N)I_N$, and since v is normalized, we have that $v^*F^*Fv = (M/N)||v|| = M/N$. This means that $\mu_i \leq M/N$, thus proving that

$|(N/M)\mu_i| \leq 1$. Equality in (5.24) holds if and only if the summation in (5.24) has sum 0, which is true if and only if v is orthogonal to all the vectors in $\Phi_E = \{\varphi_k\}_{k=e+1}^M$. Then the remaining vectors do not span the entire N-dimensional space, and hence Φ_E cannot be a frame. Due to our earlier stated assumption that Φ_E is a frame, we must then have $|(N/M)\mu_i| < 1$ and the series in (5.22) will be convergent.

We summarize the results for the MSE for different numbers of erasures in the following theorem.

Theorem 5.19 *Consider encoding with a maximally robust uniform frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ with analysis operator F and decoding with the left-inverse (4.6) under the noise model (4.3). Let E with $e = |E|$ and $e = 0, \ldots, M-N$ be the index set of erasures and F_E the corresponding analysis operator for reconstruction and let $\{\lambda_i\}_{i=1}^N$ be the eigenvalues of $F_E^* F_E$. Then* MSE *is given by (4.8):*

$$\mathrm{MSE}_E = \frac{\sigma^2}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i}.$$

- *If $e = 0$, the MSE is minimized if and only if the frame is tight. In this case the MSE is given by (4.11):*

$$\mathrm{MSE}_0 = \frac{N}{M}\sigma^2$$

- *If $e = 1$, then the average and maximum* MSE *among all posible single erasures is minimized if and only if the frame is tight, by Theorem 5.14. In this case Theorem 5.6 gives:*

$$\mathrm{MSE}_1 = \left(1 - \frac{1}{M-N}\right) \mathrm{MSE}_0,$$

for any single erasure, independently of which erasure it is.

- *If $e > 1$, then the MSE for a tight frame is by Theorem 5.17*

$$\mathrm{MSE}_E = \left(1 + \sum_{i=1}^{e} \frac{\mu_i}{M - N\mu_i}\right) \mathrm{MSE}_0,$$

where Q is an $N \times e$ matrix with $\{\varphi_k\}_{k\in E}$ in the columns and $\{\mu_i\}_{i=1}^e$ are the eigenvalues of Q^*Q.

This theorem is the main and final theoretical result of this report. It captures the behavior of frames with respect to the error from quantization and erasures. The conclusion that can be drawn from this is, that uniform tight frames may not always be the optimal choice. But due to the MSE-minimization in the 0 and 1 erasure case and simplicity in the multiple erasure case, they are well-suited for the application on robust signal transmission. The maximally robust frames, for instance the harmonic tight frames, are especially good, since they can withstand a maximal number of erasures.

The final section of the report contains a few numerical demonstrations of the use of frames in signal transmission.

6 Numerical analysis

In this section we use numerical examples to demonstrate the advantages of using frames for signal transmission where erasures can occur. Inspired by the theoretical results we will only use uniform tight frames. More specifically we will use the real harmonic tight frames from Theorem 3.24.

The main part of this project has been concerned with the theoretical results for finite dimensional frames, and numerical investigations have only played a minor role in the last phase. We want to stress that the experiments made in this section are not intended to be an accurate description of a transmission line but merely a simple demonstration of the benefits of frames, and the results should be seen for illustration purposes only.

6.1 A model of a signal transmission system

For a signal of length N the intuitive way to implement an encoding using the frame expansion is to use an analysis operator F of size $M \times N$ for some $M \geq N$. The reconstruction is performed using $(F_E^* F_E)^{-1} F_E^*$ where F_E is the operator left when the rows corresponding to the erasure set E have been removed. However, this method has a major drawback since the computational effort taken to find $(F_E^* F_E)^{-1}$ is very large, since N will be a quite large number in practice. To avoid the large matrix-inversion we will divide the signal into smaller subsignals.

Our signal transmission system consists of an encoding part, a lossy transmission and a decoding. We begin by describing the encoding. For a signal x, now of length $|x| = pN$, the following steps are performed:

1. Choose an appropriate uniform tight frame $\Phi = \{\varphi_k\}_{k=1}^M \subseteq \mathbb{R}^N$. Denote the analysis operator by F.

2. Divide the signal x into smaller parts $\{x_i\}_{i=1}^p$ each of length N and arrange the parts in the columns of an $N \times p$ matrix X

$$X = \begin{pmatrix} | & | & & | \\ | & | & & | \\ x_1 & x_2 & \cdots & x_p \\ | & | & & | \\ | & | & & | \end{pmatrix}.$$

3. Make the frame expansion of each column, $y_i = F x_i,\ i = 1, \ldots, p$. This corresponds to introducing the matrix $Y \in \mathbb{R}^{M \times p}$, given by $Y = FX$

$$Y = \begin{pmatrix} | & | & & | \\ | & | & & | \\ y_1 & y_2 & \cdots & y_p \\ | & | & & | \\ | & | & & | \end{pmatrix}.$$

4. Denote the rows of Y by $\{z_k\}_{k=1}^M$,

$$Y = \begin{pmatrix} — & z_1 & — \\ — & z_2 & — \\ & \vdots & \\ — & z_M & — \end{pmatrix}.$$

5. The vectors $\{z_k\}_{k=1}^M$ are now ready to be sent as packages through a transmission line.

During the transmission of $\{z_k\}_{k=1}^M$ the erasures described by the index set E occur. This means that the set of vectors $\{z_k\}_{k\in E}$ are lost, and the signal must be restored using the remaining information. Losing the vector z_k is equivalent to losing the k'th vector-element in every vector $\{y_i\}_{i=1}^p$. Since we are sending the vectors $\{z_k\}_{k=1}^M$, we have the same erasure set for all the $\{y_i\}_{i=1}^p$, thus enabling us to use *the same* analysis operator in the reconstruction of all $\{y_i\}_{i=1}^p$. We proceed by using the analysis operator F_E for the decoding process:

1. Stack the received row-vectors $\{z_k\}_{k\notin E}$ to obtain the matrix $\overline{Y}$.
2. Reconstruct an approximation $\overline{X}$ to the original matrix X, by computing

$$\overline{X} = (F_E^* F_E)^{-1} F_E^* \overline{Y}.$$

3. Rearrange the matrix $\overline{X}$, in order to obtain the reconstructed signal $\overline{x}$.

In the following section, we will use this method to simulate the transmission of a sound sample.

6.2 Test setup

We will use MATLAB to perform our numerical experiments. Quantization issues will be ignored, since MATLAB automatically performs quantization to 16 digits.

In our experiments we will use a 30s sound sample of the song "Gravedigger" by Dave Matthews Band. The sample rate is is 44.1 kHz, so each second contains 44100 numbers. Each second is treated as a separate signal, which is then given as input to the algorithm described above. One reason for this is size of the involved matrices. Furthermore we wish to illustrate that this method can be used to ensure a continuous data flow, for instance streaming of a music signal. In this way, the receiver can decode each second separately as they arrive, and be able to play the sound continuously.

As mentioned earlier we will encode using the harmonic tight frames. To do that we have implemented a function `harmonic` that for a given dimension N and desired number of vectors M computes the analysis operator. The code can be seen in Appendix D.2. Furthermore the rest of the scripts and functions used in the tests can be seen in Appendix D.1 - D.4.

In all the tests the analysis operator will be of size 330×300, so the frame should be able to withstand up to 30 erasures. A given second of the signal will then be reshaped to a matrix of size 300×147 as described in the previous section. This size turned out to be efficient for computations in MATLAB.

6.3 Test results

The aim of the theory described in this report was to improve the robustness of signal transmission by using frames instead of an ONB. To illustrate the sensitivity of an ONB, we have constructed the following example.

Example 6.1 *To show the advantage of a frame over an ONB, the seconds of the signal* gravedigger *are subjected to an increasing number of erasures, beginning with 0 and ending with 29. The differences in norm between the original and the reconstructed signal in the two cases are shown in Figure 16.*

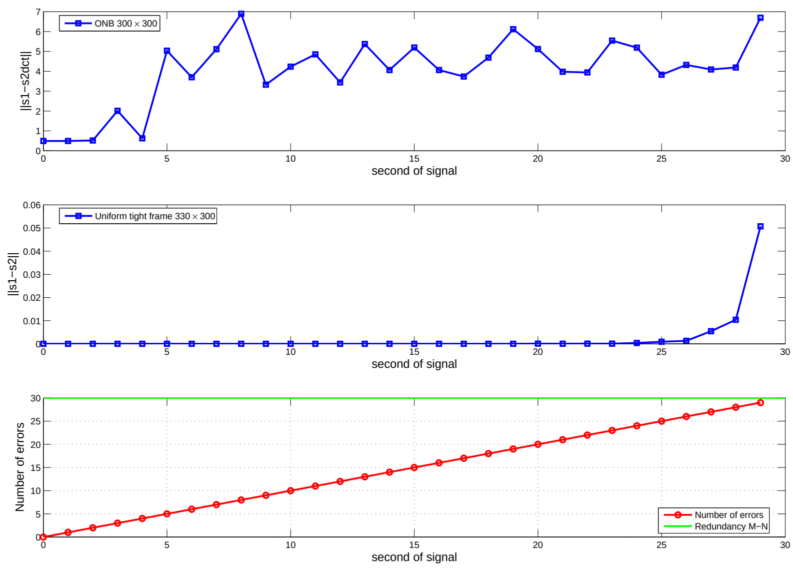

Figure 16: The seconds of the signal *gravedigger* are subjected to an increasing number of erasures as shown by the lower plot. The top plot illustrates the errors when using a 300×300 ONB and the middle plot shows the errors when using the frame expansion instead. Notice the different scales on the y-axes. The frame expansion has much better performance since the error is almost zero up to more than 25 erasures.

We can see that when using an ONB a quite large error is quickly introduced. Even for a few erasures the signal is distorted. For the frame the error is kept nearly *zero for up to around 25 erasures. This clearly demonstrates the supremacy of the frame expansion. The price of the extra accuracy is of course the extra data that must be sent. In this example an extra 10 % needs to be sent.*

A substantial error is however introduced by using the frame, when the number of erasures approaches the limit. In theory the frame operator $F_E^* F_E$ for a harmonic tight frame is invertible, as long as $|E| \leq (M-N)$. This is the content of Theorem 5.5. In practice, however, computations with $F_E^* F_E$ can be problematic, since the matrix can be numerically close to singularity. This can be described by the so-called *condition number* of a matrix, see Definition C.5. If the condition number is large, the matrix is numerically close to singular and said to be *ill-conditioned*. In our tests, reconstruction using $(F_E^* F_E)^{-1}$, when $F_E^* F_E$ was singular (or close to), it gave rise to a high level of error. This is demonstrated in the following example.

Example 6.2 *The signal is sent using the described method. Each second is subjected to a random number of erasures between 24 and 32 and the absolute difference between the original and the reconstructed signal is plotted together with the number of erasures, see Figure 17.*

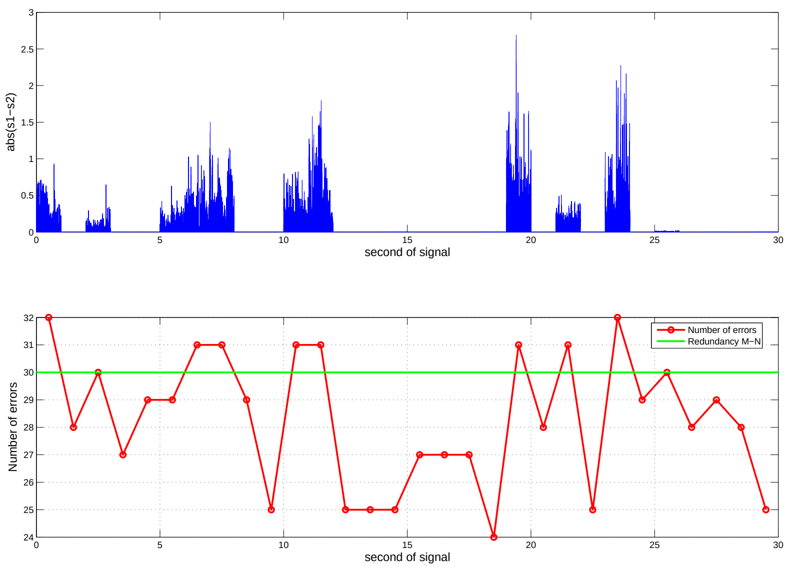

Figure 17: The signal *gravedigger*. For each second a random number of erasures between 24 and 32 is applied and the absolute difference between the reconstruction and the original signal is measured. It is clear that when the number of erasures exceeds 30, the difference is very large, since the completeness is lost. On the other hand notice the sixth second. Even though the number of erasures is only 29 there is a relatively large distortion. This is due to a high condition number for $F_E^* F_E$ used for the reconstruction.

A large error is observed whenever the number of erasures exceeds the limit of 30. This is no surprise since the remaining frame vectors do not span the entire space. Notice however that there is a relatively large error in the sixth second, even though the number of erasures is only 29. The reason is an ill-conditioned frame operator. This results in a poor reconstruction.

As seen in the example above, high levels of error can occur even for non-singular frame operators. But how often does this happen? In order to have an idea of this we perform the following experiment.

Example 6.3 *The first second of the signal is subjected to* $e = 1, \ldots, 33$ *randomly chosen erasures. For each* e *this is repeated 1000 times in order to have a large amount of data. The difference in norm between the original and the reconstructed signal is measured. For each* e *it is noted how often this error is larger than* 10^{-3}*, a tolerance chosen after multiple tests. Table 3 summarizes the results.*

Erasures	≤19	20	21	22	23
Probability for error (%)	0%	0.3%	0.5%	1.5%	1.5%
Erasures	24	25	26	27	28
Probability for error (%)	1.9%	4.0%	8.1%	13%	18.9%
Erasures	29	30	≥31		
Probability for error (%)	32.2%	46.7%	100%		

Table 3: The probability that the decoding will result in a high level of error (larger than 10^{-3}) for a given number of erasures. The table is calculated using 1000 tests on the first second of the signal for each number of random erasures with a harmonic tight frame $\Phi = \{\varphi_k\}_{k=1}^{330}$ in $\mathbb{R}^{300}$.

It is very seldom to have a large error with 20 erasures or less, and even for up to 25 the error-rate is negligible. After this the rate increases and is quite large for 29 and 30 erasures. When e *exceeds 30,* $F_E^* F_E$ *is singular and there will always be a large error when attempting to reconstruct using* $(F_E^* F_E)^{-1}$*.*

As the example demonstrates the error increases rapidly when the number of erasures approaches the limit. The reason for this is as mentioned that $F_E^* F_E$ is (close to) singular which makes the inversion unstable numerically. Is there a way to circumvent this problem? We find inspiration in the way erasures are typically handled when using the Discrete Fourier Transform (DFT). Here a zero is introduced instead of the coefficient lost in transmission and the signal is reconstructed using the same operator that was used for encoding. That is, no operator F_E is introduced. This implies that it is only necessary to invert the frame operator once for the entire signal. The problem with this approach is that some noise is introduced even for a small number of erasures, but the large errors from the inversion are avoided.

Inspired by this we develop three different methods to handle the problem: *allzeros*, *excesszeros* and *allornone*. The original method using simply F_E will be referred to as *normal*. In *allzeros* each erased coefficient is replaced by a zero and reconstructed using $(F^* F)^{-1}$. In *excesszeros* the approach is the following: The erasure set E is split into two sets, E_0 and E_1. E_0 contains the first $M - N - \delta$ erasures, where $\delta \in \mathbb{N}$, and E_1 the rest. The erasures from E_1 are handled by introduction of zeros and the erasures in E_0 by reconstruction using F_{E_0}. The parameter δ is useful since it can be seen from Table 3 that singularity problems can occur before the limit $M - N$. In *allornone* the number $M - N - \delta$ is also used. If $e < M - N - \delta$ the *normal* method is used and if $e \geq M - N - \delta$ all erasures are handled by the introduction of zeros. The methods are summarized in Table 4.

Name	Delimiter	Description	Result
normal	None	The error set E is removed from the frame expansion y, and the signal is reconstructed using the frame operator $F_E^* F_E$	Low error as long as $\|E\| \lessapprox M - N$, else a very high error.
allzeros	None	The elements from the frame expansion at the indexes from E are all set to zero, and the frame operator $F^* F$ is used for reconstruction.	Medium error for all E, the error increases with $\|E\|$.
excesszeros	$M-N-\delta$	The error set E is devided into two sets E_0 and E_1, where $\|E_0\| = M - N - \delta$. The errors in E_0 are removed from the frame expansion, end the elements in the frame expansion at the indexes E_1 are all set to zero. The signal is reconstructed using $F_{E_0}^* F_{E_0}$.	Very high error for $\|E_1\| > 0$.
allornone	$M-N-\delta$	If $\|E\| \leq M - N - \delta$ the signal is reconstructed using the approach *normal*. If $\|E\| > 0$ the method *allzeros* is used instead.	Low error for all E, difficult to determine good δ in general.
condition	Condition number of $F_E^* F_E$	For small condition numbers of the frame operator $F_E^* F_E$ the *normal* method is used. Otherwise the *allzeros* method is used.	Very low error.

Table 4: Overview of the five reconstruction methods.

The advantage of *allzeros* is that the large errors from the inversion will be avoided at the price of introducing a small noise everywhere even for few erasures. *excesszeros* is an attempt to avoid this small added noise by only inserting zeros when the number of erasures is high. As the next example will show, large errors will still occur for this approach. An alternative is *allornone* where either F_E is used (up to $M - N - \delta$ erasures) or instead zeros are introduced and F is used. The example will show that the performance is better, but large errors can still occur.

These considerations gives rise to the introduction of yet another approach, that we will denote *condition*. Since the problem is often the inversion, it could be useful to test whether $F_E^* F_E$ has a large condition number, instead of testing whether the number of erasures exceeds $M - N - \delta$. If the condition number is large, the matrix is close to singular and the method *allzeros* will be used. If the condition number is sufficiently low, the normal method with F_E will be used. As the following example will show, this method turns out to give the best results.

Example 6.4 *The 30 sec sample* gravedigger *is transmitted with a random number of erasures between 26 and 32 applied to each second and reconstructed using the five different methods* normal*,* allzeros*,* excesszeros*,* allornone *and* condition*. The results are shown in Figure 18.*

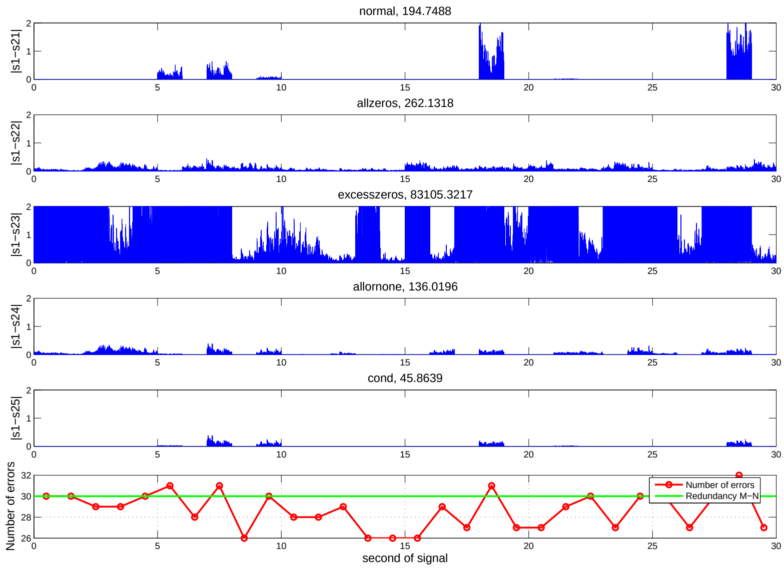

Figure 18: Comparison of the five reconstruction methods. From top to bottom: *normal*, *allzeros*, *excesszeros*, *allornone*, *condition* and the number of erasures in each second. The title of each plot gives the name of the used reconstruction method and the sum of the norm differences for each second.

The absolute difference between the original signal and the reconstructions are seen in the first five plots and the number of errors in the last. For normal *there are few but dominating errors whenever there is a large number of erasures. When there are few erasures the error is negligible. For* allzeros *there is some noise no matter how many erasures have occured. This is what we expected and this type of noise is undesired. The first attempt to fix it,* excesszeros*, fails severely, since a number of large errors are introduced. We expect the errors to be a result of a numerically unstable inversion but these errors are larger than we would have expected. This could perhaps be an indication of a mistake in the implementation. The next attempt,* allornone*, has much better performance. However noise is still introduced in a lot of the seconds. This is fixed by* condition *where the noise is* *only introduced exactly when the inversion is undesirable due to a large condition number. It is interesting to compare* normal *and* condition*. Both have (almost) no error for few erasures and when there is a large number of erasures* condition *reduces the large inversion error to a small noise.*

To summarize the results of this section, we have seen that the harmonic tight frames significantly reduces the distortion caused by erasures during the transmission, compared to the performance of an ONB. Furthermore we have developed a good method to handle the inversion problems when the frame operator is ill-conditioned in order to minimize the distortion.

7 Conclusion

In the report we have proved a number of theoretical results for frames in finite dimensional vector spaces. The highlights of the first section was the development of the frame expansion using the dual frame, the properties of the eigenvalues of the frame operator, and the construction of the harmonic tight frames, that have maximal robustness towards erasures.

The theoretical results gave rise to the application of frames on robust signal transmission. We saw that frames show stability towards quantization and erasures of the frame coefficients. The main result here was Theorem 5.19 that stated that uniform tight frames minimize quantization error, minimize the average and worst-case MSE caused by one erasure and give a relatively simple expression for the MSE for multiple erasures.

Finally we performed a few numerical experiments to illustrate the use of frames on signal transmission. We saw that the harmonic tight frames indeed do give very good robustness towards erasures in the transmission, since the error from reconstruction was surprisingly small even for a relatively large number of erasures.

During the work with this report we have met a number of different challenges: Grasping a complicated, new mathematical subject, reading and understanding a scientific paper and not least discovering and filling out all the gaps in the proofs. We have come across a wide range of mathematical subjects, especially in functional analysis and linear algebra. Furthermore we have constructed several proofs of our own and improved our ability to explain mathematical concepts.

In addition we have seen a strong interaction between mathematics and applications, and feel inspired to continue our studies in applied mathematics.

A Properties of trace

Definition A.1 *Given a square matrix A, the trace is defined by the following map:*

$$\mathrm{tr}:\ \mathbb{C}^{n\times n} \to \mathbb{C}: \quad \mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}. \tag{A.1}$$

In other words, the trace of a square matrix is the sum of the diagonal elements.

Proposition A.2 *The trace is a linear map.*

Proof. Let $A, B \in \mathbb{C}^{n\times n}$ and $c_1, c_2 \in \mathbb{C}$. We want to show that $\mathrm{tr}(c_1 A + c_2 B) = c_1 \mathrm{tr}(A) + c_2 \mathrm{tr}(B)$. Let $D = c_1 A + c_2 B$. Then we have $d_{ii} = c_1 a_{ii} + c_2 b_{ii}$ from the properties of summation and multiplication by scalar of matrices. But this implies

$$\begin{aligned}
\mathrm{tr}(c_1 A + c_2 B) = \mathrm{tr}(D) &= \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} (c_1 a_{ii} + c_2 b_{ii}) \\
&= c_1 \sum_{i=1}^{n} a_{ii} + c_2 \sum_{i=1}^{n} b_{ii} = c_1 \mathrm{tr}(A) + c_2 \mathrm{tr}(B).
\end{aligned}$$

□

Proposition A.3 *For a matrix $A \in \mathbb{C}^{n\times n}$ the following holds*

$$\mathrm{tr}(A^*) = \overline{\mathrm{tr}(A)}.$$

Proof.

$$\mathrm{tr}(A^*) = \sum_{i=1}^{n} a_{ii}^* = \sum_{i=1}^{n} \overline{a_{ii}} = \overline{\sum_{i=1}^{n} a_{ii}} = \overline{\mathrm{tr}(A)},$$

where Definition 1.2 has been used.

□

Proposition A.4 *For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$ we have*

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Proof. Notice first that AB is $n \times n$ and BA is $m \times m$. Due to the structure of matrix multiplication we have for the diagonal elements of AB and BA respectively

$$\begin{aligned}
(AB)_{ii} &= \sum_{j=1}^{m} a_{ij} b_{ji} \quad \text{for} \quad i = 1, \ldots, n, \\
(BA)_{jj} &= \sum_{i=1}^{n} b_{ji} a_{ij} \quad \text{for} \quad j = 1, \ldots, m.
\end{aligned}$$

Using that the trace is the sum of the diagonal elements by Definition A.1 we have

$$\begin{aligned}\operatorname{tr}(AB) &= \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ji} \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{m}(BA)_{jj} = \operatorname{tr}(BA),\end{aligned}$$

where we have used that the order of summation can be interchanged.

□

Proposition A.5 *Consider n complex matrices $A_1, A_2, \ldots, A_k$ such that the product $A_1A_2\cdots A_k$ exists and is square. Furthermore assume that all products of the form $A_{i+1}\cdots A_kA_1A_2\cdots A_i$ for any $i \in \mathbb{N}$, $i \leq k$ exist and are square. Then the trace of the product is invariant to cyclic permutation of the matrices, that is for any $i \in \mathbb{N}$, $i \leq k$*

$$\operatorname{tr}(A_1A_2\cdots A_k) = \operatorname{tr}(A_{i+1}\cdots A_kA_1A_2\cdots A_i).$$

Proof. Consider the matrices $A = A_1A_2\cdots A_{k-1}$ and $B = A_k$. Since by assumption the products AB and BA exist and are square, there exist numbers $n, m \in \mathbb{N}$ such that A is $n \times m$ and B is $m \times n$. By Proposition A.4 we have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ or

$$\operatorname{tr}(A_1A_2\cdots A_k) = \operatorname{tr}(A_kA_1A_2\cdots A_{k-1}).$$

Repeat this argument $k-1$ times to include all the cyclic permutations and the proof is complete.

□

Proposition A.6 *The trace is invariant to similarity transformation, that is given a square matrix A then for any invertible matrix P such that the product $P^{-1}AP$ exists*

$$\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(A).$$

Proof. By Proposition A.5 the trace is invariant to cyclic permutation, which implies

$$\operatorname{tr}(P^{-1}AP) = \operatorname{tr}(PP^{-1}A) = \operatorname{tr}(A).$$

□

Proposition A.7 *For an $n \times n$ matrix A with eigenvalues $\{\lambda_i\}_{i=1}^{n}$ counted with multiplicity, the sum of the eigenvalues equals the trace of A, that is*

$$\operatorname{tr}(A) = \sum_{k=1}^{n}\lambda_k.$$

Proof. See for instance Theorem 7.9 in [2].

□

Proposition A.8 *Let A be an $n \times n$ matrix with eigenvalues $\{\lambda_i\}_{i=1}^n$. Then for any $k \in \mathbb{N}$*

$$\operatorname{tr}\left(A^k\right) = \sum_{i=1}^n \lambda_i^k.$$

Proof. By spectral decomposition $A = U\Lambda U^*$, where $\Lambda = diag\left(\{\lambda_i\}_{i=1}^n\right)$ and U is unitary. Then $U^*U = I_n$ and

$$A^k = (U\Lambda U^*)^k = U\Lambda^k U^*.$$

$\Lambda^k = diag\left(\{\lambda_i\}_{i=1}^n\right)$ and by Proposition A.5

$$\operatorname{tr}\left(A^k\right) = \operatorname{tr}\left(U\Lambda^k U^*\right) = \operatorname{tr}\left(\Lambda^k U^* U\right) = \operatorname{tr}\left(\Lambda^k\right) = \sum_{i=1}^n \lambda_i^k.$$

□

B Special matrices and their properties

Definition B.1 *A matrix $U \in \mathbb{C}^{n\times n}$ is called unitary if*

$$UU^* = U^*U = I_n.$$

Proposition B.2 *For a matrix $U \in \mathbb{C}^{n\times n}$ the following are equivalent*

1. *U is unitary,*
2. *$U^{-1} = U^*$.*

Proof. The proof follows directly from Definition B.1.

□

Proposition B.3 *The product of two unitary $n \times n$ matrices is unitary.*

Proof. Let U_1 and U_2 be two $n \times n$ unitary matrices. Then

$$U_1{}^*U_1 = U_1U_1{}^* = I_n \quad \text{and} \quad U_2{}^*U_2 = U_2U_2{}^* = I_n.$$

From this it is easy to see that

$$\begin{aligned}(U_1U_2)(U_1U_2)^* &= U_1U_2U_2{}^*U_1{}^* = U_1U_1{}^* = I_n,\\ (U_1U_2)^*(U_1U_2) &= U_2{}^*U_1{}^*U_1U_2 = U_2{}^*U_2 = I_n,\end{aligned}$$

and hence the product U_1U_2 is unitary.

□

Proposition B.4 *Let U_1, U_2, U_3 be self-adjoint operators. If $U_1 \leq U_2$, $U_3 \geq 0$, and U_3 commutes with U_1 and U_2, then $U_1U_3 \leq U_2U_3$.*

Proof. See Theorem 68.9 in [6].

C Other useful results

Proposition C.1 *Let $f : S \to \mathbb{R}$ be a continuous function defined on a compact, connected topological space S. Then the image $f(S)$ of S under f is a closed and bounded interval $[c, d]$ in $\mathbb{R}$.*

Proof. A proof for this proposition can be found in [8] on page 50.

Proposition C.2 *For a geometric series it holds that if $x \neq 1$ then*

$$\sum_{k=0}^{n-1} ax^k = a\frac{1-x^n}{1-x}.$$

Proof. Consider the product

$$(1-x)(a + ax + ax^2 + \cdots + ax^{n-1}).$$

By carrying out the multiplication a telescoping series arises:

$$a + ax - ax + ax^2 - ax^2 + \cdots + ax^{n-1} - ax^{n-1} - ax^n = a - ax^n.$$

Now divide by $(1-x)$ on both sides and we have

$$a + ax + ax^2 + \cdots + ax^{n-1} = a\frac{1-x^n}{1-x},$$

which completes the proof.

Proposition C.3 *Let $\{\varphi_i\}_{i=1}^M$ be a frame for H^N with frame operator F^*F. If $x \in H^N$ has the representation $x = \sum_{i=1}^M c_i\varphi_i$ for some scalar coefficients $\{c_i\}_{i=1}^M$, then*

$$\sum_{i=1}^{M} |c_i|^2 = \sum_{i=1}^{M} |\left\langle x, (F^*F)^{-1}x \right\rangle|^2 + \sum_{i=1}^{M} |c_i - \left\langle x, (F^*F)^{-1}x \right\rangle|^2.$$

Proof. A proof for this proposition can be found in [1] page 5.

Proposition C.4 *The removal of a vector φ_j from a frame $\{\varphi_i\}_{i=1}^M$ for H^N leaves either a frame or an incomplete set. More precisely*

if $\left\langle \varphi_j, (F^*F)^{-1}\varphi_j \right\rangle \neq 1$, then $\{\varphi_i\}_{i\neq j}$ is a frame for H^N,
if $\left\langle \varphi_j, (F^*F)^{-1}\varphi_j \right\rangle = 1$, then $\{\varphi_i\}_{i\neq j}$ is incomplete.

Proof. A proof for this proposition can be found in [1] page 100.

Definition C.5 *For a nonsingular matrix A the condition number is*

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|.$$

From [4] Definition 8.11.1.

Proposition C.6 *For the regular matrices A and C, and some matrices B and D of appropriate size, the following holds*

$$(A - BCD)^{-1} = A^{-1} + A^{-1}B\left(C^{-1} - DA^{-1}B\right)^{-1} DA^{-1},$$

assuming that $\left(C^{-1} - DA^{-1}B\right)^{-1}$ makes sense.

Proof. A proof can be found in [13] on page 451.

D MATLAB listings

D.1 signaltrans.m

```
function [s2,n] = signaltrans(s1,F,index,mode)

% Simulation of signal transmission.
% s1:       The signal to transmit
% F:        The analysis operator for the frame expansion
% index:    Specifies the frame vectors that are kept
% mode:     Chooses the reconstruction mode to use: normal,
%           allzeros, excesszeros, allornone, condition.
%
% Returns the reconstructed signal s2 and the norm difference n
% between s1 and s2.
%
% Authors: Martin McKinnon Edwards and Jakob Heide Jørgensen
% Date: June 19, 2007.

[M,N] = size(F);
% N: package size - it is assumed to divide length of s1
% M: number of frame vectors
len = length(s1);
s1resh = reshape(s1,N,len/N);

% FRAME EXPANSION
y = F*s1resh;

switch mode
    case 'normal'
        z = y(index,:);
        FE = F(index,:);
        [M-length(index)],[cond(FE'*FE)]
        FE_PSinv = (FE'*FE)\(FE');
    case 'allzeros'
        E = setdiff(1:M, index);
        y(E,:) = 0;
        z = y;
        FE_PSinv = (F'*F)\(F');
    case 'excesszeros'
        allE = setdiff(1:M, index)
        limit = 20;
        if length(allE) <= limit
            z = y(index,:);
            FE = F(index,:);
            FE_PSinv = (FE'*FE)\(FE');
        else
            E1 = allE(1:limit)
            E2 = allE(limit+1:end)
            y(E2,:)=0;
            indexkeep = setdiff(1:M,E1);
            length(indexkeep)
            z = y(indexkeep,:);
```

```
            FE = F(indexkeep,:);
            FE_PSinv = (FE'*FE)\(FE');
            excessmax = max(max(abs(FE_PSinv)))
            excesscond = cond(FE'*FE)
        end
    case 'allornone'
        allE = setdiff(1:M, index);
        if length(allE) <= M-N - 2
            z = y(index,:);
            FE = F(index,:);
            FE_PSinv = (FE'*FE)\(FE');
        else
            E = setdiff(1:M, index);
            y(E,:) = 0;
            z = y;
            FE_PSinv = (F'*F)\(F');
        end
    case 'condition'
        FE = F(index,:);
        c = cond(FE'*FE);
        if c < 1/eps
            z = y(index,:);
            FE_PSinv = (FE'*FE)\(FE');
            cond1max = max(max(abs(FE_PSinv)));
        else
            E = setdiff(1:M, index);
            y(E,:) = 0;
            z = y;
            FE_PSinv = (F'*F)\(F');
            cond2max = max(max(abs(FE_PSinv)));
        end
    otherwise
        error('wrong input of variable mode');
end

% RECONSTRUCT
xhat = FE_PSinv*z;
s2 = reshape(xhat,1,len)';
n = norm(s1-s2);
```

D.2 harmonic.m

```
function F = harmonic(M,N)

% Creates the analysis operator for a harmonic tight frame
% in R^N with M vectors
%
% Authors: Martin McKinnon Edwards and Jakob Heide Jørgensen
% Date: June 19, 2007.

F = zeros(M,N);

if ~mod(N,2)   % N even
    for k = 0:(M-1)
```

```
        l = 1:2:(N-1);
        F(k+1,:) = sqrt(2/N)*[cos(l*k*pi/M), sin(l*k*pi/M)];
    end
else          % N odd
    for k = 0:(M-1)
        l = 2:2:(N-1);
        F(k+1,:) = sqrt(2/N)*[1/sqrt(2), cos(l*k*pi/M), sin(l*k*pi
            /M)];
    end
end
```

D.3 transmissionplot.m

```
% Script to call the function signaltrans.m for each of the
% five reconstruction modes.
%
% Authors: Martin McKinnon Edwards and Jakob Heide Jørgensen
% Date: June 19, 2007.


close all
clc
clear

% Load the sound sample
load 'lyde/gravedigger30';
s1 = (sum(s1')/2)';
f = 44100;

% Setup variables for the five reconstructions
s1resh = reshape(s1, f, 30);
s2resh1 = zeros(size(s1resh));
s2resh2 = zeros(size(s1resh));
s2resh3 = zeros(size(s1resh));
s2resh4 = zeros(size(s1resh));
s2resh5 = zeros(size(s1resh));

% Setup the size of the frame and the lower and upper bound
% for the number of erasures introduced.
M = 330;
N = 300;
lower = 26;
upper = 32;

F = harmonic(M,N);

% Simulate the transmission of each of the 30 secs using
% the five methods
for i = 1:30
    e(i) = lower + round((upper-lower)*rand);
    index = leftset(M,e(i));
    [tempsig1, n1(i)] = signaltrans(s1resh(:,i),F,index,'normal');
    s2resh1(:,i) = tempsig1;
    [tempsig2, n2(i)] = signaltrans(s1resh(:,i),F,index,'allzeros'
```

```
        );
    s2resh2(:,i) = tempsig2;
    [tempsig3, n3(i)] = signaltrans(s1resh(:,i),F,index,'
        excesszeros');
    s2resh3(:,i) = tempsig3;
    [tempsig4, n4(i)] = signaltrans(s1resh(:,i),F,index,'allornone
        ');
    s2resh4(:,i) = tempsig4;
    [tempsig5, n5(i)] = signaltrans(s1resh(:,i),F,index,'cond');
    s2resh5(:,i) = tempsig5;
end

s21 = s2resh1(:);
s22 = s2resh2(:);
s23 = s2resh3(:);
s24 = s2resh4(:);
s25 = s2resh5(:);

% Plot the results
figure
numplots = 6;
maxy = 2;

subplot(numplots,1,1), plot(linspace(0,30,44100*30),abs(s1-s21))
title(['normal, ', num2str(sum(n1))],'FontSize',13)
ylabel('|s1-s21|','FontSize',13)
axis([0,30,0,maxy]);

subplot(numplots,1,2), plot(linspace(0,30,44100*30),abs(s1-s22))
title(['allzeros, ', num2str(sum(n2))],'FontSize',13)
ylabel('|s1-s22|','FontSize',13)
axis([0,30,0,maxy]);

subplot(numplots,1,3), plot(linspace(0,30,44100*30),abs(s1-s23))
title(['excesszeros, ', num2str(sum(n3))],'FontSize',13)
ylabel('|s1-s23|','FontSize',13)
axis([0,30,0,maxy]);

subplot(numplots,1,4), plot(linspace(0,30,44100*30),abs(s1-s24))
title(['allornone, ', num2str(sum(n4))],'FontSize',13)
ylabel('|s1-s24|','FontSize',13)
axis([0,30,0,maxy]);

subplot(numplots,1,5), plot(linspace(0,30,44100*30),abs(s1-s25))
title(['cond, ', num2str(sum(n5))],'FontSize',13)
ylabel('|s1-s25|','FontSize',13)
axis([0,30,0,maxy]);

subplot(numplots,1,numplots), plot((1:30)-.5,e,'-or','LineWidth'
    ,2)
hold on, plot([0,30],[M-N,M-N],'-g','LineWidth',2)
legend('Number of errors','Redundancy M-N')
xlabel('second of signal','FontSize',13)
ylabel('Number of errors','FontSize',13)
```

```
grid on
```

D.4 leftset.m

```
function index = leftset(M,e)

% Determines the index of the frame vectors to keep from the
% number M of frame vectors and the number e of erasures.
%
% Authors: Martin McKinnon Edwards and Jakob Heide Jørgensen
% Date: June 19, 2007.


% Compute a set of e indexes taken from 1:M
E = round(M*rand(1,e));
index = setdiff(1:M,E);

% Check if doubles are present. If so, repeat the computation
while length(index) ~= M-e
    E = round(M*rand(1,e));
    index = setdiff(1:M,E);
end
```

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