$\mathbb{TIME}$ & $\mathbb{SPACE}$ in Domains

– EARLY INCOMPLETE DRAFT –

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Abstract

I muse over $\mathbb{TIME}$ and $\mathbb{SPACE}$. Presently, at the time of starting to write this report, there is no definite goal in sight. I hope one will emerge ! I am just curious as to where this might lead me – us !

Contents

1 Introduction

TIME and SPACE are unavoidable concepts of any world. We can ascribe attributes to endurants that record temporal concepts. And we can ascribe spatial attributes to any concrete, manifest endurant.

2 TIME

a moving image of eternity;
the number of the movement in respect of the before and the after;
the life of the soul in movement as it passes from one stage of act or experience to another;
a present of things past: *memory*, a present of things present: *sight*, a present of things future: *expectations*.[1]

This thing all things devours:
Birds, beasts, trees, flowers;
Gnaws iron, bites steel,
Grinds hard stones to meal;
Slays king, ruins town,
And beats high mountain down.[2]

Concepts of time continue to fascinate philosophers and scientists [14, 3, 5, 6, 12, 7, 8, 9, 10, 11, 13] and [4].

2.1 Treatments

`J.M.E. McTaggart` (1908, [5, 3, 13]) discussed theories of time around the notions of **"A-series":** with concepts like "past", "present" and "future", and **"B-series":** has terms like "precede", "simultaneous" and "follow". `Johan van Benthem` [14] and `Wayne D. Blizard` [2] relates abstracted entities to spatial points and time. A recent computer programming-oriented treatment is given in [4, `Mandrioli et al.`, 2013].

2.2 Time Motivated Philosophically

2.2.1 Indefinite Time

We motivate the abstract notion of time as follows. Two different states must necessarily be ascribed different incompatible predicates. But how can we ensure so ? Only if states stand in an asymmetric relation to one another. This state relation is also transitive. So that is an indispensable property of any world. By a transcendental deduction we say that *primary entities exist in time*. *So every possible world must exist in time*.

2.2.2 Definite Time

By a definite time we shall understand an abstract representation of time such as for example year, month, day, hour, minute, second, etc.

2.2.3 Temporal Notions of Endurants

By temporal notions of endurants we mean time properties of endurants, usually modelled as attributes.

[1] Quoted from [1, Cambridge Dictionary of Philosophy]
[2] J.R.R. Tolkien, The Hobbit

2.3 $\mathbb{TIME}$: Types and Operations

We shall not be concerned with any representation of time. That is, we leave it to the domain modeller to choose an own representation [4]. Similarly we shall not be concerned with any representation of time intervals.[3]

2.3.1 Types and Values

1. So there is an abstract type, $\mathbb{TIME}$,
2. and an abstract type $\mathbb{TI}$: $\mathbb{T}$ime $\mathbb{I}$nterval.
3. There is no $\mathbb{TIME}$ origin, but there is a "zero" $\mathbb{T}$ime $\mathbb{I}$nterval.

2.3.2 Operations on $\mathbb{TIME}$ and $\mathbb{TI}$

4. One can add (subtract) a time interval to (from) a time and obtain a time.
5. One can add and subtract two time intervals and obtain a time interval – with subtraction respecting that the subtrahend is smaller than or equal to the minuend.
6. One can subtract a time from another time obtaining a time interval respecting that the subtrahend is smaller than or equal to the minuend.
7. One can multiply a time interval with a real and obtain a time interval.
8. One can compare two times and two time intervals.

type
1 $\mathbb{T}$
2 $\mathbb{TI}$
value
3 $\mathbf{0}$:$\mathbb{TI}$
4 $+,-$: $\mathbb{T} \times \mathbb{TI} \to \mathbb{T}$
5 $+,-$: $\mathbb{TI} \times \mathbb{TI} \stackrel{\sim}{\to} \mathbb{TI}$
6 $-$: $\mathbb{T} \times \mathbb{T} \to \mathbb{TI}$
7 $*$: $\mathbb{TI} \times$ **Real** $\to \mathbb{TI}$
8 $<,\leq,=,\neq,\geq,>$: $\mathbb{T} \times \mathbb{T} \to$ **Bool**
8 $<,\leq,=,\neq,\geq,>$: $\mathbb{TI} \times \mathbb{TI} \to$ **Bool**
axiom
4 $\forall$ t:$\mathbb{T}$ • t+$\mathbf{0}$ = t

2.3.3 Recording $\mathbb{TIME}$

9. We define the signature of the meta-physical time observer.

type
9 $\mathbb{T}$
value
9 **record**_$\mathbb{TIME}$(): **Unit** $\to \mathbb{TIME}$

The time recorder applies to nothing and yields a time. **record**_$\mathbb{TIME}$() can only occur in action, event and behavioural descriptions.

[3]– but point out, that although a definite time interval may be referred to by number of years, number of days (less than 365), number of hours (less than 24), number of minutes (less than 60) number of seconds (less than 60), et cetera, this is not a time, but a time interval.

2.4 J. van Benthem: A Continuum Theory of Time

The following is taken from Johan van Benthem [14]:

- Let P be a point structure (for example, a set).
- Think of time as a continuum.
- The following axioms characterise ordering ($<$, $=$, $>$) relations between (i.e., aspects of) time points.
- The axioms listed below are not thought of as an axiom system, that is, as a set of independent axioms all claimed to hold for the time concept, which we are encircling.
- Instead van Benthem offers the individual axioms as possible "blocks" from which we can then "build" our own time system — one that suits the application at hand, while also fitting our intuition.
- Time is transitive: If $p<p'$ and $p'<p''$ then $p<p''$.
- Time may not loop, that is, is not reflexive: $p \not< p$.
- Linear time can be defined: Either one time comes before, or is equal to, or comes after another time.
- Time can be left-linear, i.e., linear "to the left" of a given time.
- One could designate a time axis as beginning at some time, that is, having no predecessor times.
- And one can designate a time axis as ending at some time, that is, having no successor times.
- General, past and future successors (predecessors, respectively successors in daily talk) can be defined.
- Time can be dense: Given any two times one can always find a time between them.
- Discrete time can be defined.

2.4.1 The Continuum Theory of Time

[TRANS: Transitivity] $\forall$ p,p$'$,p$''$:P • p $<$ p$'$ $<$ p$''$ $\Rightarrow$ p $<$ p$''$
[IRREF: Irreflexitivity] $\forall$ p:P • p $\not<$ p
[LIN: Linearity] $\forall$ p,p$'$:P • (p=p$'$ $\vee$ p$<$p$'$ $\vee$ p$>$p$'$)
[L−LIN: Left Linearity]
    $\forall$ p,p$'$,p$''$:P • (p$'$$<$p $\wedge$ p$''$$<$p) $\Rightarrow$ (p$'$$<p''$ $\vee$ p$'$=p$''$ $\vee$ p$''$$<p'$)
[BEG: Beginning] $\exists$ p:P • $\sim\exists$ p$'$:P • p$'$$<$p
[END: Ending] $\exists$ p:P • $\sim\exists$ p$'$:P • p$<$p$'$
[SUCC: Successor]
  [PAST: Predecessors] $\forall$ p:P,$\exists$ p$'$:P • p$'$$<$p
  [FUTURE: Successor] $\forall$ p:P,$\exists$ p$'$:P • p$<$p$'$
[DENS: Dense] $\forall$ p,p$'$:P (p$<$p$'$ $\Rightarrow$ $\exists$ p$''$:P • p$<$p$''$$<p'$)
[CDENS: Converse Dense] $\equiv$ [TRANS: Transitivity]

$\forall$ p,p′:P ($\exists$ p″:P • p<p″<p′ $\Rightarrow$ p<p′)
[DISC: Discrete]
$\forall$ p,p′:P • (p<p′ $\Rightarrow$ $\exists$ p″:P • (p<p″ $\wedge$ ∼$\exists$ p‴:P • (p<p‴<p″))) $\wedge$
$\forall$ p,p′:P • (p<p′ $\Rightarrow$ $\exists$ p″:P • (p″<p′ $\wedge$ ∼$\exists$ p‴:P • (p″<p‴<p′)))

A strict partial order, SPO, is a point structure satisfying TRANS and IRREF. TRANS, IRREF and SUCC imply infinite models. TRANS and SUCC may have finite, "looping time" models.

2.5 Modal Logic

Modal Logic[4] is a form of logic used to represent statements about *necessity* and *possibility.*

MORE TO COME

2.6 Temporal Logic

Temporal logic[5]

MORE TO COME

2.7 Linear Temporal Logic: LTL

Yes, LTL does not stand for *Less-than-truckload-shipping*[6] but for Linear Tempral Logic[7] !

MORE TO COME

2.8 Lamport's Temporal Logic of Actions: TLA$^+$

TO BE WRITTEN

2.9 The Duration Calculues

TO BE WRITTEN

3 𝕊ℙ𝔸ℂ𝔼

Space is just there. So we do not define an observer, **observe**_𝕊ℙ𝔸ℂ𝔼. For us – bound to model mostly artefactual worlds on this earth – there is but one space. Although 𝕊ℙ𝔸ℂ𝔼, as a type, could be thought of as defining more than one space we shall consider these to be isomorphic ! 𝕊ℙ𝔸ℂ𝔼 is considered to consist of (an infinite number of) ℙ𝕆𝕀ℕ𝕋s.

[4]https://en.wikipedia.org/wiki/Modal_logic
[5]https://en.wikipedia.org/wiki/Temporal_logic
[6]https://en.wikipedia.org/wiki/Less-than-truckload_shipping
[7]https://en.wikipedia.org/wiki/Linear_temporal_logic

10. We can assume a point observer, observe_$\mathbb{POINT}$, is a function which applies to endurants, e, and yield a point, $pt : \mathbb{POINT}$

10. **observe**_$\mathbb{POINT}$: E $\rightarrow$ $\mathbb{POINT}$

At which "point" of an endurant, e, observe_$\mathbb{POINT}(e)$, is applied, or which of the (infinitely) many points of an endurant E, observe_$\mathbb{POINT}(e)$, yields we leave up to the domain modeller to decide !

We suggest, besides $\mathbb{POINT}$s, the following spatial attribute possibilities:

11. $\mathbb{EXTENT}$ as a dense set of $\mathbb{POINT}$s;
12. Volume, of concrete type, for example, m^3, as the "volume" of an $\mathbb{EXTENT}$ such that
13. $\mathbb{SURFACE}$s as dense sets of $\mathbb{POINT}$s have no volume, but an
14. Area, of concrete type, for example, m^2, as the "area" of a dense set of $\mathbb{POINT}$s;
15. $\mathbb{LINE}$ as dense set of $\mathbb{POINT}$s with no volume and no area, but
16. Length, of concrete type, for example, m.

For these we have that

17. the *intersection*, $\bigcap$, of two $\mathbb{EXTENT}$s is an $\mathbb{EXTENT}$ of possibly nil Volume,
18. the intersection, $\bigcap$, of two $\mathbb{SURFACE}$s may be either a possibly nil $\mathbb{SURFACE}$ or a possibly nil $\mathbb{LINE}$, or a combination of these.
19. the intersection, $\bigcap$, of two $\mathbb{LINE}$s may be either a possibly nil $\mathbb{LINE}$ or a $\mathbb{POINT}$.

Similarly we can define

20. the *union*, $\bigcup$, of two not-disjoint $\mathbb{EXTENT}$s,
21. the *union*, $\bigcup$, of two not-disjoint $\mathbb{SURFACE}$s,
22. the *union*, $\bigcup$, and of two not-disjoint $\mathbb{LINE}$s.

and:

23. the *[in]equality*, $\neq, =$, of pairs of $\mathbb{EXTENT}$, pairs of $\mathbb{SURFACE}$s, and pairs of $\mathbb{LINE}$s.

We invite the reader to first express the signatures for these operations, then their pre-conditions, and finally, being courageous, appropriate fragments of axiom systems. We leave it up to the reader to introduce, and hence define, functions that add, subtract, compare, etc., $\mathbb{EXTENT}$s, $\mathbb{SURFACE}$s, $\mathbb{LINE}$s, etc.

3.1 Mathematical Models of Space

Figure 1 on the facing page diagrams some mathematical models of space. We shall hint[8] at just one of these spaces.

[8] Figure 1 is taken from `https://en.wikipedia.org/wiki/Space_(mathematics)`.

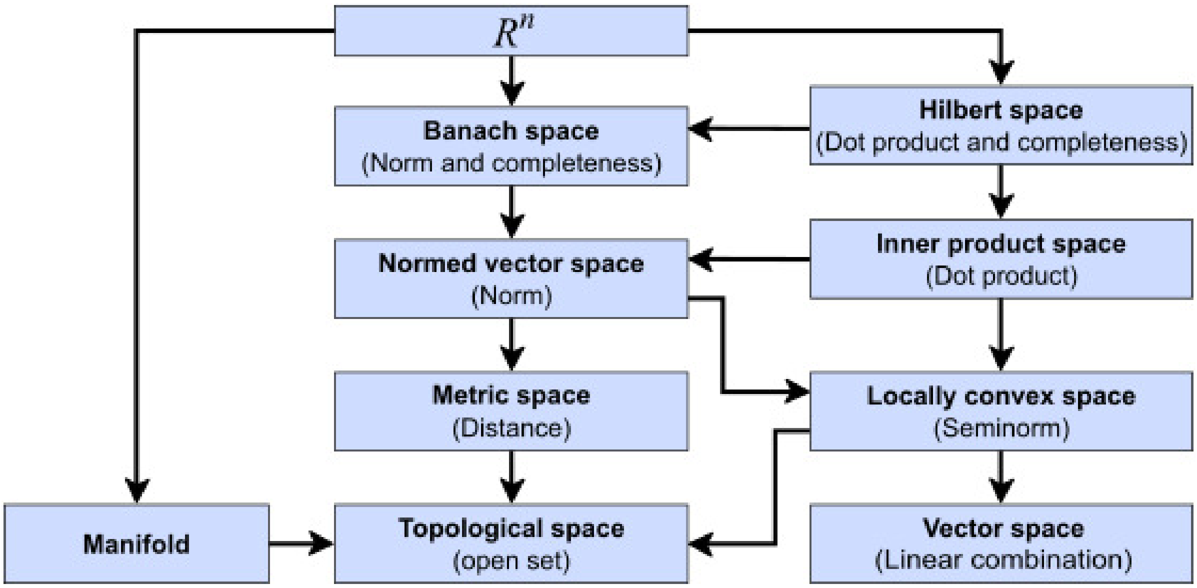

Figure 1: Variety of Abstract Spaces. An arrow from space A to space B implies that A is also a kind of B.

3.1.1 Metric Spaces

Metric Space

A metric space is an ordered pair (M,d) where M is a set and d is a metric on M, i.e., a function:

$$d : M \times M \rightarrow \textsf{Real}$$

such that for any $x,y,z \in M$, the following holds:

$$d(x,y) = 0 \equiv \; x = y \quad \text{identity of indiscernibles} \tag{1}$$

$$d(x,y) = d(y,x) \quad \text{symmetry} \tag{2}$$

$$d(x,z) \leq d(x,y) + d(y,z) \quad \text{sub-additivity or triangle inequality} \tag{3}$$

Given the above three axioms, we also have that $d(x,y) \geq \; 0$ for any $x,y \in M$. This is deduced as follows:

$$d(x,y) + d(y,x) \geq d(x,x) \quad \text{triangle inequality} \tag{4}$$

$$d(x,y) + d(y,x) \geq d(x,x) \quad \text{by symmetry} \tag{5}$$

$$2d(x,y) \geq 0 \quad \text{identity of indiscernibles} \tag{6}$$

$$d(x,y) \geq 0 \quad \text{non-negativity} \tag{7}$$

The function d is also called distance function or simply distance. Often, d is omitted and one just writes M for a metric space if it is clear from the context what metric is used.

4 Wayne D. Blizard: A Theory of $\mathbb{TIME}$–$\mathbb{SPACE}$

We shall present an axiom system [2, Wayne D. Blizard, 1980] which relate abstracted entities, endurants and perdurants, to spatial points and time. Let $A, B, \ldots$ stand for entities, $p, q, \ldots$ for spatial points, and t, τ for times. 0 designates a first, a begin time. Let t' stand for the discrete time successor of time t. Let $N(p,q)$ express that p and q are spatial neighbours. Let $=$ be an overloaded equality operator applicable, pairwise to entities, spatial locations and times, respectively. A_p^t expresses that entity A is at location p at time t. The axioms — where we omit (obvious) typings (of A, B, P, Q, and T): $'$ designates the time successor function: t'.

(I)	$\forall A \forall t \exists p$	$:$	A_p^t	
(II)	$(A_p^t \wedge A_q^t)$	$\supset$	$p = q$	
(III)	$(A_p^t \wedge B_p^t)$	$\supset$	$A = B$	
$(IV)(?)$	$(A_p^t \wedge A_p^{t'})$	$\supset$	$t = t'$	
$(V \quad i)$	$\forall p, q$	$:$	$N(p,q) \supset p \neq q$	Irreflexivity
$(V \quad ii)$	$\forall p, q$	$:$	$N(p,q) = N(q,p)$	Symmetry
$(V \quad iii)$	$\forall p \exists q, r$	$:$	$N(p,q) \wedge N(p,r) \wedge q \neq r$	No isolated locations
$(VI \quad i)$	$\forall t$	$:$	$t \neq t'$	
$(VI \quad ii)$	$\forall t$	$:$	$t' \neq 0$	
$(VI \quad iii)$	$\forall t$	$:$	$t \neq 0 \supset \exists \tau : t = \tau'$	
$(VI \quad iv)$	$\forall t, \tau$	$:$	$\tau' = t' \supset \tau = t$	
(VII)	$A_p^t \wedge A_q^{t'}$	$\supset$	$N(p,q)$	
$(VIII)$	$A_p^t \wedge B_q^t \wedge N(p,q)$	$\supset$	$\sim (A_q^{t'} \wedge B_p^{t'})$	

(II–IV,VII–VIII): The axioms are universally 'closed'; that is: We have omitted the usual $\forall A, B, p, q, t$s.

(I): For every entity, A, and every time, t, there is a location, p, at which A is located at time t.

(II): An entity cannot be in two locations at the same time.

(III): Two distinct entities cannot be at the same location at the same time.

(IV): Entities always move: An entity cannot be at the same location at different times. *This is more like a conjecture: Could be questioned.*

- (V): These three axioms define N.
- (V i): Same as $\forall p : \sim N(p,p)$. "Being a neighbour of", is the same as "being distinct from".
- (V ii): If p is a neighbour of q, then q is a neighbour of p.
- (V iii): Every location has at least two distinct neighbours.

(VI): The next four axioms determine the time successor function $'$.

- (VI i): A time is always distinct from its successor: time cannot rest. There are no time fix points.
- (VI ii): Any time successor is distinct from the begin time. Time 0 has no predecessor.
- (VI iii): Every non–begin time has an immediate predecessor.
- (VI iv): The time successor function $'$ is a one–to–one (i.e., a bijection) function.

(VII): The *continuous path axiom:* If entity A is at location p at time t, and it is at location q in the immediate next time (t'), then p and q are neighbours.

(VIII): *No "switching":* If entities A and B occupy neighbouring locations at time t them it is not possible for A and B to have switched locations at the next time (t').

Except for Axiom (IV) the system applies both to systems of entities that "sometimes" rests, i.e., do not move. These entities are spatial and occupy at least a point in space. If some entities "occupy more" space volume than others, then we interpret, in a suitable manner, the notion of the point space P (etc.). We do not show so here.

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