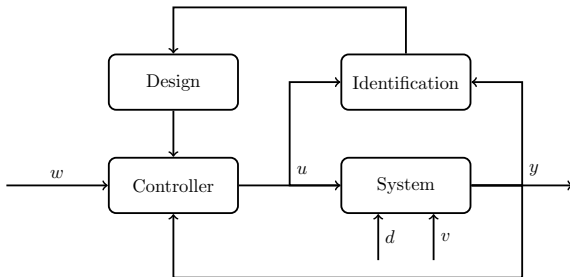


- ① ARX models
- ② ARX estimation
- ③ ARMAX estimation
- ④ ARMAX identification
- ⑤ ARMAX control
- ⑥ ARMAX adaptive control
- ⑦ Systems and control theory
- ⑧ Stochastic systems + SS estimation
- ⑨ SS estimation + control
- ⑩ SS control
- ⑪ SS parameter estimation
- ⑫ SS nonlinear estimation
- ⑬ SS nonlinear control



Today's Agenda



- Nonlinear optimal control of input-affine systems
- Assignment 3
- Suggestions for 2027
- MSc project ideas

Nonlinear optimal control

Input-affine system

$$x_{t+1} = f(x_t) + g(x_t)u_t, \quad (1)$$

$$y_t = Cx_t \quad (2)$$

$f(x)$ is a vector and $g(x)$ is a matrix

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Important assumptions

- ❶ It is reasonable to linearize f , i.e., $f(x + \Delta x) \approx f(x) + \frac{\partial f}{\partial x}(x)\Delta x$
- ❷ The input term is only mildly dependent on the state, i.e., $g(x + \Delta x) \approx g(x)$

Approximate response

Write out recursion

$$x_1 = f(x_0) + g(x_0)u_0, \quad (3)$$

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$$x_2 = f(x_1) + g(x_1)u_1 \quad (4)$$

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$$= f(f(x_0) + g(x_0)u_0) + g(f(x_0) + g(x_0)u_0)u_1 \quad (5)$$

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$$\approx f(f(x_0)) + \frac{\partial f}{\partial x}(f(x_0))g(x_0)u_0 + g(f(x_0))u_1 \quad (6)$$

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Introduce

$$z_0 = x_0, \quad z_1 = f(x_0), \quad A_1 = \frac{\partial f}{\partial x}(z_1), \quad B_0 = g(z_0), \quad B_1 = g(z_1) \quad (7)$$

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$$x_2 = f(z_1) + A_1 B_0 u_0 + B_1 u_1$$

Approximate response

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$$x_2 = f(z_1) + A_1 B_0 u_0 + B_1 u_1 = z_2 + A_1 B_0 u_0 + B_1 u_1 \quad (8)$$

Approximate response

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Continue

$$x_3 = f(z_2 + A_1 B_0 u_0 + B_1 u_1) + g(z_2 + A_1 B_0 u_0 + B_1 u_1)u_2 \quad (9)$$

Approximate response

Write out recursion

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$$x_3 = f(z_2 + A_1 B_0 u_0 + B_1 u_1) + g(z_2 + A_1 B_0 u_0 + B_1 u_1)u_2 \quad (9)$$

$$\approx f(z_2) + A_2 A_1 B_0 u_0 + A_2 B_1 u_1 + B_2 u_2 \quad (10)$$

Approximate system

Free/unforced response

$$z_{t+1} = f(z_t) \tag{11}$$

Approximate system

Free/unforced response

$$z_{t+1} = f(z_t) \quad (11)$$

Time-varying system matrices

$$A_t = \frac{\partial f}{\partial x}(z_t), \quad B_t = g(z_t) \quad (12)$$

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Forced response

$$x_t = z_t + \sum_{j=0}^{t-1} A_{t-1} \cdots A_{j+1} B_j u_j \quad (13)$$

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Notation

$$A_{t-1} \cdots A_{j+1} = A_{t-1} A_{t-2} \cdots A_{j+2} A_{j+1}, \quad A_{t-1} \cdots A_t = I \quad (14)$$

Approximate outputs

$$y_t = C z_t + \sum_{j=0}^{t-1} C A_{t-1} \cdots A_{j+1} B_j u_j \quad (15)$$

Compact notation

Vectors and matrices

$$Z_N = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}, \quad Y_N = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad U_N = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad (16)$$

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$$\Gamma_{yz}^N = \text{blkdiag}(C, C, \dots, C), \quad (17)$$

$$\Gamma_{yu}^N = \begin{bmatrix} CB_0 & & & & \\ CA_1B_0 & CB_1 & & & \\ CA_2A_1B_0 & CA_2B_1 & CB_2 & & \\ \vdots & & & \ddots & \\ CA_{N-1} \cdots A_1B_0 & CA_{N-1} \cdots A_2B_1 & \cdots & CA_{N-1}B_{N-2} & CB_{N-1} \end{bmatrix} \quad (18)$$

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Compact system

$$Y_N = \Gamma_{yz}^N Z_N + \Gamma_{yu}^N U_N \quad (19)$$

Optimal control problem

$$\min_{U_N} \phi = \frac{1}{2}(Y_N - W_N)^T Q_y (Y_N - W_N) + \frac{1}{2} U_N^T Q_u U_N \quad (20)$$

Optimal control problem

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Solution

$$U_N = -(\Gamma_{yu}^T Q_y \Gamma_{yu} + Q_u)^{-1} \Gamma_{yu}^T Q_y (\Gamma_{yz} Z_N - W_N) \quad (21)$$

Optimal control problem

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First manipulated input (e.g., for feedback control law)

$$u_0 = \Pi U_N, \quad \Pi = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \quad (22)$$

Summary

Step 1: Compute the free response (x_0 is given)

$$z_{t+1} = f(z_t), \quad z_0 = x_0, \quad t = 0, \dots, N-1 \quad (23)$$

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$$A_t = \frac{\partial f}{\partial x}(z_t), \quad t = 1, \dots, N-1, \quad (24)$$

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Assignment 3

Suggestions for 2027

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- Group assignments + oral exam (fundamental aspects)

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- Combine state space assignments
 - Design of experiments
 - System identification
 - Model validation
 - Optimal control
 - Adaptive control (disturbance/parameters)

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- Move theory to blackboard
- A little more focus on theory/derivations
 - Example: Include disturbance in GPC problem

Questions?