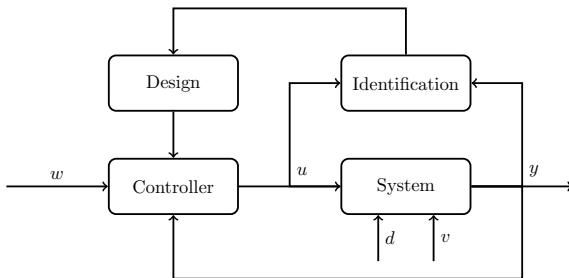


- ① ARX models
- ② ARX estimation
- ③ ARMAX estimation
- ④ ARMAX identification
- ⑤ ARMAX control
- ⑥ ARMAX adaptive control
- ⑦ Systems and control theory
- ⑧ Stochastic systems + SS estimation
- ⑨ SS estimation + control
- ⑩ SS control
- ⑪ SS parameter estimation
- ⑫ SS nonlinear estimation
- ⑬ SS nonlinear control



Today's Agenda



- Advanced topics in Kalman filtering
- Nonlinear Kalman filtering

Advanced topics in Kalman filtering

The Kalman filter is derived for the system

$$x_{t+1} = Ax_t + Bu_t + v_t \quad v_t \sim N(0, R_1), \quad (1)$$

$$y_t = Cx_t + Du_t + e_t \quad e_t \sim N(0, R_2), \quad (2)$$

$$\text{Cov}(v_t, e_t) = 0, \quad e_t, v_t \text{ white } \perp x_s \quad s \leq t \quad (3)$$

The noises are independent of the state history and each other

- ❶ $x_0 \sim N(\hat{x}_0, P_0)$
- ❷ $v_t \sim N(0, R_1)$, white
- ❸ $e_t \sim N(0, R_2)$, white
- ❹ $\text{Cov}(v_t, e_t) = 0$
- ❺ $v_t, e_t \perp x_s, \quad s \leq t$

We will now consider systems where one of these assumptions do not apply

- ① Non-zero mean process disturbances
- ② Non-zero mean output disturbances
- ③ Colored (non-white) process noise
- ④ Colored (non-white) output noise
- ⑤ Noise correlated with the state
- ⑥ Correlated noises

Uncertain Offset in the Process

System with stochastic process offset

$$x_{t+1} = Ax_t + Bu_t + Gd_t + v_t, \quad (4)$$

$$y_t = Cx_t + Du_t + e_t \quad (5)$$

Process offset

$$d_{t+1} = d_t + w_t \quad (6)$$

System in standard form

$$\begin{bmatrix} x_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} A & G \\ 0 & I \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} v_t \\ w_t \end{bmatrix}, \quad (7)$$

$$y_t = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + Du_t + e_t \quad (8)$$

Uncertain Offset in the Output

System with stochastic measurement offset

$$x_{t+1} = Ax_t + Bu_t + v_t, \quad (9)$$

$$y_t = Cx_t + Du_t + Hd_t + e_t \quad (10)$$

Output offset

$$d_{t+1} = d_t + w_t \quad (11)$$

System in standard form

$$\begin{bmatrix} x_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} v_t \\ w_t \end{bmatrix}, \quad (12)$$

$$y_t = \begin{bmatrix} C & H \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + Du_t + e_t \quad (13)$$

White noise (discrete-time):

In discrete time, a white noise signal e_t has zero mean, finite variance, and it is uncorrelated in time: $e_t \perp e_s$ for $s \neq t$

If the noise w_t of a system is colored (non-white), it can be described as a system of white noises (η_t, ξ_t)

$$z_{t+1} = A_w z_t + \eta_t, \quad (14)$$

$$w_t = C_w z_t + \xi_t \quad (15)$$

Coloured Process Noise

System with colored process noise

$$x_{t+1} = Ax_t + Bu_t + v_t, \quad (16)$$

$$y_t = Cx_t + Du_t + e_t \quad (17)$$

Process noise

$$z_{t+1} = A_v z_t + \eta_t, \quad (18)$$

$$v_t = C_v z_t + \xi_t \quad (19)$$

System in standard form

$$\begin{bmatrix} x_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} A & C_v \\ 0 & A_v \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} \xi_t \\ \eta_t \end{bmatrix}, \quad (20)$$

$$y_t = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} + Du_t + e_t \quad (21)$$

Coloured Output Noise

System with colored measurement noise

$$x_{t+1} = Ax_t + Bu_t + v_t, \quad (22)$$

$$y_t = Cx_t + Du_t + e_t \quad (23)$$

Measurement noise

$$z_{t+1} = A_e z_t + \eta_t, \quad (24)$$

$$e_t = C_e z_t + \xi_t \quad (25)$$

System in standard form

$$\begin{bmatrix} x_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} v_t \\ \eta_t \end{bmatrix}, \quad (26)$$

$$y_t = \begin{bmatrix} C & C_e \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix} + Du_t + \xi_t \quad (27)$$

System

$$x_{t+1} = Ax_t + Bu_t + Gw_t, \quad (28)$$

$$y_t = Cx_t + Du_t + \eta_t \quad (29)$$

w_t and η_t are correlated with the states

$$w_t = Hx_t + v_t, \quad v_t \sim N(0, R_1), \quad x_t \perp v_t, \quad (30)$$

$$\eta_t = Fx_t + e_t, \quad e_t \sim N(0, R_2), \quad x_t \perp e_t \quad (31)$$

System in standard form

$$x_{t+1} = (A + GH)x_t + Bu_t + Gv_t, \quad (32)$$

$$y_t = (C + F)x_t + Du_t + e_t \quad (33)$$

Kalman filter: Correlated noise

System with correlated process and measurement noise

$$x_{t+1} = Ax_t + Bu_t + v_t, \quad v_t \sim N(0, R_1), \quad (34)$$

$$y_t = Cx_t + Du_t + e_t, \quad e_t \sim N(0, R_2), \quad (35)$$

$$\text{Cov}(v_t, e_t) = R_{12}, \quad e_t, v_t \text{ white } \perp x_s, \quad s \leq t \quad (36)$$

Measurement update

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \kappa_t(y_t - C\hat{x}_{t|t-1} - Du_t), \quad (37)$$

$$P_{t|t} = (I - \kappa_t C)P_{t|t-1}, \quad (38)$$

$$\kappa_t = P_{t|t-1}C^T(CP_{t|t-1}C^T + R_2)^{-1} \quad (39)$$

Time update

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t + M(y_t - C\hat{x}_{t|t} - Du_t), \quad (40)$$

$$= (A - MC)\hat{x}_{t|t} + (B - MD)u_t + My_t, \quad (41)$$

$$P_{t+1|t} = (A - MC)P_{t|t}(A - MC)^T + R_1 - MR_{12}, \quad M = R_{12}R_2^{-1} \quad (42)$$

Derivation of Kalman filter: Correlated noise

Original system

$$x_{t+1} = Ax_t + Bu_t + v_t + M(y_t - y_t) \quad (43)$$

$$= Ax_t + Bu_t + v_t + M(y_t - Cx_t - Du_t - e_t) \quad (44)$$

$$= (A - MC)x_t + (B - MD)u_t + My_t + \tilde{v}_t, \tilde{v}_t = v_t - Me_t \quad (45)$$

Define the covariance of $\tilde{v}_t$ and e_t to be zero

$$\tilde{R}_{12} = \mathbb{E}[\tilde{v}_t e_t^T] = \mathbb{E}[(v_t - Me_t)e_t^T] = R_{12} - MR_2 = 0 \quad (46)$$

$$\Rightarrow M = R_{12}R_2^{-1}, \quad (47)$$

$$\tilde{R}_1 = \mathbb{E}[\tilde{v}_t \tilde{v}_t^T] = R_1 - R_{12}R_2^{-1}R_{12}^T \quad (48)$$

Kalman filter for systems with correlated noise: Original data update with new time update

$$\hat{x}_{t+1|t} = (A - MC)\hat{x}_{t|t} + (B - MD)u_t + My_t, \quad (49)$$

$$P_{t+1|t} = (A - MC)P_{t|t}(A - MC)^T + \tilde{R}_1 \quad (50)$$

System with correlated process and measurement noise

$$x_{t+1} = Ax_t + Bu_t + v_t, \quad v_t \in N(0, R_1), \quad (51)$$

$$y_t = Cx_t + Du_t + e_t, \quad e_t \in N(0, R_2), \quad (52)$$

$$\text{Cov}(v_t, e_t) = R_{12}, \quad e_t, v_t \text{ white } \perp x_s, \quad s \leq t \quad (53)$$

Predictive Kalman filter

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t-1} + Bu_t + K_t(y_t - C\hat{x}_{t|t-1} - Du_t), \quad (54)$$

$$P_{t+1|t} = AP_{t|t-1}A^T + R_1 - K_t(AP_{t|t-1}C^T + R_{12})^T, \quad (55)$$

$$K_t = (AP_{t|t-1}C^T + R_{12})(CP_{t|t-1}C^T + R_2)^{-1} \quad (56)$$

Relationship between gains

$$K_t = (A - MC)\kappa_t + M \quad (57)$$

Conditional distribution

$$\begin{bmatrix} x_{t+1} \\ y_t \end{bmatrix} | Y_{t-1} \sim N \left(\begin{bmatrix} A\hat{x}_{t|t-1} + Bu_t \\ C\hat{x}_{t|t-1} + Du_t \end{bmatrix}, \begin{bmatrix} AP_{t|t-1}A^T + R_1 & AP_{t|t-1}C^T + R_{12} \\ CP_{t|t-1}A^T + R_{12}^T & CP_{t|t-1}C^T + R_2 \end{bmatrix} \right) \quad (58)$$

Predictive distribution

$$x_{t+1}|Y_t \sim N(\hat{x}_{t+1|t}, P_{t+1|t}) \quad (59)$$

Use projection theorem

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t-1} + Bu_t + K_t(y_t - C\hat{x}_{t|t-1} - Du_t), \quad (60)$$

$$P_{t+1|t} = AP_{t|t-1}A^T + R_1 - K_t(AP_{t|t-1}C^T + R_{12})^T, \quad (61)$$

$$K_t = (AP_{t|t-1}C^T + R_{12})(CP_{t|t-1}C^T + R_2)^{-1} \quad (62)$$

Correlated Noise - A Special Case

Perfect correlation between process noise and measurement noise

$$v_t = Ge_t, \quad R_1 = GR_2G^T, \quad R_{12} = GR_2 \quad (63)$$

Stationary predictive Kalman filter

$$P_\infty = 0, \quad K_\infty = G \quad (64)$$

Proof: Predictive variance and gain if $P_{t|t-1} = 0$

$$P_{t+1|t} = R_1 - K_t R_{12}^T, \quad K_t = R_{12} R_2^{-1} \quad (65)$$

Substitute R_1 and R_{12}

$$P_{t+1|t} = GR_2G^T - K_t(GR_2)^T = 0, \quad K_t = GR_2R_2^{-1} = G \quad (66)$$

This proves that $P_\infty = 0$ and $K_\infty = G$ is a solution

If $(A - GC, R_1)$ is reachable and $R_2 \succ 0$, it is the only solution (from the theory of Riccati equations)

Nonlinear Kalman filtering

Linearization

Nonlinear system

$$x_{t+1} = f(x_t, u_t, v_t), \quad v_t \sim N(m_v, R_1), \quad (67)$$

$$y_t = g(x_t, u_t, e_t), \quad e_t \sim N(m_e, R_2) \quad (68)$$

Linearize right-hand side functions

$$f(x_t, u_t, v_t) \approx f(x_t^*, u_t^*, v_t^*) + \frac{\partial f}{\partial x}(x_t - x_t^*) + \frac{\partial f}{\partial u}(u_t - u_t^*) + \frac{\partial f}{\partial v}(v_t - v_t^*), \quad (69)$$

$$g(x_t, u_t, e_t) \approx g(x_t^*, u_t^*, e_t^*) + \frac{\partial g}{\partial x}(x_t - x_t^*) + \frac{\partial g}{\partial u}(u_t - u_t^*) + \frac{\partial g}{\partial e}(e_t - e_t^*) \quad (70)$$

The Jacobian matrices are evaluated in the linearization point, e.g.,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x_t^*, u_t^*, v_t^*) \quad (71)$$

Linearization

Deviation variables

$$X_t = x_t - x_t^*, \quad Y_t = y_t - y_t^*, \quad y_t^* = g(x_t^*, u_t^*, e_t^*), \quad (72)$$

$$U_t = u_t - u_t^*, \quad V_t = v_t - v_t^*, \quad E_t = e_t - e_t^* \quad (73)$$

Linearized system

$$X_{t+1} = x_{t+1} - x_{t+1}^* = f(x_t, u_t, v_t) - f(x_t^*, u_t^*, v_t^*) \quad (74)$$

$$= A_t X_t + B_t U_t + G_t V_t, \quad (75)$$

$$Y_t = y_t - y_t^* = g(x_t, u_t, e_t) - g(x_t^*, u_t^*, e_t^*) \quad (76)$$

$$= C_t X_t + D_t U_t + F_t E_t \quad (77)$$

System matrices

$$A_t = \frac{\partial f}{\partial x}(x_t^*, u_t^*, v_t^*), \quad B_t = \frac{\partial f}{\partial u}(x_t^*, u_t^*, v_t^*), \quad G_t = \frac{\partial f}{\partial v}(x_t^*, u_t^*, v_t^*), \quad (78)$$

$$C_t = \frac{\partial g}{\partial x}(x_t^*, u_t^*, e_t^*), \quad D_t = \frac{\partial g}{\partial u}(x_t^*, u_t^*, e_t^*), \quad H_t = \frac{\partial g}{\partial e}(x_t^*, u_t^*, e_t^*) \quad (79)$$

Measurement update

$$\hat{Y}_{t|t-1} = C_t \hat{X}_{t|t-1} + D_t U_t, \quad (80)$$

$$\hat{X}_{t|t} = \hat{X}_{t|t-1} + \kappa_t (Y_t - \hat{Y}_{t|t-1}), \quad (81)$$

$$\kappa_t = P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_2)^{-1}, \quad (82)$$

$$P_{t|t} = P_{t|t-1} - \kappa_t C_t^T P_{t|t-1} \quad (83)$$

Time update

$$\hat{X}_{t+1|t} = A_t \hat{X}_{t|t} + B_t U_t, \quad (84)$$

$$P_{t+1|t} = A_t P_{t|t} A_t^T + R_1 \quad (85)$$

Extended Kalman filter: Measurement update

Linearization point

$$x_t^* = \hat{x}_{t|t-1}, \quad u_t^* = u_t, \quad e_t^* = m_e \quad (86)$$

Deviation variables

$$\hat{X}_{t|t-1} = 0, \quad U_t = 0, \quad V_t = v_t - m_v, \quad E_t = e_t - m_e, \quad (87)$$

$$\hat{Y}_{t|t-1} = C_t \hat{X}_{t|t-1} + D_t U_t = 0 \quad (88)$$

$$= \hat{y}_{t|t-1} - y_t^* \quad \Rightarrow \quad \hat{y}_{t|t-1} = y_t^* = g(x_t^*, u_t^*, e_t^*) \quad (89)$$

Measurement update

$$\hat{y}_{t|t-1} = g(\hat{x}_{t|t-1}, u_t, m_e), \quad (90)$$

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \kappa_t (y_t - \hat{y}_{t|t-1}), \quad (91)$$

$$\kappa_t = P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_2)^{-1}, \quad (92)$$

$$P_{t|t} = P_{t|t-1} - \kappa_t C_t^T P_{t|t-1} \quad (93)$$

The Jacobian matrix C_t is evaluated in $\hat{x}_{t|t-1}$, u_t , and m_e

Extended Kalman filter: Time update

Linearization point

$$x_t^* = \hat{x}_{t|t}, \quad u_t^* = u_t, \quad v_t^* = m_v, \quad e_t^* = m_e \quad (94)$$

Deviation variables

$$\hat{X}_{t|t} = 0, \quad U_t = 0, \quad V_t = v_t - m_v, \quad E_t = e_t - m_e, \quad (95)$$

$$\hat{X}_{t+1|t} = A\hat{X}_{t|t} + BU_t = 0 \quad (96)$$

$$= \hat{x}_{t+1|t} - x_{t+1}^* = \hat{x}_{t+1|t} - f(x_t^*, u_t^*, v_t^*) \Rightarrow \hat{x}_{t+1|t} = f(x_t^*, u_t^*, v_t^*)$$

Time update

$$\hat{x}_{t+1|t} = f(\hat{x}_{t|t}, u_t, m_v), \quad (97)$$

$$P_{t+1|t} = A_t P_{t|t} A_t^T + R_1 \quad (98)$$

The Jacobian matrix A_t is evaluated in $\hat{x}_{t|t}$, u_t , and m_v

In general, nonlinear Kalman filters are not optimal

Questions?