Stochastic Adaptive Control (02421)

Lecture 9

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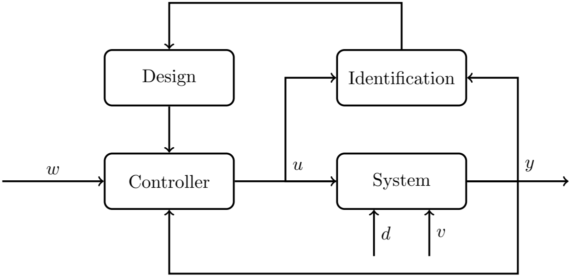
Lecture Plan

1. ARX models
2. ARX estimation
3. ARMAX estimation
4. ARMAX identification
5. ARMAX control
6. ARMAX adaptive control
7. Systems and control theory
8. Stochastic systems + SS estimation
9. SS estimation + control
10. SS control
11. SS parameter estimation
12. SS nonlinear estimation
13. SS nonlinear control



Today's Agenda

- Stationary Kalman filter
- Kalman filter errors
- Recursive parameter estimation
- Prediction
- General predictive control

Stationary Kalman filter

Stationary Kalman Filters

Stationary covariance of the predictive Kalman filter

$$\boxed{P^p_\infty = AP^p_\infty A^T + R_1 - AP^p_\infty C^T (CP^p_\infty C^T + R_2)^{-1} C P^p_\infty A^T} \tag{1}$$

Stationary covariance of the ordinary Kalman filter

$$\boxed{\begin{aligned} P^o_\infty = {} & AP^o_\infty A^T + R_1 - (AP^o_\infty A^T + R_1) C^T \\ & (C(AP^o_\infty A^T + R_1)C^T + R_2)^{-1} C(AP^o_\infty A^T + R_1) \end{aligned}} \tag{2}$$

Relation between stationary covariances

$$P^p_\infty = AP^o_\infty A^T + R_1 \tag{3}$$

$$(P^o_\infty)^{-1} = (P^p_\infty)^{-1} + C^T R_2^{-1} C \tag{4}$$

Stationary Kalman Filter - The Riccati Equation

Discrete Riccati equation

$$X_{t+1} = AX_tA^T + R_1 - AX_tC^T(CX_tC^T + R_2)^{-1}CX_tA^T \tag{5}$$

Discrete algebraic Riccati equation (DARE)

$$X = AXA^T + R_1 - AXC^T(CXC^T + R_2)^{-1}CXA^T \tag{6}$$

- If (A, C) is observable, a positive semi-definite solution X exists for each X_0
- If (A, C) is observable, (A, R) is reachable $(RR^T = R_1)$, $R_1 \succeq 0$, and $R_2 \succ 0$, the solution is unique and independent of X_0 and $A - KC$ is asymptotically stable (its eigenvalues are strictly within the unit circle)

Hint: Use Matlab's `idare` function

Kalman filter errors

Kalman Errors

Estimation errors (if the model is correct)

$$\boxed{\begin{aligned} \tilde{x}_{t|t} &= x_t - \hat{x}_{t|t}, & \tilde{x}_{t|t} &\sim N(0, P_{t|t}), \\ \tilde{x}_{t|t-1} &= x_t - \hat{x}_{t|t-1}, & \tilde{x}_{t|t-1} &\sim N(0, P_{t|t-1}), \\ \epsilon_t &= y_t - C\hat{x}_{t|t-1}, & \epsilon_t &\sim N(0, CP_{t|t-1}C^T + R_2) \end{aligned}} \qquad \begin{aligned} (7) \\ (8) \\ (9) \end{aligned}$$

The innovation errors are white ($\epsilon_s \perp \epsilon_t$ for $s \neq t$) and can be used for

1. model validation (i.e., validating estimates of A, B, $\ldots$)
2. system representation
3. fault detection

Discrete-time systems for estimation errors

$$\tilde{x}_{t+1|t+1} = (I - \kappa_{t+1}C)(A\tilde{x}_{t|t} + v_t) - \kappa_{t+1}e_{t+1}, \tag{10}$$

$$\tilde{x}_{t+1|t} = (A - K_tC)\tilde{x}_{t|t-1} - K_te_t + v_t \tag{11}$$

The relation between the Kalman gains is $K_t = A\kappa_t$

Example of Prediction Error

Discrete-time system

$$x_{t+1} = 0.5x_t + v_t, \qquad v_t \sim N(0, 0.1), \tag{12}$$

$$y_t = x_t + e_t, \qquad e_t \sim N(0, 0.5) \tag{13}$$

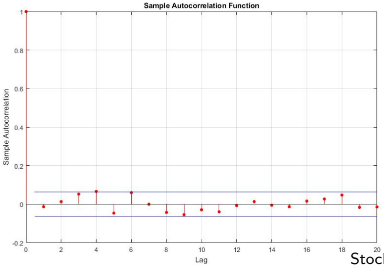
Prediction error

$$\epsilon_t \sim N(0, 0.625) \tag{14}$$

Empirical mean and variance

$$\mathbb{E}[\epsilon_t] = -0.0047, \qquad \mathrm{Var}(\epsilon_t) = 0.6218 \tag{15}$$

Empirical autocorrelation indicates that ϵ_t is white



Theory - Overview of Kalman filter Assumptions

The Kalman filter is designed for systems in the form

$$x_{t+1} = Ax_t + Bu_t + v_t, \tag{16}$$

$$y_t = Cx_t + e_t \tag{17}$$

It assumes the following noise distributions

1. $x_0 \sim N(\hat{x}_0, P_0)$
2. $v_t \sim N(0, P_v)$, white
3. $e_t \sim N(0, P_e)$, white
4. $\mathrm{Cov}(v_t, e_t) = 0$
5. $v_t, e_t \perp x_s, \quad s \leq t$

In lecture 12, we will relax some of these assumptions

Recursive parameter estimation

Augmented system: Parameter model

Parameter model

$$\theta_{t+1} = \theta_t + H\eta_t, \qquad \eta_t \sim N(0, I) \tag{18}$$

Augmented system

$$\begin{aligned} x_{t+1} &= A(\theta)x_t + B(\theta)u_t + G(\theta)v_t, & v_t &\sim N(0, I), & (19)\\ \theta_{t+1} &= \theta_t + H\eta_t, & \eta_t &\sim N(0, I), & (20)\\ y_t &= C(\theta)x_t + D(\theta)u_t + F(\theta)e_t, & e_t &\sim N(0, I) & (21) \end{aligned}$$

Compact notation

$$\bar{x}_t = \begin{bmatrix} x_t \\ \theta_t \end{bmatrix}, \qquad \bar{v}_t = \begin{bmatrix} v_t \\ \eta_t \end{bmatrix}, \tag{22}$$

$$\bar{x}_{t+1} = f(\bar{x}_t, u_t, \bar{v}_t), \quad f(\bar{x}_t, u_t, \bar{v}_t) = \begin{bmatrix} A(\theta)x_t + B(\theta)u_t + G(\theta)v_t \\ \theta_t + H\eta_t \end{bmatrix}, \tag{23}$$

$$y_t = g(\bar{x}_t, u_t, e_t), \quad g(\bar{x}_t, u_t, e_t) = C(\theta)x_t + D(\theta)u_t + F(\theta)e_t \tag{24}$$

Linearized system

Linearize around $\bar{x}^*$, u^*, $\bar{v}^* = 0$, and $e^* = 0$

$$f(\bar{x}_t, u_t, \bar{v}_t) \approx f(\bar{x}^*, u^*, \bar{v}^*) + \frac{\partial f}{\partial \bar{x}}(\bar{x}_t - \bar{x}^*) + \frac{\partial f}{\partial u}(u_t - u^*) + \frac{\partial f}{\partial \bar{v}}(\bar{v}_t - \bar{v}^*), \tag{25}$$

$$g(\bar{x}_t, u_t, e_t) \approx g(\bar{x}^*, u^*, e^*) + \frac{\partial g}{\partial \bar{x}}(\bar{x}_t - \bar{x}^*) + \frac{\partial g}{\partial u}(u_t - u^*) + \frac{\partial g}{\partial e}(e_t - e^*) \tag{26}$$

The Jacobian matrices are evaluated in the linearization point, e.g., $\frac{\partial f}{\partial \bar{x}} = \frac{\partial f}{\partial \bar{x}}(\bar{x}^*, u^*, \bar{v}^*)$

Offsets

$$d^* = f(\bar{x}^*, u^*, \bar{v}^*) - \frac{\partial f}{\partial \bar{x}}\bar{x}^* - \frac{\partial f}{\partial u}u^* - \frac{\partial f}{\partial \bar{v}}\bar{v}^*, \tag{27}$$

$$r^* = g(\bar{x}^*, u^*, e^*) - \frac{\partial g}{\partial \bar{x}}\bar{x}^* - \frac{\partial g}{\partial u}u^* - \frac{\partial g}{\partial e}e^* \tag{28}$$

Jacobians

Jacobians

$$\bar{A} = \frac{\partial f}{\partial \bar{x}} = \begin{bmatrix} A & S \\ & I \end{bmatrix}, \quad \bar{B} = \frac{\partial f}{\partial u} = \begin{bmatrix} B \\ \end{bmatrix}, \quad \bar{G} = \frac{\partial f}{\partial \bar{v}} = \begin{bmatrix} G & \\ & H \end{bmatrix}, \tag{29}$$

$$\bar{C} = \frac{\partial g}{\partial \bar{x}} = \begin{bmatrix} C & M \end{bmatrix}, \quad \bar{D} = \frac{\partial g}{\partial u} = D, \quad \bar{F} = \frac{\partial g}{\partial e} = F \tag{30}$$

Jacobians wrt. parameters

$$\boxed{\begin{aligned} S_{\cdot i} &= \frac{\partial A}{\partial \theta_i} x^* + \frac{\partial B}{\partial \theta_i} u^* + \frac{\partial G}{\partial \theta_i} v^*, && (31) \\ M_{\cdot i} &= \frac{\partial C}{\partial \theta_i} x^* + \frac{\partial D}{\partial \theta_i} u^* + \frac{\partial F}{\partial \theta_i} e^* && (32) \end{aligned}}$$

All matrices are evaluated in θ^*

Linearized system

Linearized system

$$\bar{x}_{t+1} = \bar{A}\bar{x}_t + \bar{B}u_t + \bar{G}\bar{v}_t + d_t, \tag{33}$$

$$y_t = \bar{C}\bar{x}_t + \bar{D}u_t + \bar{F}e_t + r_t \tag{34}$$

Offset models

$$d_{t+1} = d_t, \qquad d_0 \sim N(d^*, 0), \tag{35}$$

$$r_{t+1} = r_t, \qquad r_0 \sim N(r^*, 0) \tag{36}$$

Further augmented system

$$\begin{bmatrix} \bar{x}_{t+1} \\ d_{t+1} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} \bar{A} & I & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ d_t \\ r_t \end{bmatrix} + \begin{bmatrix} \bar{B} \\ \\ \end{bmatrix} u_t + \begin{bmatrix} \bar{G} \\ \\ \end{bmatrix} \bar{v}_t, \tag{37}$$

$$y_t = \begin{bmatrix} \bar{C} & 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ d_t \\ r_t \end{bmatrix} + \bar{D}u_t + \bar{F}e_t \tag{38}$$

Extended Kalman filter: Measurement update

Predicted output

$$\boxed{\hat{y}_{t|t-1} = C\hat{x}_{t|t-1} + Du_t} \tag{39}$$

Filtered estimates

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \kappa_{x,t}(y_t - \hat{y}_{t|t-1}), \tag{40}$$
$$\hat{\theta}_{t|t} = \hat{\theta}_{t|t-1} + \kappa_{\theta,t}(y_t - \hat{y}_{t|t-1}) \tag{41}$$

Kalman gains

$$\kappa_{x,t} = \left(P_{xx,t|t-1}C^T + P_{x\theta,t|t-1}M^T\right)\left(CP_{xx,t|t-1}C^T + CP_{x\theta,t|t-1}M^T\right. \tag{42}$$
$$\left. + MP_{\theta x,t|t-1}C^T + MP_{\theta\theta,t|t-1}M^T + FF^T\right)^{-1}, \tag{43}$$
$$\kappa_{\theta,t} = \left(P_{\theta x,t|t-1}C^T + P_{\theta\theta,t|t-1}M^T\right)\left(CP_{xx,t|t-1}C^T + CP_{x\theta,t|t-1}M^T\right. \tag{44}$$
$$\left. + MP_{\theta x,t|t-1}C^T + MP_{\theta\theta,t|t-1}M^T + FF^T\right)^{-1}, \tag{45}$$

Covariances

$$P_{xx,t|t} = P_{xx,t|t-1} - \kappa_{x,t}\left(CP_{xx,t|t-1} + MP_{\theta x,t|t-1}\right), \tag{46}$$
$$P_{x\theta,t|t} = P_{x\theta,t|t-1} - \kappa_{x,t}\left(CP_{x\theta,t|t-1} + MP_{\theta\theta,t|t-1}\right), \tag{47}$$
$$P_{\theta\theta,t|t} = P_{\theta\theta,t|t-1} - \kappa_{\theta,t}\left(CP_{x\theta,t|t-1} + MP_{\theta\theta,t|t-1}\right) \tag{48}$$

C, D, and F are evaluated in $\hat{\theta}_{t|t-1}$ and M is evaluated in $\hat{x}_{t|t-1}$ and $\hat{\theta}_{t|t-1}$

Extended Kalman filter: Time update

Predicted estimates

$$\boxed{\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} + Bu_t, && (49) \\ \hat{\theta}_{t+1|t} &= \hat{\theta}_{t|t}, && (50) \end{aligned}}$$

Covariances

$$\boxed{\begin{aligned} P_{xx,t+1|t} &= AP_{xx,t|t}A^T + AP_{x\theta,t|t}S^T && (51) \\ &\quad + SP_{\theta x,t|t}A^T + SP_{\theta\theta,t|t}S^T + GG^T, && (52) \\ P_{x\theta,t+1|t} &= AP_{x\theta,t|t} + SP_{\theta\theta,t|t}, && (53) \\ P_{\theta\theta,t+1|t} &= P_{\theta\theta,t|t} + HH^T && (54) \end{aligned}}$$

A, B, and G are evaluated in $\hat{\theta}_{t|t}$ and S is evaluated in $\hat{x}_{t|t}$ and $\hat{\theta}_{t|t}$

Prediction

The prediction problem: State prediction

System

$$x_{t+1} = Ax_t + Bu_t + Gv_t, \qquad v_t \sim N(0, R_1), \tag{55}$$

$$y_t = Cx_t + Du_t + Fe_t, \qquad e_t \sim N(0, R_2) \tag{56}$$

State predictions

$$x_1 = Ax_0 + Bu_0 + Gv_0, \tag{57}$$

$$x_2 = Ax_1 + Bu_1 + Gv_1 = A(Ax_0 + Bu_0 + Gv_0) + Bu_1 + Gv_1 \tag{58}$$

$$= A^2x_0 + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \begin{bmatrix} AG & G \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}, \tag{59}$$

$$x_3 = Ax_2 + Bu_2 + Gv_2 \tag{60}$$

$$= A\left(A^2x_0 + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \begin{bmatrix} AG & G \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}\right) + Bu_2 + Gv_2 \tag{61}$$

$$= A^3x_0 + \begin{bmatrix} A^2B & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} A^2G & AG & G \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \tag{62}$$

The prediction problem: State and output predictions

N-step state and output prediction

$$x_N = A^N x_0 + \begin{bmatrix} A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \tag{63}$$

$$+ \begin{bmatrix} A^{N-1}G & A^{N-2}G & \cdots & G \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \tag{64}$$

$$y_N = CA^N x_0 + \begin{bmatrix} CA^{N-1}B & CA^{N-2}B & \cdots & CB & D \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \tag{65}$$

$$+ \begin{bmatrix} CA^{N-1}G & CA^{N-2}G & \cdots & CG & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} + Fe_N \tag{66}$$

The prediction problem: N-step predictions

N-step state and output predictions

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} I \\ A \\ \vdots \\ A^N \end{bmatrix} x_0 + \begin{bmatrix} 0 & & & \\ B & 0 & & \\ \vdots & \ddots & \ddots & \\ A^{N-1}B & \cdots & B & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix} + \begin{bmatrix} 0 & & & \\ G & 0 & & \\ \vdots & \ddots & \ddots & \\ A^{N-1}G & \cdots & G & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \end{bmatrix}, \tag{67}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^N \end{bmatrix} x_0 + \begin{bmatrix} D & & & \\ CB & D & & \\ \vdots & \ddots & \ddots & \\ CA^{N-1}B & \cdots & CB & D \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix} + \begin{bmatrix} 0 & & & \\ CG & 0 & & \\ \vdots & \ddots & \ddots & \\ CA^{N-1}G & \cdots & CG & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \end{bmatrix} \tag{68}$$

$$+ \begin{bmatrix} F & & & \\ & F & & \\ & & \ddots & \\ & & & F \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_N \end{bmatrix} \tag{69}$$

The prediction problem: Compact notation

Compact notation

$$X_N = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad Y_N = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad U_N = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad V_N = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad E_N = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_N \end{bmatrix}, \tag{70}$$

$$\Phi_{xx}^N = \begin{bmatrix} I \\ A \\ \vdots \\ A^N \end{bmatrix}, \quad \Gamma_{xu}^N = \begin{bmatrix} 0 & & & \\ B & 0 & & \\ \vdots & \ddots & \ddots & \\ A^{N-1}B & \cdots & B & 0 \end{bmatrix}, \quad \Gamma_{xv}^N = \begin{bmatrix} 0 & & & \\ G & 0 & & \\ \vdots & \ddots & \ddots & \\ A^{N-1}G & \cdots & G & 0 \end{bmatrix}, \tag{71}$$

$$\Phi_{yx}^N = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^N \end{bmatrix}, \quad \Gamma_{yu}^N = \begin{bmatrix} D & & & \\ CB & D & & \\ \vdots & \ddots & \ddots & \\ CA^{N-1}B & \cdots & CB & D \end{bmatrix}, \quad \Gamma_{yv}^N = \begin{bmatrix} 0 & & & \\ CG & 0 & & \\ \vdots & \ddots & \ddots & \\ CA^{N-1}G & \cdots & CG & 0 \end{bmatrix}, \tag{72}$$

$$\Gamma_{ye} = \begin{bmatrix} F & & & \\ & F & & \\ & & \ddots & \\ & & & F \end{bmatrix}, R_V^N = \begin{bmatrix} R_1 & & & \\ & R_1 & & \\ & & \ddots & \\ & & & R_1 \end{bmatrix}, R_E^N = \begin{bmatrix} R_2 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_2 \end{bmatrix}, \tag{73}$$

$$X_N = \Phi_{xx}^N x_0 + \Gamma_{xu}^N U_N + \Gamma_{xv}^N V_N, \qquad V_N \sim N(0, R_V^N), \tag{74}$$

$$Y_N = \Phi_{yx}^N x_0 + \Gamma_{yu}^N U_N + \Gamma_{yv}^N V_N + \Gamma_{ye}^N E_N, \qquad E_N \sim N(0, R_E^N) \tag{75}$$

General predictive control

General Linear Control Theory

System

$$x_{t+1} = Ax_t + Bu_t + d \tag{76}$$

Control law

$$u_t = -Lx_t + w_t \tag{77}$$

Design control gain, L, and w_t such that

- the system is stable
- the disturbance is mitigated
- the setpoint/reference/tracking target is followed

Closed-loop system

$$x_{t+1} = (A - BL)x_t + Bw_t + d \tag{78}$$

Optimal control - General predictive control

Optimal control problem

$$\min_{U_N} \quad J_N, \qquad J_N = \mathbb{E}[(Y_N - W_N)^T Q_y (Y_N - W_N) + U_N^T Q_u U_N] \tag{79}$$

Write out the objective function

$$\begin{aligned} J_N &= \mathbb{E}[(Y_N - W_N)^T Q_y (Y_N - W_N) + U_N^T Q_u U_N] && (80) \\ &= \mathbb{E}[Y_N^T Q_y Y_N - W_N^T Q_y Y_N - Y_N^T Q_y W_N + W_N^T Q_y W_N + U_N^T Q_u U_N] && (81) \\ &= \mathbb{E}[Y_N^T Q_y Y_N - 2 W_N^T Q_y Y_N + W_N^T Q_y W_N + U_N^T Q_u U_N] && (82) \\ &= \mathbb{E}[Y_N^T Q_y Y_N] - 2 W_N^T Q_y \mathbb{E}[Y_N] + W_N^T Q_y W_N + U_N^T Q_u U_N && (83) \end{aligned}$$

Expectation of quadratic form

$$\mathbb{E}[Y_N^T Q_y Y_N] = \mathbb{E}[Y_N]^T Q_y \mathbb{E}[Y_N] + \operatorname{Tr}(Q_y \operatorname{Cov}(Y_N)) \tag{84}$$

*We omit the superscript N on the matrices for brevity of notation.

Output expectation and variance

Expectation

$$\mathbb{E}[Y_N] = \Phi_{yx}\mathbb{E}[x_0] + \Gamma_{yu}U_N + \Gamma_{yv}\mathbb{E}[V_N] + \Gamma_{ye}\mathbb{E}[E_N] \tag{85}$$

$$= \Phi_{yx}\mathbb{E}[x_0] + \Gamma_{yu}U_N \tag{86}$$

Deviation from expectation

$$Y_N - \mathbb{E}[Y_N] = \Phi_{yx}(x_0 - \mathbb{E}[x_0]) + \Gamma_{yv}V_N + \Gamma_{ye}E_N \tag{87}$$

Covariance

$$\mathrm{Cov}(Y_N) = \mathbb{E}[(Y_N - \mathbb{E}[Y_N])(Y_N - \mathbb{E}[Y_N])^T] \tag{88}$$

$$= \Phi_{yx}\mathbb{E}[(x_0 - \mathbb{E}[x_0])(x_0 - \mathbb{E}[x_0])^T]\Phi_{yx}^T \tag{89}$$

$$+ \Gamma_{yv}\mathbb{E}[V_N V_N^T]\Gamma_{yv}^T + \Gamma_{ye}\mathbb{E}[E_N E_N^T]\Gamma_{ye}^T + \cdots \tag{90}$$

$$= \Phi_{yx}P_0\Phi_{yx}^T + \Gamma_{yv}R_V\Gamma_{yv}^T + \Gamma_{ye}R_E\Gamma_{ye}^T \tag{91}$$

Important observation: $\mathrm{Cov}(Y_N)$ is independent of U_N

Optimal control - General predictive control

Disregard terms that are independent of U_N

$$
\begin{aligned}
J_N &= \mathbb{E}[Y_N^T Q_y Y_N] - 2W_N^T Q_y \mathbb{E}[Y_N] - W_N^T Q_y W_N + U_N^T Q_u U_N && (92)\\
&= \mathbb{E}[Y_N]^T Q_y \mathbb{E}[Y_N] + \mathrm{Tr}(Q_y \,\mathrm{Cov}(Y_N)) && (93)\\
&- 2W_N^T Q_y \mathbb{E}[Y_N] + W_N^T Q_y W_N + U_N^T Q_u U_N && (94)\\
&= \mathbb{E}[Y]^T Q_y \mathbb{E}[Y_N] - 2W_N^T Q_y \mathbb{E}[Y_N] + U_N^T Q_u U_N + \text{const.} && (95)
\end{aligned}
$$

Insert expectation

$$
\begin{aligned}
J_N &= (\Phi_{yx}\mathbb{E}[x_0] + \Gamma_{yu}U_N)^T Q_y (\Phi_{yx}\mathbb{E}[x_0] + \Gamma_{yu}U_N) && (96)\\
&- 2W_N^T Q_y (\Phi_{yx}\mathbb{E}[x_0] + \Gamma_{yu}U_N) + U_N^T Q_u U_N + \text{const.} && (97)\\
&= U_N^T \Gamma_{yu}^T Q_y \Gamma_{yu} U_N + 2\mathbb{E}[x_0]^T \Phi_{yx}^T Q_y \Gamma_{yu} U_N && (98)\\
&+ \mathbb{E}[x_0]^T \Phi_{yx}^T Q_y \Phi_{yx} \mathbb{E}[x_0] - 2W_N^T Q_y \Gamma_{yu} U_N + U_N^T Q_u U_N + \text{const.} && (99)\\
&= U_N^T \left(\Gamma_{yu}^T Q_y \Gamma_{yu} + Q_u\right) U_N + 2\left(\Phi_{yx}\mathbb{E}[x_0] - W_N\right)^T Q_y \Gamma_{yu} U_N + \text{const.} && (100)
\end{aligned}
$$

Optimal control - General predictive control

Gradient of objective function

$$\nabla J_N = 2\left(\Gamma_{yu}^T Q_y \Gamma_{yu} + Q_u\right) U_N + 2\left(\left(\Phi_{yx}\mathbb{E}[x_0] - W_N\right)^T Q_y \Gamma_{yu}\right)^T \quad (101)$$

$$= 2\left(\Gamma_{yu}^T Q_y \Gamma_{yu} + Q_u\right) U_N + 2\,\Gamma_{yu}^T Q_y \left(\Phi_{yx}\mathbb{E}[x_0] - W_N\right) \quad (102)$$

The gradient is zero for the optimal solution

$$\nabla J_N = 0 \quad (103)$$

Optimal solution

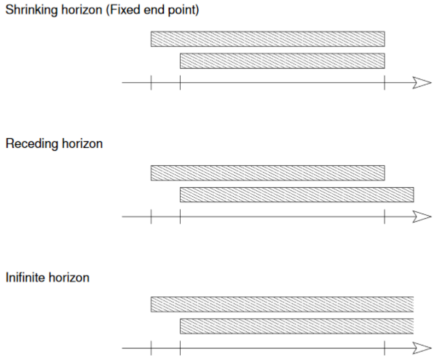
$$\boxed{U_N = -\left(\Gamma_{yu}^T Q_y \Gamma_{yu} + Q_u\right)^{-1} \Gamma_{yu}^T Q_y \left(\Phi_{yx}\mathbb{E}[x_0] - W_N\right)} \quad (104)$$

General predictive control: Prediction and control horizons

Closed-loop predictive control

$$u_t = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} U_N \tag{105}$$

U_N is updated at every time instance



Questions?