Stochastic Adaptive Control (02421)

Lecture 8

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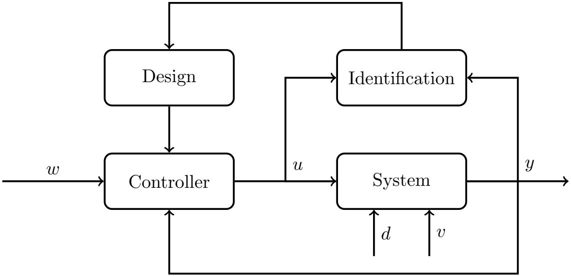
Lecture Plan

1. ARX models
2. ARX estimation
3. ARMAX estimation
4. ARMAX identification
5. ARMAX control
6. ARMAX adaptive control
7. Systems and control theory
8. Stochastic systems + SS estimation
9. SS estimation + control
10. SS control
11. To be decided ...
12. SS nonlinear estimation
13. SS nonlinear control



Today's Agenda

- Stochastic state space models
- Discretization
- State estimation
- Projection theorem
- Kalman filter

Stochastic state space models

Stochastic State-Space Models

Discrete-time system

$$x_{k+1} = Ax_k + Bu_k + Gv_k, \quad v_k \sim N(\mu_v, R_1) \tag{1a}$$
$$y_k = Cx_k + Du_k + Fe_k, \quad e_k \sim N(\mu_e, R_2) \tag{1b}$$

Mean and covariance

$$\boxed{\begin{aligned} \mu_{k+1} &= A\mu_k + Bu_k + G\mu_v, \quad \mu_0 = \mathbb{E}[x_0], \\ P_{k+1} &= AP_kA^T + GR_1G^T, \quad P_0 = \mathrm{Cov}(x_0) \end{aligned}} \quad \begin{aligned} (2a) \\ (2b) \end{aligned}$$

Note that u_k is deterministic.

Stationary mean and covariance

$$\boxed{\begin{aligned} \mu_\infty &= A\mu_\infty + Bu_\infty + G\mu_v, \\ P_\infty &= AP_\infty A^T + GR_1G^T \end{aligned}} \quad \begin{aligned} (3a) \\ (3b) \end{aligned}$$

Stationary auto-covariance (if A has full-rank and the eigenvalues lie within the unit circle)

$$r_{x,\infty}(\tau) = A^\tau P_\infty \tag{4}$$

Continuous-Time Stochastic Processes (SDE)

Continuous-time system

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{5}$$

First attempt at stochastic differential equation

$$\dot{x}(t) = f(t, x(t), u(t)) + g(t, x(t), u(t))v(t) \tag{6}$$

Process noise

- $v(t) \perp v(s)$ for any $t \neq s$ (independence)
- $v(t)$ is continuous and has bounded variance
- $\mathbb{E}[v(t)] = 0$ (zero-mean)

Theorem 4.1 in Chapter 3 of the book "Stochastic Control Theory" by Åström (1970)

Continuous-Time Stochastic Processes (SDE)

Stochastic difference equation

$$x(t+\Delta t) - x(t) = f(t, x(t))\Delta t + g(t, x(t))v(t)\Delta t + o(\Delta t). \tag{7}$$

Replace $v(t)\Delta t$ with $\Delta w(t) = w(t+\Delta t) - w(t)$, which has stationary independent zero-mean increments (Wiener process)

$$\Delta x(t) = f(t, x(t))\Delta t + g(t, x(t))\Delta w(t) + o(\Delta t). \tag{8}$$

Take the limit $\Delta t \to 0$

$$\mathrm{d}x(t) = f(t, x(t))\,\mathrm{d}t + g(t, x(t))\,\mathrm{d}w(t) \tag{9}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))\,\mathrm{d}\tau + \int_{t_0}^{t} g(\tau, x(\tau))\,\mathrm{d}w(\tau) \tag{10}$$

Continuous-Time Stochastic Processes (SDE)

The two first conditional moments of the difference process

$$\mathbb{E}\Big[\Delta x(t) \mid x(t)\Big] = f(t, x(t))\Delta t + o(\Delta t) \tag{11a}$$

$$\mathrm{Var}\Big(\Delta x(t) \mid x(t)\Big) = g^2(t, x(t))\Delta t + o(\Delta t) \tag{11b}$$

Variance of process noise increment

$$\mathbb{E}\Big[\Delta w^2(t)\Big] = \Delta t \tag{12}$$

Note that the variance is proportional to Δt and not Δt^2

Distribution of process noise increment (increment of Wiener process)

$$\Delta w(t) = w(t + \Delta t) - w(t) \sim N(0, \Delta t) \tag{13}$$

Linear Stochastic Differential Equations

Linear stochastic differential equations

$$\mathrm{d}x(t) = (A(t)x(t) + B(t)u(t))\,\mathrm{d}t + G(t)\,\mathrm{d}w(t), \quad x(t_0) \sim N(m_0, P_0) \tag{14}$$

$A(t)$ and $B(t)$ are continuous functions of time

State expectation

$$\mathbb{E}[x(t)] = \mathbb{E}[x_0] + \mathbb{E}\left[\int_{t_0}^{t} A(\tau)x(\tau) + B(\tau)u(\tau)\,\mathrm{d}\tau\right] + \mathbb{E}\left[\int_{t_0}^{t} G(\tau)\,\mathrm{d}w(\tau)\right] \tag{15}$$

$$= \mathbb{E}[x_0] + \int_{t_0}^{t} A(\tau)\mathbb{E}[x(\tau)] + B(\tau)u(\tau)\,\mathrm{d}\tau = m_x(t) \tag{16}$$

Expected value

$$\dot{m}_x(t) = Am_x(t) + B(t)u(t), \quad m_x(t_0) = m_0. \tag{17}$$

Linear Stochastic Differential Equations

State-transition matrix

$$\frac{\partial \Phi(t;t_0)}{\partial t} = A(t)\Phi(t;t_0), \qquad \Phi(t_0;t_0) = I. \tag{18}$$

Auto-covariance of x ($s \geq t$)

$$R(s,t) = \mathrm{Cov}(x(s), x(t)) = \Phi(s,t)P(t) \tag{19}$$

Covariance

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + G(t)G^T(t), \qquad P(t_0) = P_0, \tag{20}$$

Discretization

Linear stochastic differential equation

Linear continuous-time state space model

$$\begin{aligned} dx(t) &= (Ax(t) + Bu(t))\,dt + G\,dw(t), & dw(t) &\sim N(0, I\,dt), & (21a) \\ y(t) &= Cx(t) + Du(t) + Fe(t), & e(t) &\sim N(m_e, R_e) & (21b) \end{aligned}$$

Zero-order-hold parametrization of manipulated inputs

$$u(t) = u_k, \qquad t \in [t_k, t_{k+1}[\qquad (22)$$

Analytical solution

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(t_k)\,d\tau + v(t_k), \qquad (23a)$$

$$y(t_k) = Cx(t_k) + Du(t_k) + Fe(t_k), \quad e(t_k) \sim N(m_e, R_e) \qquad (23b)$$

Zero-order-hold parametrization and approximation

Discrete-time process noise

$$v(t_k) = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} G \,\mathrm{d}w(t) \tag{24}$$

Mean

$$\mathbb{E}[v(t_k)] = \mathbb{E}\left[\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} G \,\mathrm{d}w(t)\right] = 0 \tag{25}$$

Zero-order-hold parametrization and approximation

Covariance

$$\begin{aligned}
\mathrm{Cov}(v(t_k)) &= \mathbb{E}[v(t_k)v^T(t_k)] && (26)\\
&= \mathbb{E}\left[\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}G\,\mathrm{d}w(\tau)\int_{t_k}^{t_{k+1}} G^T e^{A^T(t_{k+1}-s)}\,\mathrm{d}w^T(s)\right] && (27)\\
&= \mathbb{E}\left[\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}G\,\mathrm{d}w(\tau)\,\mathrm{d}w^T(s)G^T e^{A^T(t_{k+1}-s)}\right] && (28)\\
&= \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}GG^T e^{A^T(t_{k+1}-\tau)}\,\mathrm{d}\tau && (29)
\end{aligned}$$

Stochastic discrete-time state space models

Linear discrete-time stochastic state space model

$$x_{k+1} = A_d x_k + B_d u_k + v_k, \qquad v_k \sim N(0, R_1), \tag{30}$$

$$y_k = C_d x_k + D_d u_k + e_k, \qquad e_k \sim N(0, R_2) \tag{31}$$

System matrices in the state equation

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_s \right), \tag{32}$$

$$\begin{bmatrix} A_d & \tilde{R}_1 \\ 0 & A_d^{-T} \end{bmatrix} = \exp\left(\begin{bmatrix} A & GG^T \\ 0 & -A^T \end{bmatrix} T_s \right), \qquad R_1 = \tilde{R}_1 A_d^T \tag{33}$$

$$\begin{bmatrix} A_d^{-1} & \tilde{R}_1^T \\ 0 & A_d^T \end{bmatrix} = \exp\left(\begin{bmatrix} -A & GG^T \\ 0 & A^T \end{bmatrix} T_s \right), \qquad R_1 = A_d \tilde{R}_1^T \tag{34}$$

System matrices in the measurement equation

$$\boxed{C_d = C, \qquad D_d = D, \qquad R_2 = F R_e F^T} \tag{35}$$

Proof of discretization

Exponential form

$$M = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \exp\left(\begin{bmatrix} F & G \\ 0 & H \end{bmatrix} t \right) = \exp(Kt) \tag{36}$$

Differential form

$$\dot{M} = KM, \quad M(t_0) = I \tag{37}$$

Individual differential equations

$$\dot{X} = FX, \qquad X(t_0) = I, \tag{38}$$

$$\dot{Y} = FY + GZ, \qquad Y(t_0) = 0, \tag{39}$$

$$\dot{Z} = HZ, \qquad Z(t_0) = I \tag{40}$$

Solutions

$$X = e^{F(t-t_0)} X(t_0) = e^{F(t-t_0)}, \tag{41}$$

$$Z = e^{H(t-t_0)} Z(t_0) = e^{H(t-t_0)}, \tag{42}$$

$$Y = e^{F(t-t_0)} Y(t_0) + \int_{t_0}^{t} e^{F(t-\tau)} G e^{H(\tau-t_0)} Z(t_0)\,\mathrm{d}\tau = \int_{t_0}^{t} e^{F(t-\tau)} G e^{H(\tau-t_0)}\,\mathrm{d}\tau$$

Proof of discretization

Let $F = A$, $H = -A^T$, $G = GG^T$, $t_0 = t_k$, and $t = t_{k+1}$ $(t_{k+1} - t_k = T_s)$

$$X = e^{AT_s} = A_d, \tag{43}$$

$$Z = e^{-A^T T_s} = A_d^{-T}, \tag{44}$$

$$Y = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} GG^T e^{A^T(t_{k+1}-\tau)} \, \mathrm{d}\tau \, e^{-A^T T_s} = \tilde{R}_1 = R_1 A_d^T \tag{45}$$

The proof of the other approach is similar, but has one more step:
A change of variables in the integral

State estimation

Filters and Estimation

Objective: Obtain estimate, $\hat{x}_t$, of the signal x_t based on measurements

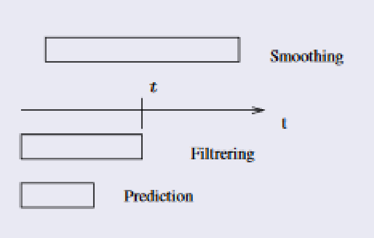
$$Y_{0:N} = \begin{bmatrix} y_0 & y_1 & \cdots & y_N \end{bmatrix} \tag{46}$$

and a state-output relation, e.g.,

$$y_t = Cx_t + e_t \tag{47}$$

Different types of estimation

1. Smoothing ($t < t_N$): Use both past and future data to estimate the states
2. Filtering: ($t = t_N$): Estimate the current states based on current and past data
3. Prediction: ($t > t_N$): Predict future states based on past data



Filter Theory

Stochastic discrete-time system

$$x_{t+1} = Ax_t + Bu_t + v_t, \qquad x_0 \sim N(m_0, P_0), \qquad v_t \sim N(0, R_1), \tag{48}$$

$$y_t = Cx_t + e_t, \qquad e_t \sim N(0, R_2) \tag{49}$$

We will only consider filtering in this lecture

Filter Theory - The Good Estimate

Core concepts of filter design

1. Characteristics of the signal and noise
2. Observation model (relation between y, x, e)
3. Criterion (what is a good estimate)
4. Restrictions (what information is available)

Characteristics: Nature of the states, dynamics, and noises

Observation: Relation between the output, y, the state x, and the noise

The criterion: A good estimate minimizes the expected squared deviation

$$\mathbb{E}[||x - \hat{x}||^2] \tag{50}$$

Restrictions: What data, Y, is available (filter, predict, or smoothe?)

The filter problem: A good estimator

The law of total expectation

$$\mathbb{E}[g(x)] = \mathbb{E}_Y[\mathbb{E}[g(x)|Y]] \tag{51}$$

Introduce "inner" objective function

$$\begin{aligned} J &= \mathbb{E}[\|x - \hat{x}\|^2] && (52) \\ &= \mathbb{E}[(x - \hat{x})^T(x - \hat{x})] && (53) \\ &= \mathbb{E}_Y[\mathbb{E}[(x - \hat{x})^T(x - \hat{x})|Y]] = \mathbb{E}_Y[J_{in}] && (54) \end{aligned}$$

Inner objective function

$$\begin{aligned} J_{in} &= \mathbb{E}[x^T x - \hat{x}^T x - x^T \hat{x} + \hat{x}^T \hat{x}|Y], && (55) \\ &= \mathbb{E}[x^T x|Y] - \hat{x}^T \mathbb{E}[x|Y] - \mathbb{E}[x|Y]^T \hat{x} + \hat{x}^T \hat{x} && (56) \end{aligned}$$

Optimal estimate

$$\begin{aligned} \nabla_{\hat{x}} J_{in} &= 2\hat{x} - 2\mathbb{E}[x|Y] = 0, && (57) \\ \hat{x} &= \mathbb{E}[x|Y] && (58) \end{aligned}$$

Projection theorem

Filter Theory - Projection Theorem

Normally distributed vector

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & P_{xy} \\ P_{xy}^T & P_y \end{bmatrix} \right) \tag{59}$$

Projection theorem: The conditional distribution $X|Y \sim N(m_{x|y}, P_{x|y})$ is

$$m_{x|y} = m_x + P_{xy} P_y^{-1} (y - m_y), \tag{60}$$

$$P_{x|y} = P_x - P_{xy} P_y^{-1} P_{xy}^T, \tag{61}$$

$$X - m_{x|y} \perp Y \tag{62}$$

Filter Theory - Proof of Projection Theorem

Probability density functions

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^{n_x+n_y}\sqrt{\det(P_z)}} e^{-\frac{1}{2}(z-m_z)^T P_z^{-1}(z-m_z)}, \tag{63}$$

$$f_Y(y) = \frac{1}{(2\pi)^{n_y}\sqrt{\det(P_y)}} e^{-\frac{1}{2}(y-m_y)^T P_y^{-1}(y-m_y)} \tag{64}$$

Probability density function of conditional normal distribution

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \tag{65}$$

$$= \sqrt{\frac{\det(P_y)}{(2\pi)^{n_x}\det(P_z)}} e^{-\frac{1}{2}(z-m_z)^T P_z^{-1}(z-m_z)+\frac{1}{2}(y-m_y)^T P_y^{-1}(y-m_y)}$$

$$= \kappa e^{-\frac{1}{2}\alpha} \tag{66}$$

Filter Theory - Proof of Projection Theorem

Schur complement

$$D = P_x - P_{xy} P_y^{-1} P_{xy}^T \tag{67}$$

Use Woodbury matrix identity on P_z^{-1}

$$P_z^{-1} = \begin{bmatrix} D^{-1} & -D^{-1} P_{xy} P_y^{-1} \\ -P_y^{-1} P_{xy}^T D^{-1} & P_y^{-1} + P_y^{-1} P_{xy}^T D^{-1} P_{xy} P_y^{-1} \end{bmatrix} \tag{68}$$

Determinant

$$\det(P_z) = \det(P_y) \det(D) \qquad \Leftrightarrow \qquad \frac{\det(P_y)}{\det(P_z)} = \frac{1}{\det(D)} \tag{69}$$

Factor

$$\kappa = \sqrt{\frac{\det(P_y)}{(2\pi)^{n_x} \det(P_z)}} = \frac{1}{\sqrt{(2\pi)^{n_x} \det(D)}} \tag{70}$$

Exponent

$$\alpha = (z - m_z)^T P_z^{-1} (z - m_z) - (y - m_y)^T P_y^{-1} (y - m_y) \tag{71}$$

$$= [x - (m_x + P_{xy} P_y^{-1} (y - m_y))]^T D^{-1} [x - (m_x + P_{xy} P_y^{-1} (y - m_y))] \tag{72}$$

Filter Theory - Proof of Projection Theorem

Mean and covariance of conditional distribution

$$\mathbb{E}[X|Y] = m_{x|y} = m_x + P_{xy}P_y^{-1}(y - m_y) \tag{73}$$

$$\mathrm{Cov}(X|Y) = P_{x|y} = D = P_x - P_{xy}P_y^{-1}P_{xy}^T \tag{74}$$

Covariance (are the variables independent?)

$$\mathrm{Cov}(X - m_{x|y}, Y) = \mathrm{Cov}(X, Y) - P_{xy}P_y^{-1}\,\mathrm{Cov}(Y, Y) \tag{75}$$

$$= P_{xy} - P_{xy}P_y^{-1}P_y = 0 \tag{76}$$

As X and Y are Gaussian, $X - m_{x|y}$ and Y are independent

Kalman filter

State estimation

Stochastic discrete-time system

$$x_{t+1} = Ax_t + Bu_t + v_t, \qquad x_0 \sim N(\hat{x}_0, P_0), \qquad v_t \sim N(0, R_1), \tag{77}$$

$$y_t = Cx_t + e_t, \qquad e_t \sim N(0, R_2) \tag{78}$$

$v_t \perp x_s$ for all $s \leq t$ and $e_t \perp x_s$ for all s

Mean and covariance of joint distribution

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} |Y_{t-1} \sim N\left(\begin{bmatrix} \times \\ \times \end{bmatrix}, \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \right), \quad Y_t = \begin{bmatrix} Y_{t-1} \\ y_t \end{bmatrix} \tag{79}$$

Conditional state distributions

$$x_t|Y_{t-1} \sim N(\hat{x}_{t|t-1}, P_{t|t-1}) \tag{80}$$

$$x_t|Y_t = x_t|y_t, Y_{t-1} \sim N(\hat{x}_{t|t}, P_{t|t}) \tag{81}$$

State estimation: A recursion

Measurement equation

$$y_t = Cx_t + e_t, \qquad e_t \sim N(0, R_2), \qquad e_t \perp x_s \tag{82}$$

Mean

$$\mathbb{E}[y_t|Y_{t-1}] = C\mathbb{E}[x_t|Y_{t-1}] + \mathbb{E}[e_t|Y_{t-1}] = C\hat{x}_{t|t-1} \tag{83}$$

Covariance

$$\begin{aligned}\mathrm{Cov}(y_t|Y_{t-1}) &= C\,\mathrm{Cov}(x_t|Y_{t-1})C^T + C\,\mathrm{Cov}(x_t, e_t|Y_{t-1}) \\ &+ \mathrm{Cov}(e_t, x_t|Y_{t-1})C^T + \mathrm{Cov}(e_t|Y_{t-1}) = CP_{t|t-1}C^T + R_2\end{aligned} \tag{84}$$

Cross-covariance

$$\mathrm{Cov}(y_t, x_t|Y_{t-1}) = C\,\mathrm{Cov}(x_t|Y_{t-1}) + C\,\mathrm{Cov}(x_t, e_t|Y_{t-1}) \tag{85}$$

$$= CP_{t|t-1} \tag{86}$$

Conditional distribution

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} |Y_{t-1} \sim N\left(\begin{bmatrix} \hat{x}_{t|t-1} \\ C\hat{x}_{t|t-1} \end{bmatrix}, \begin{bmatrix} P_{t|t-1} & P_{t|t-1}C^T \\ CP_{t|t-1} & CP_{t|t-1}C^T + R_2 \end{bmatrix}\right) \tag{87}$$

State Estimation 4: The Current Estimate

Conditional distribution

$$x_t|y_t, Y_{t-1} = x_t|Y_t \sim N(\hat{x}_{t|t}, P_{t|t}) \tag{88}$$

Use projection theorem

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + P_{t|t-1}C^T(CP_{t|t-1}C^T + R_2)^{-1}(y_t - C\hat{x}_{t|t-1}), \tag{89}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}C^T(CP_{t|t-1}C^T + R_2)^{-1}CP_{t|t-1} \tag{90}$$

State Estimation 5: Prediction Estimate

State equation

$$x_{t+1} = Ax_t + Bu_t + v_t, \quad v_t \sim N(0, R_1), \quad v_t \perp x_s \text{ for all } s \leq t \tag{91}$$

Mean

$$\begin{aligned} \mathbb{E}[x_{t+1}|Y_t] &= A\mathbb{E}[x_t|Y_t] + B\mathbb{E}[u_t|Y_t] + \mathbb{E}[v_t|Y_t] && (92) \\ &= A\hat{x}_{t|t} + Bu_t && (93) \end{aligned}$$

Covariance

$$\begin{aligned} \operatorname{Cov}(x_{t+1}|Y_t) = A\operatorname{Cov}(x_t|Y_t)A^T + \operatorname{Cov}(v_t|Y_t) + A\operatorname{Cov}(x_t, v_t|Y_t) && (94) \\ + \operatorname{Cov}(v_t, x_t|Y_t)A^T = AP_{t|t}A^T + R_1 && (95) \end{aligned}$$

Prediction estimate

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} + Bu_t, && (96) \\ P_{t+1|t} &= AP_{t|t}A^T + R_1 && (97) \end{aligned}$$

Filter Theory - Kalman Filter

Data/measurement-update (inference)

$$\boxed{\begin{aligned} \hat{x}_{t|t} &= \hat{x}_{t|t-1} + \kappa_t(y_t - C\hat{x}_{t|t-1}), \\ \kappa_t &= P_{t|t-1}C^T(CP_{t|t-1}C^T + R_2)^{-1}, \\ P_{t|t} &= P_{t|t-1} - \kappa_t C P_{t|t-1} \end{aligned}} \quad \begin{aligned} &(98) \\ &(99) \\ &(100) \end{aligned}$$

Time-update (prediction)

$$\boxed{\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} + Bu_t, & \hat{x}_{0|0} &= \hat{x}_0, \\ P_{t+1|t} &= AP_{t|t}A^T + R_1, & P_{0|0} &= P_0 \end{aligned}} \quad \begin{aligned} &(101) \\ &(102) \end{aligned}$$

Example: Pseudocode - Kalman Filter/Simulation Implementation

Initial values: $x_{0|-1}$, $P_{0|-1}$, x_0

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for $t = 0, \dots, N$

Measurement from true system:

$y_t = \texttt{Measurement}(x_t, e_t)$

Data update:

$[\hat{x}_{t|t}, P_{t|t}, \kappa_t] = \texttt{DataUpdate}(y_t, \hat{x}_{t|t-1}, P_{t|t-1}; C, R_2)$

Compute control:

$u_t = \texttt{Actuator}(\hat{x}_{t|t})$

Apply control:

$x_{t+1} = \texttt{Simulator}(x_t, u_t, v_t)$

Time update:

$[\hat{x}_{t+1|t}, P_{t+1|t}] = \texttt{TimeUpdate}(\hat{x}_{t|t}, P_{t|t}, u_t; A, B, R_1)$

end

Example: Estimation of constant

Estimate scalar constant

$$x_{t+1} = x_t, \tag{103}$$

$$y_t = x_t + e_t, \quad e_t \in N(0, r_2) \tag{104}$$

Define $q_t = p_t^{-1}$

$$\kappa_t = \frac{p_t}{p_t + r_2} = \frac{1}{1 + r_2 q_t}, \tag{105}$$

$$p_{t+1} = (1 - \kappa_t) p_t = \left(1 - \frac{1}{1 + r_2 q_t}\right) p_t = \frac{r_2 q_t}{1 + r_2 q_t} p_t = \frac{r_2}{1 + r_2 q_t}, \tag{106}$$

$$q_{t+1} = \frac{1}{p_{t+1}} = \frac{1 + r_2 q_t}{r_2} = q_t + \frac{1}{r_2} = q_0 + \frac{t+1}{r_2}, \tag{107}$$

$$\hat{x}_{t+1} = \hat{x}_t + \kappa_t (y_t - \hat{x}_t) \tag{108}$$

If $q_0 = 0$ $(p_0 = \infty)$,

$$\hat{x}_{t+1} = \hat{x}_t + \frac{1}{1+t}(y_t - \hat{x}_t) \qquad \text{or} \qquad \hat{x}_t = \frac{1}{t} \sum_{i=0}^{t-1} y_i \tag{109}$$

Two different forms of the Kalman filter

Ordinary Kalman filter

$$\underbrace{\begin{bmatrix}\hat{x}_{t|t}\\ P_{t|t}\end{bmatrix} \to \begin{bmatrix}\hat{x}_{t+1|t}\\ P_{t+1|t}\end{bmatrix} \to \begin{bmatrix}\hat{x}_{t+1|t+1}\\ P_{t+1|t+1}\end{bmatrix}}_{\text{Ordinary Kalman Filter}} \tag{110}$$

Predictive Kalman filter

$$\underbrace{\begin{bmatrix}\hat{x}_{t|t-1}\\ P_{t|t-1}\end{bmatrix} \to \begin{bmatrix}\hat{x}_{t|t}\\ P_{t|t}\end{bmatrix} \to \begin{bmatrix}\hat{x}_{t+1|t}\\ P_{t+1|t}\end{bmatrix}}_{\text{Predictive Kalman Filter}} \tag{111}$$

Ordinary form of the Kalman filter

Time-update (prediction)

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t, \qquad \hat{x}_{0|0} = \hat{x}_0, \tag{112}$$

$$P_{t+1|t} = AP_{t|t}A^T + R_1, \qquad P_{0|0} = P_0 \tag{113}$$

Data-update (inference)

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \kappa_t(y_t - C\hat{x}_{t|t-1}), \tag{114}$$

$$\kappa_t = P_{t|t-1}C^T(CP_{t|t-1}C^T + R_2)^{-1}, \tag{115}$$

$$P_{t|t} = P_{t|t-1} - \kappa_t CP_{t|t-1} \tag{116}$$

Ordinary form of the Kalman filter

$$\boxed{\begin{aligned} \hat{x}_{t|t} &= (I - \kappa_t C)(A\hat{x}_{t-1|t-1} + Bu_{t-1}) + \kappa_t y_t, \\ P_{t|t} &= AP_{t-1|t-1}A^T + R_1 - \kappa_t C(AP_{t-1|t-1}A^T + R_1), \\ \kappa_t &= (AP_{t-1|t-1}A^T + R_1)C^T(C(AP_{t-1|t-1}A^T + R_1)C^T + R_2)^{-1} \end{aligned}}$$

(117)

(118)

(119)

Predictive form of the Kalman filter

Time-update (prediction)

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t, \qquad \hat{x}_{0|0} = \hat{x}_0, \tag{120}$$

$$P_{t+1|t} = AP_{t|t}A^T + R_1, \qquad P_{0|0} = P_0 \tag{121}$$

Data-update (inference)

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \kappa_t(y_t - C\hat{x}_{t|t-1}), \tag{122}$$

$$\kappa_t = P_{t|t-1}C^T(CP_{t|t-1}C^T + R_2)^{-1}, \tag{123}$$

$$P_{t|t} = P_{t|t-1} - \kappa_t CP_{t|t-1} \tag{124}$$

Predictive form of the Kalman filter

$$\boxed{\begin{aligned} \hat{x}_{t+1|t} &= (A - K_tC)\hat{x}_{t|t-1} + Bu_t + K_ty_t, && (125)\\ P_{t+1|t} &= AP_{t|t-1}A^T + R_1 - K_tCP_{t|t-1}A^T, && (126)\\ K_t &= A\kappa_t = AP_{t|t-1}C^T(CP_{t|t-1}C^T + R_2)^{-1} && (127)\end{aligned}}$$

Questions?