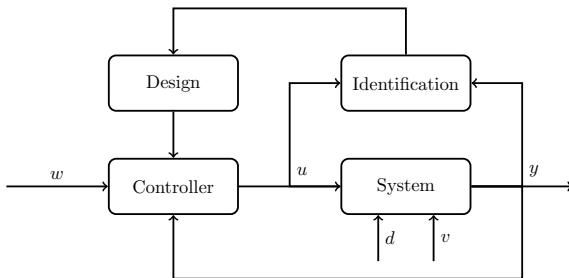


- ① ARX models
- ② ARX estimation
- ③ ARMAX estimation
- ④ ARMAX identification
- ⑤ ARMAX control
- ⑥ ARMAX adaptive control
- ⑦ Systems and control theory
- ⑧ Stochastic systems + SS estimation
- ⑨ SS estimation + control
- ⑩ SS control
- ⑪ To be decided ...
- ⑫ SS nonlinear estimation
- ⑬ SS nonlinear control



Systems Theory

- Continuous- and discrete-time internal and external models
- Linearization
- Discretization
- Transformations, stability, reachability, and observability

Dynamical systems

Dynamical Systems: External and Internal Models

We describe dynamical systems in two ways:

Internal Models

- States of the system
- Differential equations



External Models

- Transfer functions
- Zeros and poles



Focus in this course

- Discrete-time state space models
- Discrete-time external models

Nonlinear/continuous-time systems

- Linearize
- Discretize

State space model

$$x(t_0) = x_0, \quad (1a)$$

$$\dot{x}(t) = \frac{\partial x}{\partial t}(t) = f(x(t), u(t); \theta) = A(\theta)x(t) + B(\theta)u(t) \quad (1b)$$

Analytical solution

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t f(x(\tau), u(\tau); \theta) d\tau \\ &= e^{A(\theta)(t-t_0)} x_0 + \int_{t_0}^t e^{A(\theta)(\tau-t_0)} B(\theta) u(\tau) d\tau. \end{aligned} \quad (2)$$

Output equation

$$y(t) = g(x(t), u(t); \theta) = C(\theta)x(t) + D(\theta)u(t) \quad (3)$$

Steady state (x^*, u^*)

$$f(x^*, u^*; \theta) = 0. \quad (4)$$

Dynamical systems: ODE (External)

Inhomogeneous N -th order linear time-invariant external model

$$\sum_{k=0}^N \alpha_k \frac{\partial^k y}{\partial t^k} = \sum_{k=0}^M \beta_k \frac{\partial^k u}{\partial t^k}, \quad \alpha_k, \beta_k \in \mathbb{R} \quad (5)$$

Analytical solution

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(s) u(t-s) \, ds, \quad (6)$$

$h(t)$ is the impulse response

Laplace transformed variables

$$Y(s) = H(s)U(s), \quad U(s) = \mathcal{L}(u), \quad (7)$$

where

$$\begin{aligned} H(s) &= \mathcal{L}(h) = \int_{-\infty}^{\infty} h(t) e^{-st} \, dt \\ &= C(\theta) (sI - A(\theta))^{-1} B(\theta) + D(\theta). \end{aligned} \quad (8)$$

Continuous-time time-domain

$$t$$
$$y(t) = h(t) * u(t)$$
$$\frac{dy}{dt}(t) = sY(s)$$

Discrete-time time-domain

$$t_k = kT_s$$
$$y_k = H_d(q)u_k$$
$$u_k = u(t_k) = u(kT_s)$$
$$u_{k-1} = q^{-1}u_k$$

where T_s is the sampling time

Continuous-time frequency-domain

$$s = a + iw$$
$$Y(s) = H(s)U(s)$$
$$H(s) = \frac{\sum_{k=0}^M \beta_k s^k}{\sum_{k=0}^N \alpha_k s^k}$$

Discrete-time frequency-domain

$$z = e^{T_s s}$$
$$Y(z) = H_z(z)U(z)$$
$$H_d(q) = H_z(q)$$

Linearization

Linearize around steady state (x^*, u^*)

$$f(x^*, u^*; \theta) = 0 \quad (9)$$

Linearization (truncated Taylor expansion)

$$\dot{x} = f(x^*, u^*; \theta) + \frac{\partial f}{\partial x}(x^*, u^*; \theta)(x - x^*) + \frac{\partial f}{\partial u}(x^*, u^*; \theta)(u - u^*), \quad (10a)$$

$$y = g(x^*, u^*; \theta) + \frac{\partial g}{\partial x}(x^*, u^*; \theta)(x - x^*) + \frac{\partial g}{\partial u}(x^*, u^*; \theta)(u - u^*) \quad (10b)$$

$$X = x - x^*, \quad (11a)$$

$$U = u - u^*, \quad (11b)$$

$$Y = y - y^*, \quad y^* = g(x^*, u^*; \theta) \quad (11c)$$

System matrices

$$A(\theta, x^*, u^*) = \frac{\partial f}{\partial x}(x^*, u^*; \theta), \quad B(\theta, x^*, u^*) = \frac{\partial f}{\partial u}(x^*, u^*; \theta), \quad (12a)$$

$$C(\theta, x^*, u^*) = \frac{\partial g}{\partial x}(x^*, u^*; \theta), \quad D(\theta, x^*, u^*) = \frac{\partial g}{\partial u}(x^*, u^*; \theta) \quad (12b)$$

Linear time invariant (LTI) system

$$\dot{X} = A(\theta, x^*, u^*)X + B(\theta, x^*, u^*)U, \quad (13a)$$

$$Y = C(\theta, x^*, u^*)X + D(\theta, x^*, u^*)U \quad (13b)$$

Discretization

Discretization: Sampling of Continuous Systems

Sampling

$$x_k = x(t_0 + T_s k), \quad y_k = y(t_0 + T_s k) \quad (14)$$

Zero-order hold (ZOH) parametrization: Piecewise constant input, u

$$u(t) = u_k, \quad kT_s \leq t < (k+1)T_s \quad (15)$$

Shannon's Sampling Theorem: If the highest frequency of the system is w_0 , then a sampling frequency of at least the double is needed for reconstruction

$$w_s \geq 2w_0, \quad w_s = \frac{2\pi}{T_s} \quad (16)$$

Choosing based on desired samples per rise time:

$$T_s = t_r / N_r, \quad N_r \in [2, 4] \quad (17)$$

Discretization of state space models

Analytical solution for continuous-time state space models

$$x(t_{k+1}) = e^{A(\theta)(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(\theta)(t_{k+1}-\tau)}B(\theta)u(\tau) d\tau, \quad (18a)$$

$$y(t_k) = C(\theta)x(t_k) + D(\theta)u(t_k). \quad (18b)$$

Discrete-time state space models

$$x_{k+1} = A_d(\theta, T_s)x_k + B_d(\theta, T_s)u_k, \quad (19)$$

$$y_k = C(\theta)x_k + D(\theta)u_k \quad (20)$$

Discrete-time matrices

$$A_d(\theta, T_s) = e^{A(\theta)T_s}, \quad B_d(\theta, T_s) = \int_0^{T_s} e^{A(\theta)\tau}B(\theta) d\tau \quad (21)$$

Matrix exponential

$$\begin{bmatrix} A_d(\theta, T_s) & B_d(\theta, T_s) \\ 0 & I \end{bmatrix} = \exp \left(\begin{bmatrix} A(\theta) & B(\theta) \\ 0 & 0 \end{bmatrix} T_s \right) \quad (22)$$

Discretization of transfer function models

Continuous-time transfer function model (frequency domain)

$$y(s) = H(s)u(s), \quad H(s) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} \quad (23)$$

Discretization with Z-transform (use look-up tables)

$$H_z(z) = (1 - z^{-1}) \mathcal{Z} \left(\frac{H(s)}{s} \right), \quad z \in \mathbb{C} \quad (24)$$

Discrete-time transfer function model (frequency domain)

$$y(z) = H_z(z)u(z) = \frac{\bar{b}_0 z^n + \bar{b}_1 z^{n-1} + \dots + \bar{b}_n}{z^n + \bar{a}_1 z^{n-1} + \dots + \bar{a}_n} u(z) \quad (25)$$

Discrete-time transfer function model (time domain) - recall that

$$H_d(q) = H_z(q)$$

$$y_t = H_d(q)u_t = \frac{\bar{b}_0 + \bar{b}_1 q^{-1} + \dots + \bar{b}_n q^{-n}}{1 + \bar{a}_1 q^{-1} + \dots + \bar{a}_n q^{-n}} u_t \quad (26)$$

Discrete-time transfer function model (time domain) - difference equations

$$y_t + \bar{a}_1 y_{t-1} + \dots + \bar{a}_n y_{t-n} = \bar{b}_0 u_t + \bar{b}_1 u_{t-1} + \dots + \bar{b}_n u_{t-n} \quad (27)$$

Transforms, stability, reachability, observability, etc.

Consider the factor terms of transfer functions:

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} = K_0 \frac{\prod_i (s - z_i)}{\prod_i (s - p_i)} \quad (28)$$

$$H_d(q) = \frac{B_d(q^{-1})}{A_d(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \dots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \dots + a_{n_a} q^{-n_a}} = K_{d,0} \frac{\prod_i (q - z_{d,i})}{\prod_i (q - p_{d,i})}$$

Transfer function properties

$$\textbf{Zeros: } H(z_i) = 0, \quad (29)$$

$$\textbf{Poles: } |H(p_i)| = \infty, \quad (30)$$

$$\textbf{DC-gain: } H(s=0), H_z(z=1) = H_d(q=1) \quad (31)$$

Poles of external models = eigenvalues of internal models

$$\mathcal{C}(A) = A(s) \quad (32)$$

Instability criteria

$$\textbf{Continuous: } 0 < \text{Re}(p_c) \quad (33a)$$

$$\textbf{Discrete: } 1 < |p_d| \quad (33b)$$

Poles of discrete- (p_d) and continuous-time (p_c) systems are related

$$p_d = e^{p_c T_s} \quad (34)$$

Number of zeros m and poles n

$$\textbf{Continuous: } m \leq n \quad (35)$$

$$\textbf{Discrete: } \begin{cases} m = n - 1 & (\text{for } D = 0) \\ m = n & \text{otherwise} \end{cases} \quad (36)$$

Zeros of discrete- (p_d) and continuous-time (p_c) systems are related

$$z_d = e^{z_c T_s} \quad (37)$$

Zero-pole cancellation

$$z_i = p_i \quad \Rightarrow \quad H(s) = \frac{s - z_i}{(s - p_i)(s - p_1)} = \frac{1}{(s - p_1)} \quad (38)$$

Transforms - Similarity Transform and Diagonal Transform

Change internal state variables

$$z_t = \Upsilon x_t, \quad (39)$$

$$z_{t+1} = \Upsilon A \Upsilon^{-1} z_t + \Upsilon B u_t, \quad (40)$$

$$y_t = C \Upsilon^{-1} z_t + D u_t \quad (41)$$

The external model is unaffected by the transformation

$$H(q) = C \Upsilon^{-1} (qI - \Upsilon A \Upsilon^{-1})^{-1} \Upsilon B + D = C (qI - A)^{-1} B + D \quad (42)$$

Example: Diagonal transform

$$\Lambda = Q^{-1} A Q = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad (43)$$

The columns of Q are the right eigenvectors of A

Transform external to internal models

External system

$$y_t + a_1 y_{t-1} + \cdots + a_n y_{t-n} = b_0 u_t + b_1 u_{t-1} + \cdots + b_n u_{t-n} \quad (44)$$

Transfer function

$$H(q) = \frac{B(q^{-1})}{A(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} = \sum_{i=0}^{\infty} h_i q^{-i} \quad (45)$$

Minimal representation: An internal model with minimum number of states, e.g., the 4 canonical forms

Controller canonical form

$$A_c = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (46)$$

$$C_c = [b_1 - b_0 a_1, b_2 - b_0 a_2, \dots, b_n - b_0 a_n] \quad D_c = b_0 \quad (47)$$

Observer canonical form

$$A_o = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} \quad B_o = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ \vdots \\ b_n - b_0 a_n \end{bmatrix} \quad (48)$$

$$C_o = [1, 0, \dots, 0] \quad D_o = b_0 \quad (49)$$

Transforms - Canonical forms

Controllability canonical form

$$A_{co} = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & \cdots & 0 & -a_{n-1} \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & -a_1 \end{bmatrix} \quad B_{co} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (50)$$

$$C_{co} = (h_1, h_2, \dots, h_n) \quad D_{co} = h_0 \quad (51)$$

Observability canonical form

$$A_{ob} = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \quad B_{ob} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \quad (52)$$

$$C_{ob} = (1, 0, \dots, 0) \quad D_{ob} = h_0 \quad (53)$$

Relations between canonical forms

$$A_c = A_o^T, \quad A_{co} = A_{ob}^T, \quad (54)$$

$$B_c = C_o^T, \quad B_{co} = C_{ob}^T, \quad (55)$$

$$B_o = C_c^T, \quad B_{ob} = C_{co}^T, \quad (56)$$

$$D_c = D_o = D_{co} = D_{ob} = b_0 = h_0 \quad (57)$$

Transforms - Direct realization

General external model

$$y_t + a_1 y_{t-1} + \cdots + a_{n_a} y_{t-n_a} = b_0 u_t + b_1 u_{t-1} + \cdots + b_{n_b} u_{t-n_b} \quad (58)$$

State

$$x_t = \begin{bmatrix} -y_{t-1} & \cdots & -y_{t-n_a} & u_{t-1} & \cdots & u_{t-n_b} \end{bmatrix} \quad (59)$$

Non-minimal internal model

$$A_d = \begin{bmatrix} -a_1 & \cdots & -a_{n_a-1} & -a_{n_a} & -b_1 & \cdots & -b_{n_b-1} & -b_{n_b} \\ 1 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & & 1 & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} -b_0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (60)$$

$$C_d = (a_1, \dots, a_{n_a}, b_1, \dots, b_{n_b}), \quad D_d = b_0 \quad (61)$$

Definition

A system is said to be controllable, if it is possible to move the system from an arbitrary state value to the origin in finite time.

Definition

A system is said to be reachable, if it is possible to move the system from one arbitrary state value to another arbitrary state in finite time.

Reachability $\Rightarrow$ controllability, not the reverse

An n -state system is reachable if and only if the reachability matrix W_c has full rank ($k \geq n$)

$$W_c(k) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{k-1}B \end{bmatrix} \quad (62)$$

The reachability Gramian is given by $\Sigma_k^c = W_c(k)W_c^T(k)$

k -step input sequence (not unique)

$$x_k = A^k x_0 + W_c(k) U_{k-1}, \quad (63)$$

$$U_{k-1} = \begin{bmatrix} u_{k-1} & u_{k-2} & \cdots & u_0 \end{bmatrix}^T \quad (64)$$

brings the system from any x_0 to a desired state, $\bar{x}$

Sequence with minimal control usage

$$\min_{u_{k-1}, \dots, u_0} \sum_{j=0}^{k-1} u_j^T u_j \quad (65a)$$

Solution

$$U_{k-1}^* = W_c^T(k) (\Sigma_k^c)^{-1} [\bar{x} - A^k x_0] \quad (66)$$

Continuous-time system

$$\dot{x} = Ax + Bu, \quad (67)$$

Reachability Gramian

$$\dot{\Sigma}^c = A\Sigma^c + \Sigma^c A^T + BB^T \quad (68a)$$

$$\Sigma^c(t_0) = 0. \quad (68b)$$

The system is reachable if Σ^c is invertible for any $t \geq t_0$

Note: For continuous-time systems, reachability $\Leftrightarrow$ controllability

Observability and constructability

Definition:

A system is observable if any initial state can be estimated using only the information from the following outputs and inputs.

Definition:

A system is constructable if, for any possible evolution of the state and control variables, the current state can be estimated using only the information from outputs.

Observability $\Rightarrow$ constructability, but the reverse is not true

An n -state system is observable if and only if the observability matrix W_o has full rank ($k \geq n$)

$$W_o(k) = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \dots & (CA^{k-1})^T \end{bmatrix}^T \quad (69)$$

Observability Gramian: $\Sigma_k^o = W_o(k)W_o^T(k)$

Continuous-time system

$$\dot{x} = Ax + Bu, \quad (70a)$$

$$y = Cx + Du \quad (70b)$$

Observability Gramian

$$\dot{\Sigma}^o = A\Sigma^o + \Sigma^o A^T + C^T C \quad (71a)$$

$$\Sigma^o(t_0) = 0. \quad (71b)$$

The system is observable if Σ^o is invertible for any $t \geq t_0$

Several definitions of stability exist: e.g., marginal and asymptotic stability

Consider a steady state x_s of the system

- **Marginally stable:** x_s is said to be (marginally) stable if any solution trajectory $\{x(t), t \in [t_0, \infty]\}$ is bounded.
- **Asymptotically stable:** x_s is said to be asymptotically stable if any solution trajectory converges to x_s ($x(t) \rightarrow x_s$) as time progresses ($t \rightarrow \infty$).

A system which is not stable (i.e., not marginally stable) is unstable

A system is BIBO stable if the output is bounded for any bounded input

Note: Asymptotic stability $\Rightarrow$ BIBO stability

A state space model is stable if and only if all of the following requirements are fulfilled

Continuous-time

Marginally stable:

- $\text{Re}\{\text{eig}(A)\} \leq 0$
- $\forall \text{Re}\{\text{eig}(A)_i\} = 0$, the AM=GM

Asymptotically stable:

- $\text{Re}\{\text{eig}(A)\} < 0$

Discrete-time

- $|\text{eig}(A)| \leq 1$
- $\forall |\text{eig}(A)_i| = 1$, the AM=GM

- $|\text{eig}(A)| < 1$

*AM = Algebraic multiplicity (# of identical eigenvalues)

**GM = geometric multiplicity (# of associated eigenvectors)

Steady state of nonlinear system

$$\dot{x} = f(x_s, u_s) = 0 \qquad x_s = f(x_s, u_s), \qquad (72)$$

Approximate behavior around steady state using linearization

$$A = \frac{\partial f}{\partial x}(x_s, u_s). \qquad (73)$$

The system is locally stable (marginal or asymptotic) around the steady state if the requirements on the previous slide are fulfilled

Questions?