Stochastic Adaptive Control (02421)

Lecture 6

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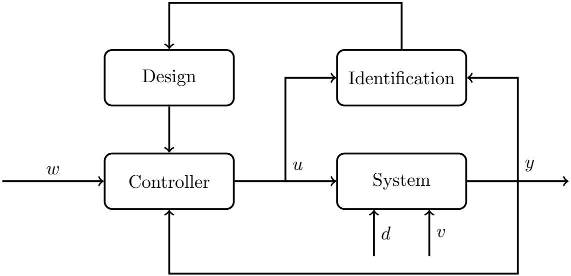
Lecture Plan

1. ARX models
2. ARX estimation
3. ARMAX estimation
4. ARMAX identification
5. ARMAX control
6. ARMAX adaptive control
7. Systems and control theory
8. Stochastic systems + SS estimation
9. SS estimation + control
10. SS control
11. To be decided ...
12. SS nonlinear estimation
13. SS nonlinear control

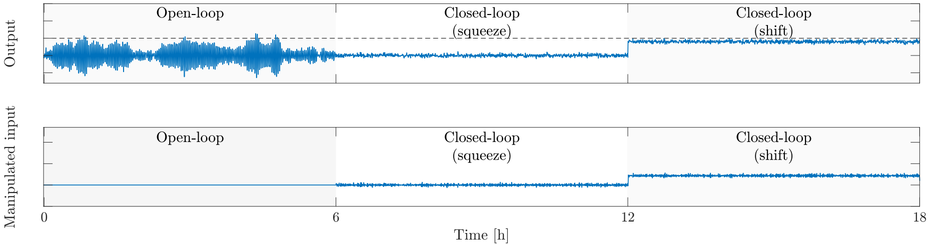


Today's Agenda

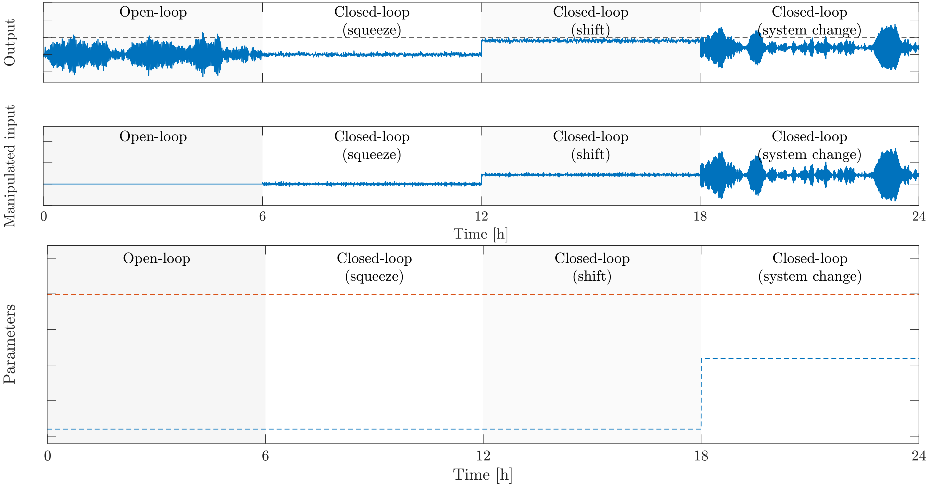
- Adaptive control – general introduction
- Explicit self-tuner
- Example system
- Cautious controllers
- Dual adaptive controllers
- Sub-optimal dual adaptive controllers

Adaptive control

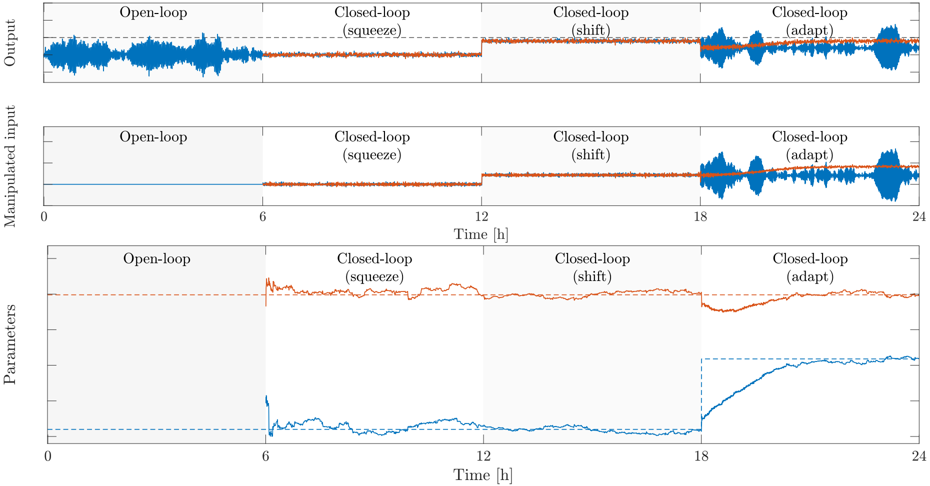
Squeeze and shift principle



Motivation for adaptive control



Motivation for adaptive control



Controller test quantities (sanity checks)

Test of tracking error (against stationary distribution)

$$J_{r,t} = \sum_{j=1}^{t} (y_j - w_j)^2, \qquad J_{r,t} \approx \mathbb{E}[(y-w)^2]t \tag{1}$$

Test of manipulated inputs (against stationary distribution)

$$J_{u,t} = \sum_{j=1}^{t} u_j^2, \qquad J_{u,t} \approx \mathbb{E}[u^2]t \tag{2}$$

Test of residuals (model accuracy)

$$J_{e,t} = \sum_{j=1}^{t} \epsilon_j^2, \qquad J_{e,t} \approx \sigma^2 t \tag{3}$$

Adaptive Control

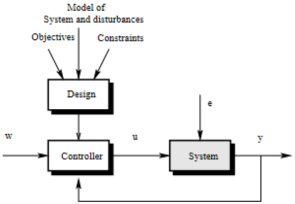
Stochastic control relies on a detailed model which might not be available

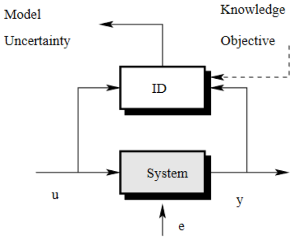
1. Parameter values cannot be measured
2. The underlying physics is not understood sufficiently well

Approach 1:
A model can be created using identification methods and a stochastic controller can be designed

If the system varies in time, e.g., due to aging or wear, the identification will have to be repeated occasionally

Approach 2:
Alternatively, we can combine online identification and control.





Alternative to adaptive control: Robust control

If the model of a system is uncertain, there also exist other methods than the adaptive control. One such method is the robust control

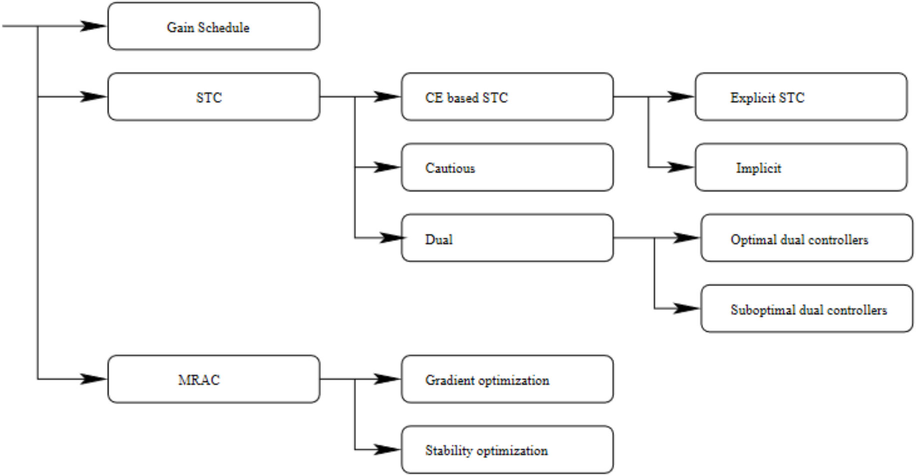
1. Robust control: Low sensitivity to the effect of uncertain system parts, a control that, in some sense, operates after worst-case scenario
2. Adaptive control: Monitors/estimates the uncertain parts, a control law that changes with the identified system.

In some sense, robust control can be seen as the opposite method to adaptive control: Adapting the control usage (sensitivity) vs. adapting the control design

That is the subject of the course 34746 Robust and fault-tolerant control

Adaptive control - Method Overview

Several schools exist within adaptive control

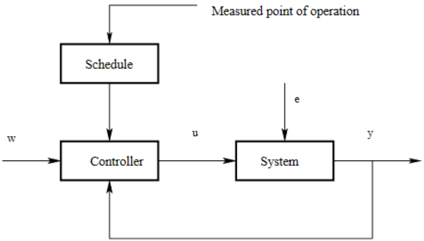


Adaptive control - Gain Scheduling Control

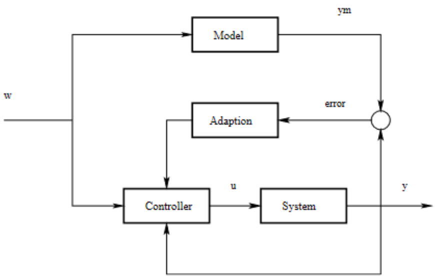
A simple approach is to manually change the model based on the operating point

1. Linear control of non-linear system: Airplanes/robots
2. Piecewise systems: Laws for behaviour at night vs. day

Adaptation is manual, so no performance feedback to the adaptations



Adaptive control - Model Reference Adaptive Control (MRAC)



Another approach is to adapt the control until the output follows a desired transfer function with the least possible deviation

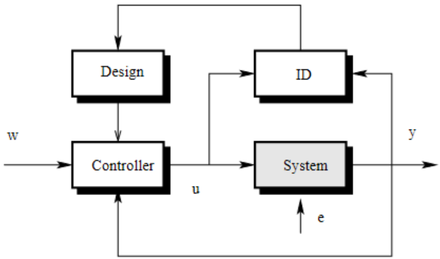
The focus is on the control problem and the adaptation is feedback on the model deviations

The concept is similarly to that of an observer/Kalman filter

Adaptive control - Self-tuning Control

The self-tuning methods are based on the combination of an identification algorithm, a design method, and a controller

It is further assumed that the certainty equivalence principle holds



Adaptive control - Certainty Equivalence Principle

Certainty equivalence principle: Replace true parameters by an estimate

$$\theta \to \hat{\theta} \tag{4}$$

For linear systems with additive noise, the principle holds

$$u_t = -Lx_t \quad \to \quad u_t = -L\hat{x}_t \tag{5}$$

In adaptive control, the principle is an assumption (minimum variance control example)

$$C(q^{-1}) = AG + q^{-k}S \qquad \to \qquad \hat{C} = \hat{A}G + q^{-k}S \tag{6}$$

$$BGu_t = -Sy_t \qquad \to \qquad \hat{B}Gu_t = -Sy_t \tag{7}$$

The principle does not guarantee optimality – it is assumed for convenience

Explicit self-tuner

Basic Self-tuning

Let us now discuss the self-tuning methods in terms of the minimum variance controller, the so-called basic self-tuning controller

We combine a recursive estimation approach for

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t, \tag{8}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}, \quad b_0 \neq 0, \tag{9}$$

$$e_t \sim \mathcal{F}(0, \sigma^2) \text{ and white} \tag{10}$$

with the design of the minimum variance controller for the objective

$$J = \mathbb{E}[y_{t+k}^2] \tag{11}$$

$$u_t = \text{func}(Y_t) \tag{12}$$

Self-tuning methods come in two variants: Explicit and implicit

1. Explicit: Estimation of model used to design the control
2. Implicit: Estimation of the controller parameters + $C(q^{-1})$

The Basic Self tuner - Explicit

In the explicit method, we are interested in identifying the model

$$\hat{A}(q^{-1})y_t = q^{-k}\hat{B}(q^{-1})u_t + \hat{C}(q^{-1})\epsilon_t \tag{13}$$

to use in the control

We do this using a chosen estimation method

$$y_t = \phi_t^T \theta_{t-1} + e_t \tag{14}$$

$$\hat{\theta}_t = \arg\min \sum_{i=1}^{t} \epsilon_i^2 \tag{15}$$

Using the estimate, we compute the control as

$$u_t = \arg\min \mathbb{E}[y_{t+k}^2] \tag{16}$$

and we repeat at the next sampling time.

Explicit MV_0 Control

Identified system (general method)

$$\hat{A}(q^{-1})y_t = q^{-k}\hat{B}(q^{-1})u_t + \hat{C}(q^{-1})\epsilon_t \tag{17}$$

Controller optimality criteria:

$$J = \mathbb{E}[(y_{t+k} - w_t)^2] \tag{18}$$

Controller design:

$$\hat{B}(q^{-1})G(q^{-1})u_t = \hat{C}(q^{-1})w_t - S(q^{-1})y_t, \tag{19}$$

$$\hat{C}(q^{-1}) = \hat{A}(q^{-1})G(q^{-1}) + q^{-k}S(q^{-1}) \tag{20}$$

QRS form:

$$Q(q^{-1}) = \hat{C}(q^{-1}), \quad R(q^{-1}) = \hat{B}(q^{-1})G(q^{-1}) \tag{21}$$

Example system

Known Systems and Control

ARX model with 1-step delay

$$A(q^{-1})y_t = B(q^{-1})u_{t-1} + e_t, \qquad e_t \sim N(0, \sigma^2) \tag{22}$$

Cost function

$$J = \mathbb{E}\left[\sum_{i=1}^{N}(y_{t+i} - w_{t+i})^2\right] \tag{23}$$

Example: MV_0 controller $(N = 1)$

$$u_{t-1} = \frac{1}{B}w_t - \frac{S}{B}y_{t-1} = \frac{1}{B}w_t - \frac{q(1-A)}{B}y_{t-1} \tag{24}$$

$$y_t = w_t + e_t \tag{25}$$

Known Systems and Control

Alternative form: System

$$y_t = \phi_t^T \theta + e_t = b_0 u_{t-1} + \varphi_t^T \vartheta + e_t \tag{26}$$

$$\phi_t^T = (-y_{t-1}, -y_{t-2}, \dots, u_{t-1}, u_{t-2}, \dots), \quad \theta^T = (a_1, a_2, \dots, b_0, b_1, \dots) \tag{27}$$

$$\varphi_t^T = (-y_{t-1}, -y_{t-2}, \dots, 0, u_{t-2}, \dots), \quad \vartheta^T = (a_1, a_2, \dots, 0, b_1, \dots) \tag{28}$$

Alternative form: Controller

$$u_{t-1} = \frac{1}{b_0} w_t - \frac{\varphi_t^T \vartheta}{b_0} \tag{29}$$

$$y_t = w_t + e_t \tag{30}$$

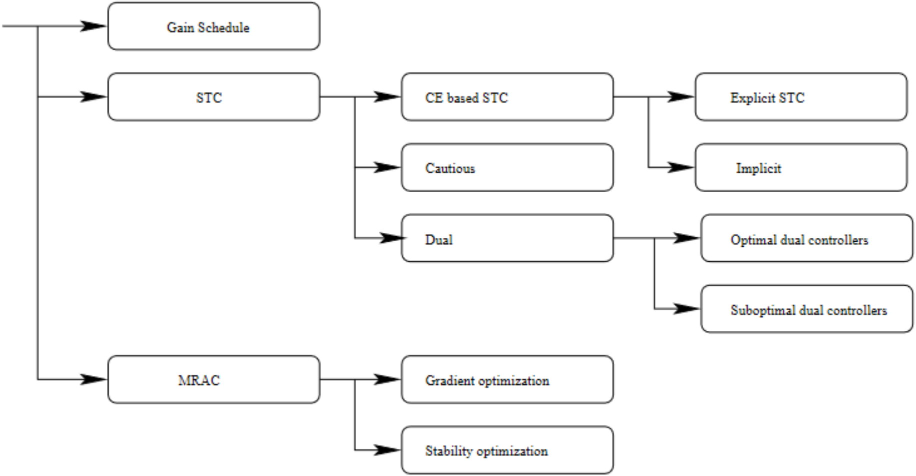
Relation between ϕ and φ

$$\varphi = \phi - \mathrm{diag}(l)\phi, \qquad \vartheta = \theta - \mathrm{diag}(l)\theta \tag{31}$$

$$l^T = (0, 0, \dots, 1, 0, \dots) \tag{32}$$

The nonzero entry corresponds to the position of b_0 and u_{t-1}

Adaptive control - Method Overview



Adaptive Control - CE Control (Explicit)

Certainty equivalent self-tuner: Use the parameter estimate as the true parameters

$$y_t = \phi_t^T \theta + \epsilon_t, \quad \theta \to \hat{\theta} \tag{33}$$

Update parameter estimate

$$\epsilon_t = y_t - \phi_t^T \hat{\theta}_{t-1}, \tag{34}$$

$$s_t = 1 + \phi_t^T P_{t-1} \phi_t, \tag{35}$$

$$K_t = \frac{P_{t-1}\phi_t}{s_t}, \tag{36}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t \epsilon_t, \tag{37}$$

$$P_t = P_{t-1} - K_t s_t K_t^T \tag{38}$$

Redesign the control law

$$u_t = \frac{1}{\hat{b}_0} w_t - \frac{\varphi_t^T \hat{\vartheta}}{\hat{b}_0} \tag{39}$$

Cautious controllers

Adaptive Control - Cautious Control

Cautious adaptive control: Take estimation uncertainty into account

Conditional cost function

$$
\begin{aligned}
J &= \mathbb{E}[(y_{t+1} - w_{t+1})^2 | Y_t] && (40)\\
&= (\mathbb{E}[y_{t+1} - w_{t+1} | Y_t])^2 + \mathrm{Var}(y_{t+1} - w_{t+1} | Y_t) && (41)
\end{aligned}
$$

Uncertainty of parameter estimate

$$
\begin{aligned}
\hat{\theta}_t &\sim N(\theta, P_t) && (42)\\
\hat{b}_{0,t} &= l^T \hat{\theta}_t && (43)\\
p_{b,t} &= l^T P_t l && (44)
\end{aligned}
$$

Control law

$$
u_t = \frac{\hat{b}_{0,t}^2}{\hat{b}_{0,t}^2 + p_{b,t}} \left(\frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} - \frac{\varphi_t^T P_t l}{\hat{b}_{0,t}^2} \right) \qquad (45)
$$

Adaptive Control - Cautious Control

If $P_t \to 0$, cautious control is equivalent to certainty equivalent control

$$u_t = \frac{\hat{b}_{0,t}^2}{\hat{b}_{0,t}^2 + p_{b,t}} \left(\frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} - \frac{\varphi_t^T P_t l}{\hat{b}_{0,t}^2} \right) \quad \to \quad u_t = \frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} \tag{46}$$

If $\hat{\theta}_t \to \theta$, certainty equivalent control is equivalent to the known control

1. $P_t \to 0 :$ Cautious $=$ CE $\neq$ known
2. $\hat{\theta}_t \to \theta :$ Cautious $\neq$ CE $=$ known
3. $P_t \to 0,\ \hat{\theta}_t \to \theta :$ Cautious $=$ CE $=$ known

Adaptive Control - Cautious Control - Usage

Cautious controller

$$u_t = \frac{\hat{b}_{0,t}^2}{\hat{b}_{0,t}^2 + p_{b,t}} \left(\frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} - \frac{\varphi_t^T P_t l}{\hat{b}_{0,t}^2} \right) \tag{47}$$

- *Turn-off phenomenon:* Control is dampened due to high uncertainty of b_0
- Consequence: Less information about b_0 for the next estimate, i.e., the uncertainty increases
- Turn-off usually occurs if b_0 or the control signal is small
- Conclusion: The cautious controller is useful for systems with constant or almost constant parameters, but unsuitable for general time-varying systems

Dual adaptive controllers

Adaptive Control - Optimal Dual Control

Dual control: Conditional expectation of the cost

$$J = \min_{U_t} \mathbb{E}\left[\sum_{i=1}^{N} (y_{t+i} - w_{t+i})^2\right] = \mathbb{E}_{Y_t}\left[\min_{U_t} \mathbb{E}\left[\sum_{i=1}^{N} (y_{t+i} - w_{t+i})^2 \middle| Y_t\right]\right] \tag{48}$$

If the parameter uncertainty is Gaussian, the conditional expectation is Gaussian (even if y_t is not)

$$\xi_t = [\varphi_{t-1}, \quad \hat{\theta}_t, \quad P_t] \tag{49}$$

contains the necessary information

If not Gaussian, it becomes computationally challenging to compute the hyper space and storage requirements increase

Adaptive Control - Optimal Dual Control

Bellman equation

$$V(\xi_t, t) = \min_{u_{t-1}} \mathbb{E}[(y_t - w_t)^2 + V(\xi_{t+1}, t) | Y_{t-1}] \tag{50}$$

The last (N'th) step is identical to the cautious controller

$$V(\xi_N, N) = \min_{u_{N-1}} \mathbb{E}[(y_N - w_N)^2 | Y_{N-1}] \tag{51}$$

$$= (\varphi_{N-1}^T \theta_N - w_N)^2 + \sigma^2 + \varphi_{N-1}^T P_N \varphi_{N-1} - \frac{\hat{b}_{0,N} w_N - \varphi_{N-1}^T (\hat{b}_{0,N} \hat{\theta}_N + P_N l)}{\hat{b}_{0,N}^2 - p_{b,N}} \tag{52}$$

substituting into $V(\xi_{N-1}, N-1)$, the second last control can be computed, and so on

This is similar to the LQR – however, it does not have an analytical solution and must be solved numerically

Adaptive Control - Optimal Dual control

Fundamental paradox of adaptive control

1. Control objective: Small signals (control action)
2. Estimation: Large signals (probing action)

For the optimal N-step dual control problem, the solution is a compromise between these goals

1. Improved long-term estimation accuracy; sacrificing short-term loss
2. Probing adds active learning to the method

Cautious control ($N = 1$): The probing effects diminishes and any learning is "accidental"

Issue with dual control: Curse of dimensionality – the computational cost increases drastically with increasing hyperspace dimension and horizon

Sub-optimal dual adaptive controllers

Adaptive Control - Sub-Optimal Dual control

As optimal dual control is impractical, sub-optimal dual controllers exist. They are based on the cautious controller and fix the issue with turn-off

Various approaches

1. Constrain the uncertainty
2. Extend the loss function
3. Serial expansion of the loss function
4. Add perturbation signals to the control

Sub-Optimal Dual Control - Constrained One-step Controller

Constrained one-step controller (minimum distance to zero control)

$$u_t = \begin{cases} u_c & \text{if } |u_c| \geq |u_l| \\ u_l \operatorname{sign}(u_c) & \text{if } |u_c| < |u_l| \end{cases} \tag{53}$$

u_c is the cautious controller input and u_l is a lower limit determined by us

The constraints do not prevent turn-off, but add extra perturbation when it happens

Sub-Optimal Dual Control - Constrained Uncertainty

Alternatively, constrain the uncertainty

$$\mathrm{Tr}(P_{t+1}^{-1}) \geq M \tag{54}$$

or constrain only p_b

$$p_{b,t+1} \leq \begin{cases} \gamma \hat{b}_{0,t+1}^2 & \text{if } p_{b,t} \leq \hat{b}_{0,t}^2 \\ \alpha p_{b,t} & \text{otherwise} \end{cases} \tag{55}$$

Sub-Optimal Dual control - extended loss function

Add uncertainty to the cost function

$$J = \mathbb{E}[(y_{t+1} - w_{t+1})^2 + \rho f(P_{t+1})] \tag{56}$$

f can be formulated in many ways

1. $f(P_{t+1}) = p_{b,t+1}$
2. $f(P_{t+1}) = R_2 \frac{p_{b,t+1}}{p_{b,t}}$
3. $f(P_{t+1}) = -\frac{\det(P_t)}{\det(P_{t+1})}$
4. $f(P_{t+1}) = -\epsilon_{t+1}^2$

This might lead to multiple local minima, and numerical optimization is required. Alternatively, use a second order serial expansion (e.g., a Taylor expansion)

Sub-Optimal Dual control - Extended Loss Function - Example

Third version of f

$$J = \mathbb{E}\left[(y_{t+1} - w_{t+1})^2 - \rho \frac{\det(P_t)}{\det(P_{t+1})} \middle| Y_t \right] \tag{57}$$

Ratio between determinants

$$\frac{\det(P_t)}{\det(P_{t+1})} = 1 + \phi_{t+1}^T P_t \phi_{t+1} \tag{58}$$

Analytical control law

$$u_t = \frac{\hat{b}_0(w_{t+1} - \varphi_{t+1}^T \hat{\vartheta}_t) + \rho (P_t l)^T \varphi_{t+1}}{\hat{b}_0^2 - \rho p_{b,t}} \tag{59}$$

Depending on ρ, we get specific controllers

1. $\rho = 0$: the CE controller
2. $\rho = -1$: the cautious controller
3. $\rho > 0$: an active learning controller

Sub-Optimal Dual control - Probing

Add probing/perturbation signal

$$u_t = u_t^c + u_t^x \tag{60}$$

Possible probing/perturbation signals include

1. PRBS
2. DOX: Design of excitation signal

They can be applied both at certain points in time (low uncertainty) or constantly

Questions?