Stochastic Adaptive Control (02421)

Lecture 5

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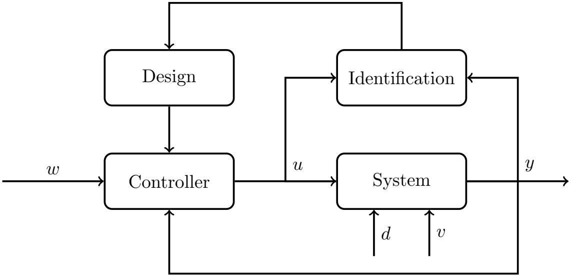
Lecture Plan

1. ARX models
2. ARX estimation
3. ARMAX estimation
4. ARMAX identification
5. ARMAX control
6. ARMAX adaptive control
7. Systems and control theory
8. Stochastic systems + SS estimation
9. SS estimation + control
10. SS control
11. To be decided . . .
12. SS nonlinear estimation
13. SS nonlinear control



Today's Agenda

- Diophantine equations
- Prediction
- What is stochastic control?
- Minimum variance control
- Pole-zero control
- Generalized minimum variance (GMV) control
- Closed-loop simulation

Diophantine equations

Polynomials and Transfer functions

Polynomials

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_n q^{-n} \tag{1}$$

The polynomial is order n if $b_n \neq 0$ and $b_i = 0$ for $i > n$

If $b_0 = 1$, the polynomial is *monic*

A transfer function $H(q)$ can be written in infinitely many ways

$$H(q) = \frac{B(q^{-1})}{A(q^{-1})} = \frac{C(q^{-1})B(q^{-1})}{C(q^{-1})A(q^{-1})} \tag{2}$$

Polynomials and Transfer functions

$$\frac{B(q^{-1})}{A(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} \tag{3}$$

$$= b_0 - b_0 + \frac{b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} \tag{4}$$

$$= b_0 - \frac{b_0(1 + a_1 q^{-1} + \cdots + a_n q^{-n})}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} + \frac{b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} \tag{5}$$

$$= b_0 + q^{-1} \frac{(b_1 - b_0 a_1) + (b_2 - b_0 a_2) q^{-1} + \cdots + (b_n - b_0 a_n) q^{-(n-1)}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} \tag{6}$$

Define the transfer function

$$H(q) = \frac{B(q^{-1})}{A(q^{-1})} = g_0 + q^{-1} \frac{S_1(q^{-1})}{A(q^{-1})}, \tag{7}$$

$$S_1(q^{-1}) = s_0 + s_1 q^{-1} + \ldots + s_{n_1} q^{-n_1}, \tag{8}$$

$$g_0 = b_0, \quad s_i = b_{i+1} - b_0 a_{i+1} \tag{9}$$

$n_1 = n - 1$ is the order of S_1

Polynomials and Transfer functions

Repeat the rewriting for $\frac{S_1}{A}$, $\frac{S_2}{A}$, etc.

$$H(q) = \frac{B(q)}{A(q)} = g_0 + g_1 q^{-1} + \cdots + g_{m-1} q^{-(m-1)} + q^{-m} \frac{S_m(q^{-1})}{A(q^{-1})} \quad (10)$$

$$= G_m(q^{-1}) + q^{-m} \frac{S_m(q^{-1})}{A(q^{-1})} \quad (11)$$

Diophantine equation

$$B(q^{-1}) = A(q^{-1}) G_m(q^{-1}) + q^{-m} S_m(q^{-1}) \quad (12)$$

The order of S_m is $\max(n_a - 1, n_b - m)$ and the order of G_m is $m - 1$

Solving simple Diophantine equations

This (simple) Diophantine equation can be solved iteratively

```
% Initialize
G = [];
S = [B, 0]; % Pad B with zeros to make S as long as A

for i = 1:m
    % Augment with first element of S
    G = [G, S(1)];

    % Update S
    S = [S(2:end) - S(1)*A(2:end), 0];
end

% Remove last element
S = S(1:end-1);
```

Prediction

Prediction in the ARMA Structure

ARMA process

$$A(q^{-1})y_t = C(q^{-1})e_t \tag{13}$$

Diophantine equation

$$C(q^{-1}) = A(q^{-1})G_m(q^{-1}) + q^{-m}S_m(q^{-1}) \tag{14}$$

m-step prediction based on the solution to the Diophantine equation

$$y_{t+m} = \frac{C(q^{-1})}{A(q^{-1})}e_{t+m} = G_m(q^{-1})e_{t+m} + \frac{S_m(q^{-1})}{A(q^{-1})}e_t \tag{15}$$

Prediction and error

$$\hat{y}_{t+m|t} = \frac{S_m(q^{-1})}{A(q^{-1})}e_t = \frac{S_m(q^{-1})}{A(q^{-1})}\left(\frac{A(q^{-1})}{C(q^{-1})}y_t\right) = \frac{S_m(q^{-1})}{C(q^{-1})}y_t, \tag{16}$$

$$\tilde{y}_{t+m|t} = G_m(q^{-1})e_{t+m} \tag{17}$$

$\hat{y}_t$ and $\tilde{y}_t$ are independent

This approach requires that $C(q^{-1})$ is inversely stable

Prediction in the ARMAX structure

System

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{18}$$

k is the control delay

m-step prediction

$$\hat{y}_{t+m|t} = \frac{1}{C(q^{-1})}(B(q^{-1})G_m(q^{-1})u_{t+m-k} + G_m(1)d + S_m(q^{-1})y_t), \tag{19}$$

$$\tilde{y}_{t+m|t} = G_m(q^{-1})e_{t+m} \tag{20}$$

Diophantine equation

$$C(q^{-1}) = A(q^{-1})G_m(q^{-1}) + q^{-m}S_m(q^{-1}) \tag{21}$$

The order of G_m and S_m are $m-1$ and $\max(n_a-1, n_c-m)$ and $G_m(0)=1$

Proof of ARMAX prediction

Rewrite future output using the Diophantine equation

$$y_{t+m} = \frac{C(q^{-1})}{C(q^{-1})} y_{t+m} \tag{22}$$

$$= \frac{A(q^{-1})G_m(q^{-1}) + q^{-m}S_m(q^{-1})}{C(q^{-1})} y_{t+m} \tag{23}$$

$$= \frac{G_m(q^{-1})}{C(q^{-1})} A(q^{-1}) y_{t+m} + \frac{S_m(q^{-1})}{C(q^{-1})} y_t \tag{24}$$

Substitute system description

$$y_{t+m} = \frac{G_m(q^{-1})}{C(q^{-1})} (B(q^{-1})u_{t+m-k} + C(q^{-1})e_{t+m} + d) + \frac{S_m(q^{-1})}{C(q^{-1})} y_t \tag{25}$$

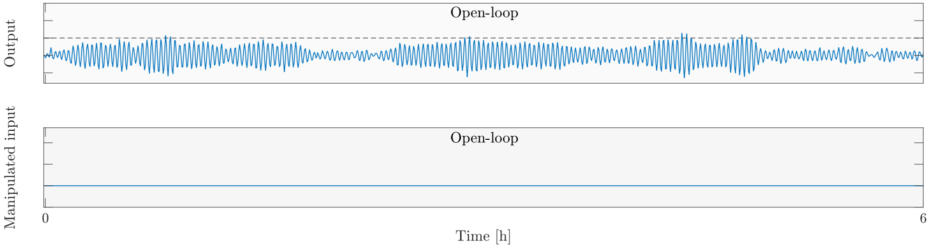
$$= \frac{1}{C(q^{-1})} \left(G_m(q^{-1})B(q^{-1})u_{t+m-k} + G_m(1)d + S_m(q^{-1})y_t \right)$$

$$+ G_m(q^{-1})e_{t+m} \tag{26}$$

$$= \hat{y}_{t+m|t} + \tilde{y}_{t+m|t} \tag{27}$$

Control objectives

Control objectives



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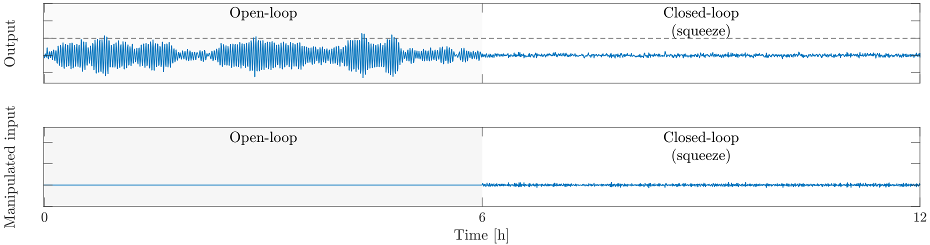

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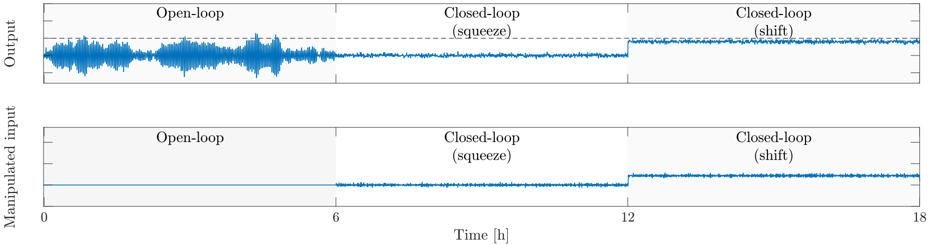

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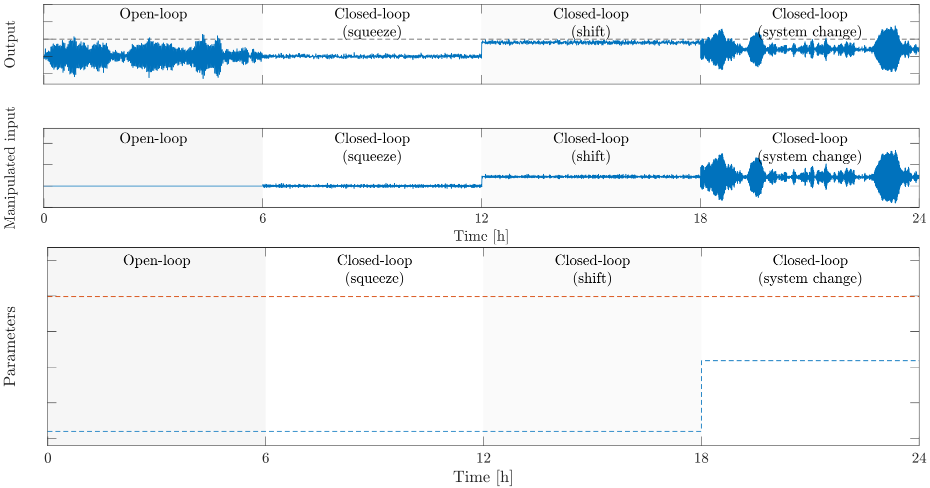
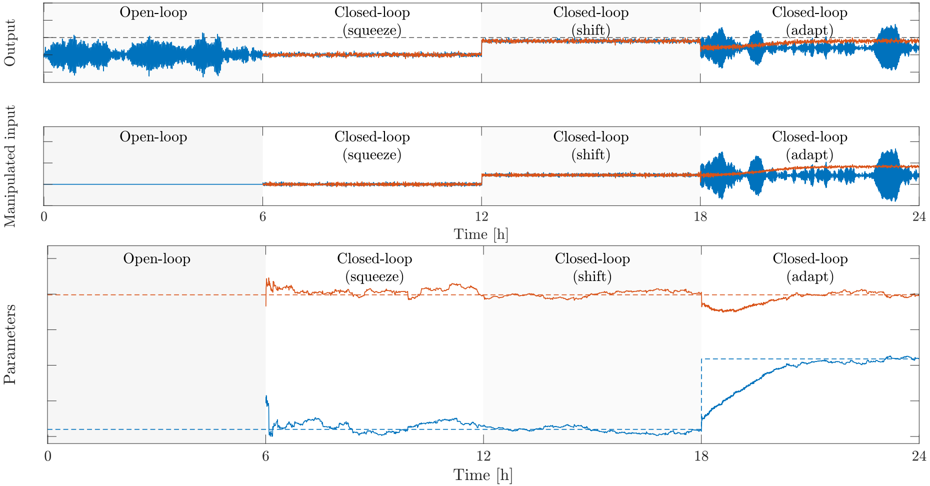


Control objectives



Control objectives

We are trying to achieve all of the following objectives

1. Reduce the variance of the output, y, under stationary conditions
2. Ensure a proper transient response to changes in the setpoint, w
3. Ensure that manipulated input, u, satisfies operational constraints (amplitude, rate of change)

Remarks on testing

- Typically, we will test the stationary variance for $w = 0$ and $d = 0$
  - Test in simulation
  - Analyze the stationary state mathematically
- Typically, we will test the transient response without stochasticity, $\sigma^2 = 0$
  - Test in simulation
  - Use a periodic setpoint, e.g., a square wave

Control of ARMAX processes

ARMAX process

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{28}$$

General control law ($\bar{d}$ depends on d)

$$R(q^{-1})u_t = Q(q^{-1})w_t - S(q^{-1})y_t + \bar{d} \tag{29}$$

Criterion used to derive optimal control laws

$$\min_{u_t} J_t(y_{t+k}, u_t) \tag{30}$$

Controller test quantities (sanity checks)

Test of tracking error

$$J_{r,t} = \sum_{j=1}^{t} (y_j - w_j)^2 \tag{31}$$

Tracking error target (based on stationary distribution of $y \sim N(\bar{y}, P_y)$)

$$J_{r,t} \approx \mathbb{E}[(y - w)^2]t \tag{32}$$

Test of manipulated inputs

$$J_{u,t} = \sum_{j=1}^{t} u_j^2 \tag{33}$$

Manipulated input target (based on stationary distribution of $u \sim N(\bar{u}, P_u)$)

$$J_{u,t} \approx \mathbb{E}[u^2]t \tag{34}$$

Minimum variance control

Minimum Variance Control

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t \tag{35}$$

B is stable

Minimize the variance

$$J_t = \mathbb{E}[y_{t+k}^2] \tag{36}$$

$G_k(q^{-1})$ and $S_k(q^{-1})$ are the solution to the simple Diophantine equation

$$C(q^{-1}) = A(q^{-1})G_k(q^{-1}) + q^{-k}S_k(q^{-1}) \tag{37}$$

Minimum Variance Control

m-step prediction and error

$$\hat{y}_{t+m} = \frac{1}{C(q^{-1})}\left(B(q^{-1})G_m(q^{-1})u_{t+m-k} + S_m(q^{-1}y_t)\right), \tag{38}$$

$$\tilde{y}_{t+m} = G_m(q^{-1})e_{t+m} \tag{39}$$

Cost function (we exploit that $\hat{y}_{t+m} \perp \tilde{y}_{t+m}$)

$$J_t = \mathbb{E}[y_{t+k}^2] = \mathbb{E}\left[\left(\frac{1}{C(q^{-1})}\left(B(q^{-1})G_k(q^{-1})u_t + S_k(q^{-1})y_t\right)\right)^2\right] \tag{40}$$

$$+\,\mathbb{E}\left[\left(G_k(q^{-1})e_{t+k}\right)^2\right] \tag{41}$$

Optimal control law

$$B(q^{-1})G_k(q^{-1})u_t = -S_k(q^{-1})y_t \tag{42}$$

Minimum Variance Control – Closed-loop analysis

Closed-loop system

$$
\begin{aligned}
A(q^{-1})y_t &= q^{-k}B(q^{-1})u_t + C(q^{-1})e_t && (43)\\
&= -q^{-k}B(q^{-1})\frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})}y_t + C(q^{-1})e_t && (44)\\
&= -q^{-k}\frac{S_k(q^{-1})}{G_k(q^{-1})}y_t + C(q^{-1})e_t && (45)
\end{aligned}
$$

Simplify

$$
\begin{aligned}
\overbrace{\left(G_k(q^{-1})A(q^{-1}) + q^{-k}S_k(q^{-1})\right)}^{=C(q^{-1})\ \text{(Diophantine equation)}} y_t &= G_k(q^{-1})C(q^{-1})e_t && (46)\\
C(q^{-1})y_t &= G_k(q^{-1})C(q^{-1})e_t && (47)\\
y_t &= G_k(q^{-1})e_t && (48)
\end{aligned}
$$

Control law

$$
u_t = \frac{-S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})}y_t = \frac{-S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})}G_k(q^{-1})e_t = \frac{-S_k(q^{-1})}{B(q^{-1})}e_t \qquad (49)
$$

Minimum Variance Control – Closed-loop analysis

Closed-loop system

$$y_t = G_k(q^{-1})e_t, \qquad B(q^{-1})u_t = -S_k(q^{-1})e_t \tag{50}$$

The closed loop poles are determined by $B(q^{-1})$

The minimum variance controller has the following shortcomings

1. No possibility for setpoints
2. Large control effort
3. Undamped zeros (zeros outside of the unit circle)

$\mathbf{MV}_0$ controller

MV_0 controller

ARMAX process

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{51}$$

Cost function

$$J_t = \mathbb{E}[(y_{t+k} - w_t)^2] \tag{52}$$

Optimal control law

$$B(q^{-1})G_k(q^{-1})u_t = C(q^{-1})w_t - S_k(q^{-1})y_t - G_k(q^{-1})d \tag{53}$$

The Diophantine equation is the same as for the minimum variance controller

$$C(q^{-1}) = A(q^{-1})G_k(q^{-1}) + q^{-k}S_k(q^{-1}) \tag{54}$$

MV_0 controller – Closed-loop analysis

Closed-loop system

$$y_t = q^{-k} w_t + G_k(q^{-1}) e_t, \tag{55}$$

$$B(q^{-1}) u_t = A(q^{-1}) w_t - S_k(q^{-1}) e_t - d \tag{56}$$

The poles are determined by $B(q^{-1})$

If $w_t = 0$ and $d = 0$, the MV$_0$ control becomes the minimum variance control

The MV_0 controller still has the following shortcomings.

1. Large control effort.
2. Undamped zeros.

Example of MV_0 controller

ARMAX model

$$y_t - 1.7y_{t-1} + 0.7y_{t-2} = u_{t-1} + 0.5u_{t-2} + e_t + 1.5e_{t-1} + 0.9e_{t-2} \tag{57}$$

Criterion

$$\min J_t = \mathbb{E}[(y_{t+1} - 1)^2] \tag{58}$$

Polynomials, etc.

$$A(q^{-1}) = 1 - 1.7q^{-1} + 0.7q^{-2}, \quad B(q^{-1}) = 1 + 0.5q^{-1}, \tag{59}$$

$$C(q^{-1}) = 1 + 1.5q^{-1} + 0.9q^{-2}, \qquad d = 0, \quad w_t = 1, \quad k = 1 \tag{60}$$

Example of MV_0 controller

Diophantine equation

$$1 + 1.5q^{-1} + 0.9q^{-2} = \left(1 - 1.7q^{-1} + 0.7q^{-2}\right) G(q^{-1}) + q^{-1} S(q^{-1}) \quad (61)$$

Furthermore, $G(0) = 1$, ord$[G] = k - 1$ and ord$[S] = \max(n_a - 1, n_c - k)$

$$\text{ord}[G] = 0, \qquad \text{ord}[S] = 1 \quad (62)$$

Rewrite

$$1 + 1.5q^{-1} + 0.9q^{-2} = 1 - 1.7q^{-1} + 0.7q^{-2} + s_1 q^{-1} + s_2 q^{-2} \quad (63)$$

$$1.5 = -1.7 + s_1, \qquad 0.9 = 0.7 + s_2 \quad (64)$$

Match coefficients

$$G(q^{-1}) = 1, \quad (65)$$

$$S(q^{-1}) = s_1 + s_2 q^{-1} = 3.2 + 0.2q^{-1} \quad (66)$$

Example of MV_0 controller

MV_0 control law

$$u_t = \frac{C(q^{-1})}{B(q^{-1})G_k(q^{-1})} w_t - \frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})} y_t - \frac{1}{B(q^{-1})} d \tag{67}$$

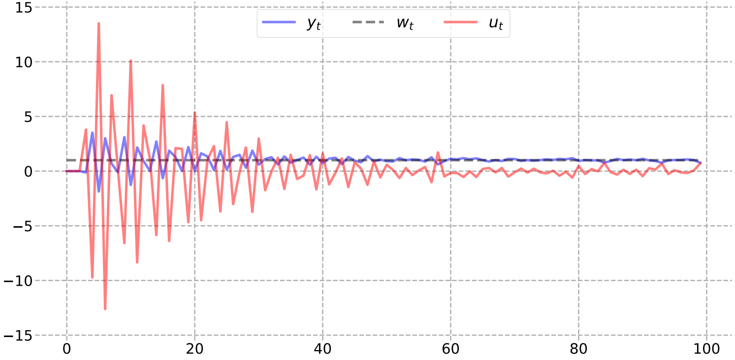
Optimal control law

$$(1 + 0.5q^{-1})u_t = (1 + 1.5q^{-1} + 0.9q^{-2})w_t - (3.2 + 0.2q^{-1})y_t \tag{68}$$

Substitute $w_t = 1$

$$u_t = 3.4 - 0.5u_{t-1} - 3.2y_t - 0.2y_{t-1} \tag{69}$$

Example of MV_0 controller



The control input is large compared to the output

MV_1 control

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{70}$$

Objective function

$$J_t = \mathbb{E}[(y_{t+k} - w_t)^2 + \rho u_t^2] \tag{71}$$

Optimal control law

$$\left(B(q^{-1})G_k(q^{-1}) + \frac{\rho}{b_0}C(q^{-1})\right)u_t = C(q^{-1})w_t - S_k(q^{-1})y_t - G_k(q^{-1})d \tag{72}$$

The Diophantine equation is the same as for the minimum variance controller

$$C(q^{-1}) = A(q^{-1})G_k(q^{-1}) + q^{-k}S_k(q^{-1}) \tag{73}$$

MV_1 control – Closed-loop analysis

Closed-loop system

$$y_t = q^{-k}\frac{B(q^{-1})}{B(q^{-1})+\frac{\rho}{b_0}A(q^{-1})}w_t + \frac{B(q^{-1})G_k(q^{-1})+\frac{\rho}{b_0}C(q^{-1})}{B(q^{-1})+\frac{\rho}{b_0}A(q^{-1})}e_t \tag{74}$$

$$+\frac{\frac{\rho}{b_0}}{B(q^{-1})+\frac{\rho}{b_0}A(q^{-1})}d, \tag{75}$$

$$u_t = \frac{A(q^{-1})}{B(q^{-1})+\frac{\rho}{b_0}A(q^{-1})}w_t - \frac{S_k(q^{-1})}{B(q^{-1})+\frac{\rho}{b_0}A(q^{-1})}e_t \tag{76}$$

$$-\frac{1}{B(q^{-1})+\frac{\rho}{b_0}A(q^{-1})}d \tag{77}$$

If $A(1) \neq 0$ (no pure integrator), MV_1 contains a stationary error for nonzero setpoints

MV_{1a} control

Penalize changes in the control action

$$J_t = \mathbb{E}[(y_{t+k} - w_t)^2 + \rho(u_t - u_{t-1})^2] \tag{78}$$

Optimal control law

$$\left(B(q^{-1})G_k(q^{-1}) + \frac{\rho}{b_0}C(q^{-1})\Delta\right)u_t = C(q^{-1})w_t - S_k(q^{-1})y_t - G_k(q^{-1})d \tag{79}$$

$$\Delta = 1 - q^{-1} \tag{80}$$

The Diophantine equation is the same as for the minimum variance controller

$$C(q^{-1}) = A(q^{-1})G_k(q^{-1}) + q^{-k}S_k(q^{-1}) \tag{81}$$

MV_{1a} control – Closed-loop analysis

Closed-loop system

$$y_t = q^{-k} \frac{B(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})} w_t + \frac{B(q^{-1})G_k(q^{-1}) + \frac{\rho}{b_0}\Delta C(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})} e_t, \tag{82}$$

$$u_t = \frac{A(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})} w_t - \frac{S_k(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})} e_t \tag{83}$$

$$- \frac{1}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})} d \tag{84}$$

Pole-zero control

Pole-zero control

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{85}$$

Relax the objective of following the setpoint

$$\tilde{w}_t = q^{-k}\frac{B_m(q^{-1})}{A_m(q^{-1})}w_t \tag{86}$$

Minimize

$$y_{t+k} - \tilde{w}_{t+k} \qquad \text{or} \qquad A_m(q^{-1})y_{t+k} - B_m(q^{-1})w_t \tag{87}$$

Cost function

$$J_t = \mathbb{E}\left[\left(A_m(q^{-1})y_{t+k} - B_m(q^{-1})w_t\right)^2\right] \tag{88}$$

Optimal control law

$$B(q^{-1})G_k(q^{-1})u_t = C(q^{-1})B_m(q^{-1})w_t - S_k(q^{-1})y_t - G_k(q^{-1})d \tag{89}$$

Diophantine equation

$$A_m(q^{-1})C(q^{-1}) = A(q^{-1})G_k(q^{-1}) + q^{-k}S_k(q^{-1}) \tag{90}$$

Pole-zero control - Closed-loop analysis

Stationary closed-loop system

$$A_m(q^{-1})y_t = q^{-k}B_m(q^{-1})w_t + G_k(q^{-1})e_t, \tag{91}$$

$$B(q^{-1})A_m(q^{-1})u_t = A(q^{-1})B_m(q^{-1})w_t - S_k(q^{-1})e_t - A_m(q^{-1})d \tag{92}$$

The pole-zero controller has the following shortcomings

1. Undamped zeros (zeros outside of the unit circle)

Generalized minimum variance control

Generalized Minimum Variance Strategy

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{93}$$

Objective function

$$J_t = \mathbb{E}\Big[(\tilde{y}_t - \tilde{w}_t)^2 + \rho \tilde{u}_t^2\Big] \tag{94}$$

Filtered variables

$$\tilde{y}_t = H_y(q)y_t = \frac{B_y(q^{-1})}{A_y(q^{-1})}y_t, \tag{95}$$

$$\tilde{w}_t = H_w(q)w_t = \frac{B_w(q^{-1})}{A_w(q^{-1})}w_t, \tag{96}$$

$$\tilde{u}_t = H_u(q)u_t = \frac{B_u(q^{-1})}{A_u(q^{-1})}u_t \tag{97}$$

$\rho > 0$ is a regularization parameter

$$A_y(0) = A_w(0) = A_u(0) = B_u(0) = 1 \tag{98}$$

Generalized Minimum Variance Strategy

Control law (q^{-1} is omitted for brevity)

$$\Big[A_u B G + \alpha C B_u\Big] u_t = A_u \Big[C \frac{B_w}{A_w} w_t - \frac{S}{A_y} y_t - G d\Big], \quad \alpha = \frac{\rho}{b_0} \tag{99}$$

G and S are solutions to the Diophantine equation

$$B_y(q^{-1}) C(q^{-1}) = A_y(q^{-1}) A(q^{-1}) G(q^{-1}) + q^{-k} S(q^{-1}), \tag{100}$$

$G(0) = B_y(0)$, ord$[G] = k - 1$ and
ord$[S] = \max(n_a + n_{a_y} - 1, n_{b_y} + n_c - k)$

Note: The Diophantine equation is independent of the control filter $\frac{B_u}{A_u}$

Generalized Minimum Variance Strategy – Closed-loop analysis

Closed-loop system

$$\Big[BA_uB_y + \alpha AB_uA_y\Big]y_t = q^{-k}\frac{B_w}{A_w}BA_uA_yw_t + \Big[A_uBG + \alpha CB_u\Big]A_ye_t + \alpha B_uA_yd, \tag{101}$$

$$\Big[BA_uB_y + \alpha AB_uA_y\Big]u_t = \frac{B_w}{A_w}AA_uA_yw_t + SA_ue_t - A_uB_yd \tag{102}$$

We can affect the closed-loop poles using the control filter because it doesn't affect the Diophantine equation

GMV – Special cases

Variant of MV_0 control

$$H_y(q) = 1, \quad H_w(q) = 1, \quad H_u(q) = 1, \quad \rho = 0 \tag{103}$$

MV_1 control

$$H_y(q) = 1, \quad H_w(q) = 1, \quad H_u(q) = 1, \quad \rho \neq 0 \tag{104}$$

MV_{1a} control

$$H_y(q) = 1, \quad H_w(q) = 1, \quad H_u(q) = 1 - q^{-1}, \quad \rho \neq 0 \tag{105}$$

PZ-control

$$H_y(q) = A_m(q^{-1}), \quad H_w(q) = B_m(q^{-1}), \quad H_u(q) = 1, \quad \rho = 0 \tag{106}$$

MV_3 control:

$$H_y(q) = \frac{A_e(q^{-1})}{B_e(q^{-1})}, \quad H_w(q) = \frac{A_e(q^{-1})B_m(q^{-1})}{B_e(q^{-1})A_m(q^{-1})}, \quad H_u(q) = 1, \quad \rho = 0 \tag{107}$$

MV_3 control

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{108}$$

Objective function

$$J_t = \mathbb{E}\left[\left(\frac{A_e(q^{-1})}{B_e(q^{-1})}y_{t+k} - \frac{A_e(q^{-1})B_m(q^{-1})}{B_e(q^{-1})A_m(q^{-1})}w_t\right)^2\right] \tag{109}$$

Control law

$$BGu_t = C\frac{A_eB_m}{B_eA_m}w_t - \frac{S}{B_e}y_t - Gd \tag{110}$$

Closed-loop system

$$y_t = q^{-k}\frac{B_m}{A_m}w_t + G\frac{B_e}{A_e}e_t \tag{111}$$

$$u_t = \frac{AB_m}{BA_m}w_t - \frac{SB_e}{BA_e}e_t - \frac{1}{B}d \tag{112}$$

Closed-loop simulation

Closed-loop simulation

For each time, t $(a_0 = 1)$

1. Evaluate y_t as a function of y_{t-j} for $j = 1, \ldots, n_a$, u_{t-k-j} for $j = 0, \ldots, n_b$, and e_{t-j} for $j = 0, \ldots, n_c$

$$\begin{aligned} y_t = {} & b_0 u_{t-k} + \cdots + b_{n_b} u_{t-k-n_b} + c_0 e_t + \cdots + c_{n_c} e_{t-n_c} \\ & - a_1 y_{t-1} - \cdots - a_{n_a} y_{t-n_a} \end{aligned} \tag{113}$$

2. Evaluate u_t as a function of w_{t-j} for $j = 0, \ldots, n_q$, y_{t-j} for $j = 0, \ldots, n_s$, and u_{t-j} for $j = 1, \ldots, n_r$

$$\begin{aligned} u_t = {} & \big(q_0 w_t + \cdots + \cdots q_{n_q} w_{t-n_q} - s_0 y_t - \cdots - s_{n_s} y_{t-n_s} \\ & - r_1 u_{t-1} - r_{n_r} u_{t-n_r} \big) / r_0 \end{aligned} \tag{114}$$

Closed-loop simulation

Example code

```
    % Extract noise and outputs
    Yt = y(t     - (1:na)); % na = numel(A) - 1;
    Ut = u(t - k - (0:nb)); % nb = numel(B) - 1;
    Et = e(t     - (0:nc)); % nc = numel(C) - 1;

    % Compute new outputs
    y(t) = B*Ut' - A(2:end)*Yt' + C*Et';

    % Extract noise and outputs
    Yt = y(t - (0:ns)); % ns = numel(S) - 1;
    Ut = u(t - (1:nr)); % nr = numel(R) - 1;
    Wt = w(t - (0:nq)); % nq = numel(Q) - 1;

    % Compute new inputs
    u(t) = (Q*Wt' - S*Yt' - R(2:end)*Ut')/R(1);
```

Questions?