Stochastic Adaptive Control (02421)

Lecture 13

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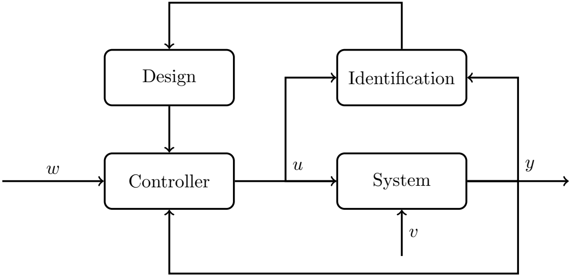
Lecture Plan

1. ARX models
2. ARX prediction + control
3. ARX estimation
4. ARX model validation
   + adaptive control
5. ARMAX control
6. ARMAX estimation
   + adaptive control
7. Systems and control theory
8. Stochastic systems + Kalman filtering
9. SS estimation (recursive) + control
10. SS control
11. SS estimation (batch)
12. SS estimation (recursive)
13. SS nonlinear control



Today's Agenda

- Nonlinear optimal control of input-affine systems

Nonlinear optimal control

System

Input-affine system

$$x_{t+1} = f(x_t) + g(x_t)u_t, \tag{1}$$
$$y_t = Cx_t \tag{2}$$

$f(x)$ is a vector and $g(x)$ is a matrix

Important assumptions

1. It is reasonable to linearize f, i.e., $f(x + \Delta x) \approx f(x) + \frac{\partial f}{\partial x}(x)\Delta x$
2. The input term is only mildly dependent on the state, i.e., $g(x + \Delta x) \approx g(x)$

Approximate response

Write out recursion

$$x_1 = f(x_0) + g(x_0)u_0, \tag{3}$$
$$x_2 = f(x_1) + g(x_1)u_1 \tag{4}$$
$$= f(f(x_0) + g(x_0)u_0) + g(f(x_0) + g(x_0)u_0)u_1 \tag{5}$$
$$\approx f(f(x_0)) + \frac{\partial f}{\partial x}(f(x_0))g(x_0)u_0 + g(f(x_0))u_1 \tag{6}$$

Introduce

$$z_0 = x_0, \quad z_1 = f(x_0), \quad A_1 = \frac{\partial f}{\partial x}(z_1), \quad B_0 = g(z_0), \quad B_1 = g(z_1) \tag{7}$$

Then

$$x_2 = f(z_1) + A_1 B_0 u_0 + B_1 u_1 \tag{8}$$

Continue

$$x_3 = f(f(z_1) + A_1 B_0 u_0 + B_1 u_1) + g(f(z_1) + A_1 B_0 u_0 + B_1 u_1)u_2 \tag{9}$$
$$\approx f(z_2) + A_2 A_1 B_0 u_0 + A_2 B_1 u_1 + B_2 u_2 \tag{10}$$

Approximate system

Free/unforced response

$$z_{t+1} = f(z_t) \tag{11}$$

Time-varying system matrices

$$A_t = \frac{\partial f}{\partial x}(z_t), \qquad B_t = g(z_t) \tag{12}$$

Forced response

$$x_t = z_t + \sum_{j=0}^{t-1} A_{t-1} \cdots A_{j+1} B_j u_j \tag{13}$$

Notation

$$A_{t-1} \cdots A_{j+1} = A_{t-1} A_{t-2} \cdots A_{j+2} A_{j+1}, \qquad A_{t-1} \cdots A_t = I \tag{14}$$

Approximate outputs

$$y_t = C z_t + \sum_{j=0}^{t-1} C A_{t-1} \cdots A_{j+1} B_j u_j \tag{15}$$

Compact notation

Vectors and matrices

$$Z_N = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}, \qquad Y_N = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \qquad U_N = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \tag{16}$$

$$\Gamma_{yz}^N = \text{blkdiag}(C, C, \dots, C), \tag{17}$$

$$\Gamma_{yu}^N = \begin{bmatrix} CB_0 & & & & \\ CA_1B_0 & CB_1 & & & \\ CA_2A_1B_0 & CA_2B_1 & CB_2 & & \\ \vdots & & & \ddots & \\ CA_{N-1}\cdots A_1B_0 & CA_{N-1}\cdots A_2B_1 & \cdots & CA_{N-1}B_{N-2} & CB_{N-1} \end{bmatrix} \tag{18}$$

Compact system

$$Y_N = \Gamma_{yz}^N Z_N + \Gamma_{yu}^N U_N \tag{19}$$

Optimal control problem

Optimal control problem

$$\min_{U_N} \quad \phi = \frac{1}{2}(Y_N - W_N)^T Q_y (Y_N - W_N) + \frac{1}{2} U_N^T Q_u U_N \tag{20}$$

Solution

$$U_N = -(\Gamma_{yu}^T Q_y \Gamma_{yu} + Q_u)^{-1} \Gamma_{yu}^T Q_y (\Gamma_{yz} Z_N - W_N) \tag{21}$$

First manipulated input (e.g., for feedback control law)

$$u_0 = -\Pi U_N, \qquad \Pi = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \tag{22}$$

Summary

Step 1: Compute the free response (x_0 is given)

$$z_{t+1} = f(z_t), \qquad z_0 = x_0, \qquad t = 0, \dots, N-1 \tag{23}$$

Step 2: Evaluate the system matrices

$$A_t = \frac{\partial f}{\partial x}(z_t), \qquad t = 1, \dots, N-1, \tag{24}$$

$$B_t = g(z_t), \qquad t = 0, \dots, N-1 \tag{25}$$

Step 3: Form the large matrices (must be done every time)

$$\Gamma_{yz}^N = \mathrm{blkdiag}(C, C, \dots, C), \tag{26}$$

$$\Gamma_{yu}^N = \begin{bmatrix} CB_0 & & & \\ CA_1B_0 & CB_1 & & \\ \vdots & & \ddots & \\ CA_{N-1}\cdots A_1B_0 & \cdots & CA_{N-1}B_{N-2} & CB_{N-1} \end{bmatrix} \tag{27}$$

Step 4: Solution

$$u_0 = -\Pi U_N, \quad U_N = -(\Gamma_{yu}^T Q_y \Gamma_{yu} + Q_u)^{-1}\Gamma_{yu}^T Q_y(\Gamma_{yz}Z_N - W_N) \tag{28}$$

Questions?