Stochastic Adaptive Control (02421)

Lecture 7

Tobias K. S. Ritschel

Assistant Professor

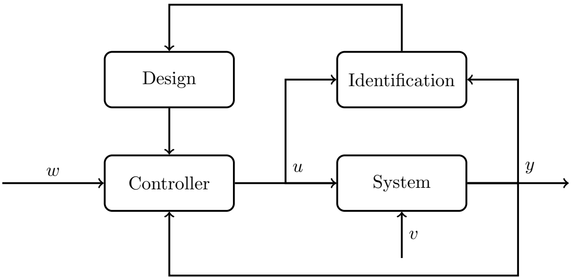
Section for Dynamical Systems, DTU Compute

DTU Compute

Department of Applied Mathematics and Computer Science

Lecture Plan

1. ARX models
2. ARX prediction + control
3. ARX estimation
4. ARX model validation + adaptive control
5. ARMAX control
6. ARMAX estimation + adaptive control
7. Systems and control theory
8. Stochastic systems + Kalman filtering
9. SS estimation (recursive) + control
10. SS control
11. SS estimation (batch)
12. SS estimation (recursive)
13. SS nonlinear control



Today's Agenda

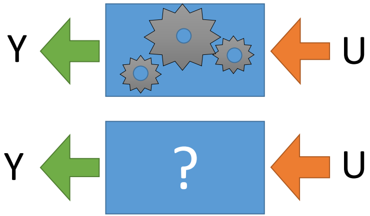
Systems Theory

- Continuous- and discrete-time internal and external models
- Linearization
- Discretization
- Transformations, stability, reachability, and observability

Dynamical systems

Dynamical Systems: External and Internal Models

We describe dynamical systems in two ways:



Internal Models

- States of the system
- Differential equations

External Models

- Transfer functions
- Zeros and poles

Focus in this course

- Discrete-time state space models
- Discrete-time external models

Nonlinear/continuous-time systems

- Linearize
- Discretize

Dynamical Systems: ODE and Algebraic Equation (Internal)

State space model

$$x(t_0) = x_0, \tag{1a}$$

$$\dot{x}(t) = \frac{\partial x}{\partial t}(t) = f(x(t), u(t); \theta) = A(\theta)x(t) + B(\theta)u(t) \tag{1b}$$

Analytical solution

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f(x(\tau), u(\tau); \theta)\, \mathrm{d}\tau \\ &= e^{A(\theta)(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(\theta)(\tau - t_0)} B(\theta) u(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{2}$$

Output equation

$$y(t) = g(x(t), u(t); \theta) = C(\theta)x(t) + D(\theta)u(t) \tag{3}$$

Steady state (x^*, u^*)

$$f(x^*, u^*; \theta) = 0. \tag{4}$$

Dynamical systems: ODE (External)

Inhomogeneous N-th order linear time-invariant external model

$$\sum_{k=0}^{N} \alpha_k \frac{\partial^k y}{\partial t^k} = \sum_{k=0}^{M} \beta_k \frac{\partial^k u}{\partial t^k}, \qquad \alpha_k, \beta_k \in \mathbb{R} \tag{5}$$

Analytical solution

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(s) u(t-s)\, \mathrm{d}s, \tag{6}$$

$h(t)$ is the impulse response

Laplace transformed variables

$$Y(s) = H(s)U(s), \qquad U(s) = \mathcal{L}(u(t)), \tag{7}$$

where

$$\boxed{\begin{aligned} H(s) = \mathcal{L}(h(t)) &= \int_{-\infty}^{\infty} h(s) e^{-st}\, \mathrm{d}s \\ &= C(\theta)\left(sI - A(\theta)\right)^{-1} B(\theta) + D(\theta). \end{aligned}} \tag{8}$$

Time and Frequency Domain

Continuous-time time-domain

$$t$$

$$y(t) = h(t) * u(t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = sY(s)$$

Continuous-time frequency-domain

$$s = a + iw$$

$$Y(s) = H(s)U(s)$$

$$H(s) = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

Discrete-time time-domain

$$t_k = kT_s$$

$$y_k = H_d(q)u_k$$

$$u_k = u(t_k) = u(kT_s)$$

$$u_{k-1} = q^{-1}u_k$$

where T_s is the sampling time

Discrete-time frequency-domain

$$z = e^{T_s s}$$

$$Y(z) = H_z(z)U(z)$$

$$H_d(q) = H_z(q)$$

Linearization

Linearization

Linearize around steady state (x^*, u^*)

$$f(x^*, u^*; \theta) = 0 \tag{9}$$

Linearization (truncated Taylor expansion)

$$\dot{x} = f(x^*, u^*; \theta) + \frac{\partial f}{\partial x}(x^*, u^*; \theta)(x - x^*) + \frac{\partial f}{\partial u}(x^*, u^*; \theta)(u - u^*), \tag{10a}$$

$$y = g(x^*, u^*; \theta) + \frac{\partial g}{\partial x}(x^*, u^*; \theta)(x - x^*) + \frac{\partial g}{\partial u}(x^*, u^*; \theta)(u - u^*) \tag{10b}$$

Linearization

Deviation variables

$$X = x - x^*, \tag{11a}$$
$$U = u - u^*, \tag{11b}$$
$$Y = y - y^*, \qquad y^* = g(x^*, u^*; \theta) \tag{11c}$$

System matrices

$$A(\theta, x^*, u^*) = \frac{\partial f}{\partial x}(x^*, u^*; \theta), \qquad B(\theta, x^*, u^*) = \frac{\partial f}{\partial u}(x^*, u^*; \theta), \tag{12a}$$
$$C(\theta, x^*, u^*) = \frac{\partial g}{\partial x}(x^*, u^*; \theta), \qquad D(\theta, x^*, u^*) = \frac{\partial g}{\partial u}(x^*, u^*; \theta) \tag{12b}$$

Linear time invariant (LTI) system

$$\dot{X} = A(\theta, x^*, u^*)X + B(\theta, x^*, u^*)U, \tag{13a}$$
$$Y = C(\theta, x^*, u^*)X + D(\theta, x^*, u^*)U \tag{13b}$$

Discretization

Discretization: Sampling of Continuous Systems

Sampling

$$x_k = x(t_0 + T_s k), \qquad y_k = y(t_0 + T_s k) \tag{14}$$

Zero-order hold (ZOH) parametrization: Piecewise constant input, u

$$u(t) = u_k, \qquad kT_s \leq t < (k+1)T_s \tag{15}$$

Shannon's Sampling Theorem: If the highest frequency of the system is w_0, then a sampling frequency of at least the double is needed for reconstruction

$$w_s \geq 2w_0, \quad w_s = \frac{2\pi}{T_s} \tag{16}$$

Choosing based on desired samples per rise time:

$$T_s = t_r/N_r, \quad N_r \in [2, 4] \tag{17}$$

Discretization of state space models

Analytical solution for continuous-time state space models

$$x(t_{k+1}) = e^{A(\theta)(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(\theta)(t_{k+1}-\tau)}B(\theta)u(\tau)\,\mathrm{d}\tau, \tag{18a}$$

$$y(t_k) = C(\theta)x(t_k) + D(\theta)u(t_k). \tag{18b}$$

Discrete-time state space models

$$\boxed{\begin{aligned} x_{k+1} &= A_d(\theta, T_s)x_k + B_d(\theta, T_s)u_k, && (19)\\ y_k &= C(\theta)x_k + D(\theta)u_k && (20)\end{aligned}}$$

Discrete-time matrices

$$A_d(\theta, T_s) = e^{A(\theta)T_s}, \qquad B_d(\theta, T_s), = \int_0^{T_s} e^{A(\theta)\tau}B(\theta)\,\mathrm{d}\tau \tag{21}$$

Matrix exponential

$$\boxed{\begin{bmatrix} A_d(\theta, T_s) & B_d(\theta, T_s) \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A(\theta) & B(\theta) \\ 0 & 0 \end{bmatrix} T_s\right)} \tag{22}$$

Discretization of transfer function models

Continuous-time transfer function model (frequency domain)

$$y(s) = H(s)u(s), \quad H(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{23}$$

Discretization with Z-transform (use look-up tables)

$$H_z(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{H(s)}{s}\right), \quad z \in \mathbb{C} \tag{24}$$

Discrete-time transfer function model (frequency domain)

$$y(z) = H_z(z)u(z) = \frac{\bar{b}_0 z^n + \bar{b}_1 z^{n-1} + \cdots + \bar{b}_n}{z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n} u(z) \tag{25}$$

Discrete-time transfer function model (time domain) - recall that $H_d(q) = H_z(q)$

$$y_t = H_d(q)u_t = \frac{\bar{b}_0 + \bar{b}_1 q^{-1} + \cdots + \bar{b}_n q^{-n}}{1 + \bar{a}_1 q^{-1} + \cdots + \bar{a}_n q^{-n}} u_t \tag{26}$$

Discrete-time transfer function model (time domain) - difference equations

$$y_t + \bar{a}_1 y_{t-1} + \cdots + \bar{a}_n y_{t-n} = \bar{b}_0 u_t + \bar{b}_1 u_{t-1} + \cdots + \bar{b}_m u_{t-n} \tag{27}$$

Transforms, stability, reachability, observability, etc.

Poles and Zeros

Consider the factor terms of transfer functions:

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = K_0 \frac{\Pi_i (s - z_i)}{\Pi_i (s - p_i)} \tag{28}$$

$$H_d(q) = \frac{B_d(q^{-1})}{A_d(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}} = K_{d,0} \frac{\Pi_i (q - z_{d,i})}{\Pi_i (q - p_{d,i})}$$

Transfer function properties

$$\textbf{Zeros: } H(z_i) = 0, \tag{29}$$

$$\textbf{Poles: } |H(p_i)| = \infty, \tag{30}$$

$$\textbf{DC-gain: } H(s = 0), H_z(z = 1) = H_d(q = 1) \tag{31}$$

Poles

Poles of external models = eigenvalues of internal models

$$\mathcal{C}(A) = A(s) \tag{32}$$

Instability criteria

$$\textbf{Continuous: } 0 < \text{Re}(p_c) \tag{33a}$$
$$\textbf{Discrete: } 1 < |p_d| \tag{33b}$$

Poles of discrete- (p_d) and continuous-time (p_c) systems are related

$$p_d = e^{p_c T_s} \tag{34}$$

Zeros

Number of zeros m and poles n

$$\textbf{Continuous: } m \leq n \tag{35}$$

$$\textbf{Discrete: } \begin{cases} m = n - 1 & (\text{for } D = 0) \\ m = n & \text{otherwise} \end{cases} \tag{36}$$

Zeros of discrete- (p_d) and continuous-time (p_c) systems are related

$$z_d = e^{z_c T_s} \tag{37}$$

Zero-pole cancellation

$$z_i = p_i \quad \Rightarrow \quad H(s) = \frac{s - z_i}{(s - p_i)(s - p_1)} = \frac{1}{(s - p_1)} \tag{38}$$

Transforms - Similarity Transform and Diagonal Transform

Change internal state variables

$$z_t = \Upsilon x_t, \tag{39}$$

$$z_{t+1} = \Upsilon A \Upsilon^{-1} z_t + \Upsilon B u_t, \tag{40}$$

$$y_t = C \Upsilon^{-1} z_t + D u_t \tag{41}$$

The external model is unaffected by the transformation

$$H(q) = C\Upsilon^{-1}(qI - \Upsilon A \Upsilon^{-1})^{-1} \Upsilon B + D = C(qI - A)^{-1} B + D \tag{42}$$

Example: Diagonal transform

$$\Lambda = Q^{-1} A Q = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{43}$$

The columns of Q are the right eigenvectors of A

Transform external to internal models

External system

$$y_t + a_1 y_{t-1} + \cdots + a_n y_{t-n} = b_0 u_t + b_1 u_{t-1} + \cdots + b_n u_{t-n} \tag{44}$$

Transfer function

$$H(q) = \frac{B(q^{-1})}{A(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} = \sum_{i=0}^{\infty} h_i q^{-i} \tag{45}$$

Minimal representation: An internal model with minimum number of states, e.g., the 4 canonical forms

Transforms - Canonical forms

Controller canonical form

$$A_c = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{46}$$

$$C_c = [b_1 - b_0 a_1, b_2 - b_0 a_2, \ldots, b_n - b_0 a_n] \qquad D_c = b_0 \tag{47}$$

Observer canonical form

$$A_o = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} \qquad B_o = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ \vdots \\ b_n - b_0 a_n \end{bmatrix} \tag{48}$$

$$C_o = [1, 0, \cdots, 0] \qquad D_o = b_0 \tag{49}$$

Transforms - Canonical forms

Controllability canonical form

$$A_{co} = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & \cdots & 0 & -a_{n-1} \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & -a_1 \end{bmatrix} \qquad B_{co} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{50}$$

$$C_{co} = (h_1, h_2, \cdots, h_n) \qquad D_{co} = h_0 \tag{51}$$

Observability canonical form

$$A_{ob} = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{ob} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \tag{52}$$

$$C_{ob} = (1, 0, \ldots, 0) \qquad D_{ob} = h_0 \tag{53}$$

Transforms - Canonical forms

Relations between canonical forms

$$A_c = A_o^T, \qquad A_{co} = A_{ob}^T, \tag{54}$$

$$B_c = C_o^T, \qquad B_{co} = C_{ob}^T, \tag{55}$$

$$B_o = C_c^T, \qquad B_{ob} = C_{co}^T, \tag{56}$$

$$D_c = D_o = D_{co} = D_{ob} = b_0 = h_0 \tag{57}$$

Transforms - Direct realization

General external model

$$y_t + a_1 y_{t-1} + \cdots + a_{n_a} y_{t-n_a} = b_0 u_t + b_1 u_{t-1} + \cdots + b_{n_b} u_{t-n_b} \tag{58}$$

State

$$x_t = \begin{bmatrix} -y_{t-1} & \cdots & -y_{t-n_a} & u_{t-1} & \cdots & u_{t-n_b} \end{bmatrix} \tag{59}$$

Non-minimal internal model

$$A_d = \left[\begin{array}{cccccccc} -a_1 & \cdots & -a_{n_a-1} & -a_{n_a} & -b_1 & \cdots & -b_{n_b-1} & -b_{n_b} \\ 1 & & 0 & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & & 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 1 & & 0 & 0 \\ \vdots & & \vdots & \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & & 1 & 0 \end{array}\right], \quad B_d = \left[\begin{array}{c} -b_0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right], \tag{60}$$

$$C_d = (a_1, \ldots, a_{n_a}, b_1, \ldots, b_{n_b}), \qquad D_d = b_0 \tag{61}$$

Controllability and reachability

Definition
A system is said to be controllable, if it is possible to move the system from an arbitrary state value to the origin in finite time.

Definition
A system is said to be reachable, if it is possible to move the system from one arbitrary state value to another arbitrary state in finite time.

Reachability $\Rightarrow$ controllability, not the reverse

An n-state system is reachable if and only if the reachability matrix W_c has full rank $(k \geq n)$

$$W_c(k) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{k-1}B \end{bmatrix} \tag{62}$$

The reachability Gramian is given by $\Sigma_k^c = W_c(k)W_c^T(k)$

Controllability and reachability - Control

k-step input sequence (not unique)

$$x_k = A^k x_0 + W_c(k) U_{k-1}, \tag{63}$$

$$U_{k-1} = \begin{bmatrix} u_{k-1} & u_{k-2} & \cdots & u_0 \end{bmatrix}^T \tag{64}$$

brings the system from any x_0 to a desired state, $\bar{x}$

Sequence with minimal control usage

$$\min_{u_{k-1},\dots,u_0} \quad \sum_{j=0}^{k-1} u_j^T u_j \tag{65a}$$

Solution

$$U_{k-1}^* = W_c^T(k)(\Sigma_k^c)^{-1}\left[\bar{x} - A^k x_0\right] \tag{66}$$

Controllability and reachability in continuous-time

Continuous-time system

$$\dot{x} = Ax + Bu, \tag{67}$$

Reachability Gramian

$$\dot{\Sigma}^c = A\Sigma^c + \Sigma^c A^T + BB^T \tag{68a}$$
$$\Sigma^c(t_0) = 0. \tag{68b}$$

The system is reachable if Σ^c is invertible for any $t \geq t_0$

Note: For continuous-time systems, reachability $\Leftrightarrow$ controllability

Observability and constructability

Definition:
A system is observable if any initial state can be estimated using only the information from the following outputs and inputs.

Definition:
A system is constructable if, for any possible evolution of the state and control variables, the current state can be estimated using only the information from outputs.

Observability $\Rightarrow$ constructability, but the reverse is not true

An n-state system is observable if and only if the observability matrix W_o has full rank $(k \geq n)$

$$W_o(k) = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \cdots & (CA^{k-1})^T \end{bmatrix}^T \tag{69}$$

Observability Gramian: $\Sigma_k^o = W_o(k) W_o^T(k)$

Observability and constructability - Continuous-Time

Continuous-time system

$$\dot{x} = Ax + Bu, \tag{70a}$$
$$y = Cx + Du \tag{70b}$$

Observability Gramian

$$\dot{\Sigma}^o = A\Sigma^o + \Sigma^o A^T + C^T C \tag{71a}$$
$$\Sigma^o(t_0) = 0. \tag{71b}$$

The system is observable if Σ^o is invertible for any $t \geq t_0$

Stability

Several definitions of stability exist: e.g., marginal and asymptotic stability

Consider a steady state x_s of the system

- **Marginally stable**: x_s is said to be (marginally) stable if any solution trajectory $\{x(t),\ t \in [t_0, \infty]\}$ is bounded.
- **Asymptotically stable**: x_s is said to be asymptotically stable if any solution trajectory converges to x_s ($x(t) \to x_s$) as time progresses ($t \to \infty$).

A system which is not stable (i.e., not marginally stable) is unstable

A system is BIBO stable if the output is bounded for any bounded input

Note: Asymptotic stability $\Rightarrow$ BIBO stability

Stability of LTI Systems

A state space model is stable if and only if all of the following requirements are fulfilled

Continuous-time	Discrete-time
Marginally stable:	
• $\text{Re}\{\text{eig}(A)\} \leq 0$	• $\lvert\,\text{eig}(A)\rvert \leq 1$
• $\forall\,\text{Re}\{\text{eig}(A)_i\} = 0$, the AM=GM	• $\forall\lvert\,\text{eig}(A)_i\rvert = 1$, the AM=GM
Asymptotically stable:	
• $\text{Re}\{\text{eig}(A)\} < 0$	• $\lvert\,\text{eig}(A)\rvert < 1$

*AM = Algebraic multiplicity (# of identical eigenvalues)
**GM = geometric multiplicity (# of associated eigenvectors)

Local Stability of Nonlinear Systems

Steady state of nonlinear system

$$\dot{x} = f(x_s, u_s) = 0 \qquad\qquad x_s = f(x_s, u_s), \tag{72}$$

Approximate behavior around steady state using linearization

$$A = \frac{\partial f}{\partial x}(x_s, u_s). \tag{73}$$

The system is locally stable (marginal or asymptotic) around the stationary point if the requirements on the previous slide are fulfilled

Questions?