Stochastic Adaptive Control (02421)

Lecture 6

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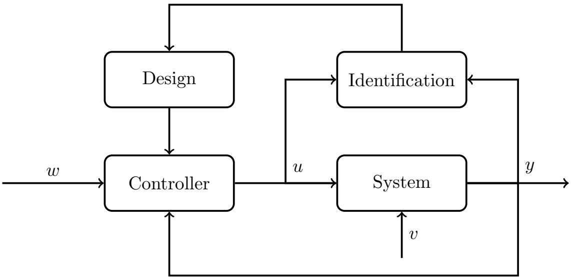
Lecture Plan

1. ARX models
2. ARX prediction + control
3. ARX estimation
4. ARX model validation + adaptive control
5. ARMAX control
6. ARMAX estimation + adaptive control
7. Systems and control theory
8. Stochastic systems + Kalman filtering
9. SS estimation (recursive) + control
10. SS control
11. SS estimation (batch)
12. SS estimation (recursive)
13. SS nonlinear control



Today's Agenda

- Pole-zero control
- Recursive extended least-squares (RELS)
- Estimation of time-varying parameters

Pole-zero control

Pole-zero control

Relax the objective of following the setpoint

$$\tilde{w}_t = q^{-k} \frac{B_m(q^{-1})}{A_m(q^{-1})} w_t \tag{1}$$

Minimize

$$y_{t+k} - \tilde{w}_{t+k} \qquad \text{or} \qquad A_m(q^{-1}) y_{t+k} - B_m(q^{-1}) w_t \tag{2}$$

Cost function

$$J_t = \mathbb{E}\left[\left(A_m(q^{-1}) y_{t+k} - B_m(q^{-1}) w_t\right)^2\right] \tag{3}$$

ARMAX model

$$A(q^{-1}) y_t = q^{-k} B(q^{-1}) u_t + C(q^{-1}) e_t \tag{4}$$

Optimal control law

$$u_t = \frac{C(q^{-1}) B_m(q^{-1})}{B(q^{-1}) G_k(q^{-1})} w_t - \frac{S_k(q^{-1})}{B(q^{-1}) G_k(q^{-1})} y_t \tag{5}$$

Diophantine equation

$$A_m(q^{-1}) C(q^{-1}) = A(q^{-1}) G_k(q^{-1}) + q^{-k} S_k(q^{-1}) \tag{6}$$

Pole-zero control

Stationary closed-loop system

$$A_m(q^{-1})y_t = q^{-k}B_m(q^{-1})w_t + G_k(q^{-1})e_t, \tag{7}$$

$$B(q^{-1})A_m(q^{-1})u_t = A(q^{-1})B_m(q^{-1})w_t - S_k(q^{-1})e_t \tag{8}$$

The pole-zero controller has the following shortcomings

1. Undamped zeros (zeros outside of the unit circle)

Recursive estimation

RELS - Recursive Extended Least Squares

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t, \tag{9}$$

$$y_t = \phi_t^T\theta + e_t, \tag{10}$$

$$\phi_t = [-y_{t-1}, \dots, -y_{t-n_a}, u_{t-k}, \dots, u_{t-n_b-k}, e_{t-1}, \dots, e_{t-n_c}]^T, \tag{11}$$

$$\theta = [a_1, \dots, a_{n_a}, b_0, \dots, b_{n_b}, c_1, \dots, c_{n_c}]^T \tag{12}$$

Least squares estimator (approximate e_i by ϵ_i in ϕ)

$$\epsilon_t = y_t - \phi_t^T\hat{\theta}_{t-1}, \tag{13}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + P_t\phi_t\epsilon_t, \tag{14}$$

$$P_t^{-1} = P_{t-1}^{-1} + \phi_t\phi_t^T \tag{15}$$

Time-varying parameters

Time-varying estimation - first example

ARX model

$$A(q^{-1})y_t = B(t, q^{-1})u_t + e_t, \quad (16)$$

$$b_1(t) = b_{1,0} + b_{1,1}t \quad (17)$$

Treat time-varying coefficient as two coefficients with their own inputs

$$y_t = \phi^T\theta + e_t \quad (18)$$

$$\theta^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_a} & b_{1,0} & b_{1,1} & b_2 & \cdots & b_{n_b} \end{bmatrix} \quad (19)$$

$$\phi^T = \begin{bmatrix} -y_{t-1} & -y_{t-2} & \cdots & -y_{t-n_a} & u_{t-1} & tu_{t-1} & u_{t-2} & \cdots & u_{t-n_b} \end{bmatrix} \quad (20)$$

Time-varying estimation

For deterministic time varying systems, rearrange the parameters

$$y_t = \phi_t^T \theta_t + e_t \tag{21}$$

$$\theta_t = \alpha + f(t)\beta = \begin{bmatrix} I & f(t) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{22}$$

$$y_t = \begin{bmatrix} \phi_t^T & \phi_t^T f(t) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + e_t \tag{23}$$

For piece-wise linear parameters, rearrange the parameters

$$y_t = \phi_t^T \theta_t + e_t, \tag{24}$$

$$\theta_t = \alpha_i + (t - T_i)\beta_i, \quad T_i \leq t \leq T_{i+1}, \tag{25}$$

$$y_t = \begin{bmatrix} \phi_t^T & \phi_t^T (t - T_i) \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} + e_t \tag{26}$$

Time-varying systems

System with general time-varying parameters

$$\theta_{t+1} = f(t, \theta_t, v_t) \tag{27}$$

- The methods discussed so far cannot estimate the time-varying dynamics and were not designed to do it
- In practice, the problem is that the correction factor diminishes over time

$$P_t \to 0 \tag{28}$$

Time-varying systems - Forgetting methods: Resetting

Reset the covariance after some time, t_i

$$P_{t_i} = P_i > P_{t_i - 1}, \quad \hat{\theta}_{t_i} = \hat{\theta}_{t_i - 1} \tag{29}$$

The appropriate restarting time depends on the application

For instance, restart at fixed intervals

$$t_i = Ni \tag{30}$$

This can be useful for periodic systems

Time-varying systems - Forgetting methods: Constant Gain

Another method: Keep the correction term large

For instance, keep the correction term κ constant

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \kappa \epsilon_t \tag{31}$$

$$\tilde{\theta}_t = (I - \kappa \phi_t^T) \tilde{\theta}_t - \kappa e_t \tag{32}$$

Alternatively, keep the variance constant

$$P_t = P \tag{33}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \kappa \epsilon_t \tag{34}$$

$$\kappa_t = \frac{P \phi_t}{1 + \phi_t^T P \phi_t} \tag{35}$$

Time-varying systems - Forgetting methods: Exponential Forgetfulness

Another approach: Forget a little bit all the time (exponential forgetfulness)

$$J_t = \frac{1}{2}\sum_{i=1}^{t} \lambda^{t-i}\epsilon_i^2 = \lambda J_{t-1} + \frac{1}{2}\epsilon_t^2 \tag{36}$$

The recursion is similar to the previous methods

$$\epsilon_t = y_t - \phi_t^T \hat{\theta}_{t-1}, \tag{37}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + P_t \phi_t \epsilon_t, \tag{38}$$

$$P_t^{-1} = \lambda P_{t-1}^{-1} + \phi_t \phi_t^T \tag{39}$$

The forgetting factor λ can be expressed in terms of a horizon, N_∞

$$\lambda = 1 - \frac{1}{N_\infty} \tag{40}$$

Time-varying systems - Fortescue's Method

Improve with a time-varying forgetting factor depending on the prediction error, ϵ_t

$$\lambda_t = 1 - \frac{1}{N_0} \frac{\epsilon_t^2}{\sigma^2 s_t} \tag{41}$$

N_0 is the approx. horizon over which the parameter is roughly constant

Recursion

$$\epsilon_t = y_t - \phi_t^T \hat{\theta}_{t-1} \tag{42}$$

$$s_t = 1 + \phi_t^T P_{t-1} \phi_t \tag{43}$$

$$K_t = \frac{P_{t-1}\phi_t}{\lambda_t + s_t} \tag{44}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t \epsilon_t \tag{45}$$

$$P_t = (I - K_t \phi_t^T) P_{t-1} \frac{1}{\lambda_t} \tag{46}$$

Time-varying systems - Fortescue's Method

If the variance is unknown, we can introduce an estimate

$$\lambda_t = 1 - \frac{1}{N_0}\frac{\epsilon_t^2}{r_t s_t} \tag{47}$$

$$r_t = r_{t-1} + \frac{1}{t}\left(\frac{\epsilon_t^2}{s_t} - r_{t-1}\right), \qquad r_0 = \epsilon_0^2 \tag{48}$$

Time-varying systems - Model Estimators

Introduce model of parameters

$$\theta_{t+1} = \theta_t + v_t, \qquad v_t \sim N(0, R_1\sigma^2) \tag{49}$$

$$y_t = \phi_t^T \theta_t + e_t, \qquad e_t \sim N(0, \sigma^2) \tag{50}$$

Estimate parameters using the Kalman filter

Data update

$$\hat{\theta}_{t|t} = \hat{\theta}_{t|t-1} + P_{t|t-1}\phi_t(y_t - \phi_t^T \hat{\theta}_{t|t-1}) \tag{51}$$

$$P_{t|t}^{-1} = P_{t|t-1}^{-1} + \phi_t \phi_t^T \tag{52}$$

Time update

$$\hat{\theta}_{t+1|t} = \hat{\theta}_{t|t} \tag{53}$$

$$P_{t+1|t} = P_{t|t} + R_1 \tag{54}$$

Questions?