Stochastic Adaptive Control (02421)

Lecture 13

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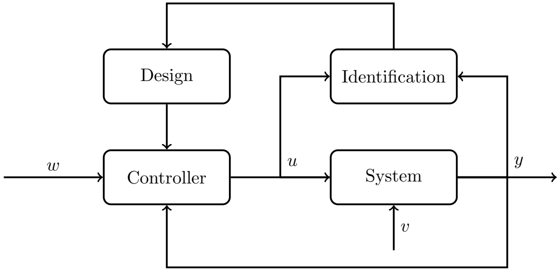
Lecture Plan

1. System theory
2. Stochastics
3. State estimation 1
4. State estimation 2
5. Optimal control 1
6. System identification 1 + adaptive control 1
7. External models + prediction
8. Optimal control 2
9. Optimal control 3
10. System identification 2
11. System identification 3 + model validation
12. System identification 4 + adaptive control 2
13. **Adaptive control 3**



Today's Agenda

- Cautious adaptive control
- Dual adaptive control
- Ideas for MSc projects/special courses
- Presentation by Emil Skov Martinsen

Follow-up from Last Lecture

CE Self-tuner Adaptive methods

- Explicit adaptive control: Estimate model parameters and then design controller *(explicit design)*
- Implicit adaptive control: Estimate the controller parameters directly *(implicit design)*

Other terms discussed

- CE: certainty equivalence principle; θ replaced by $\hat{\theta}$
- $J_r = \sum_{i=1}^{t} (y_i - w_i)^2 \approx \mathbb{E}[(y_i - w_i)^2]t$
- $J_u = \sum_{i=1}^{t} u_i^2 \approx \mathbb{E}[u_i^2]t$ (requires oscillation around 0)
- $J_e = \sum_{i=1}^{t} \epsilon_i^2 \approx \sigma^2 t$, for correct estimation $\epsilon_i = e_i$
- $J_e \approx J_r$

Follow-up from Last Lecture

Questions?

Known Systems and Control

ARX model with 1-step delay

$$A(q^{-1})y_t = B(q^{-1})u_{t-1} + e_t, \quad e_t \in N_{iid}(0, \sigma^2) \tag{1}$$

Cost function

$$J = \mathbb{E}\left[\sum_{i=1}^{N}(y_{t+i} - w_{t+i})^2\right] \tag{2}$$

Example: MV_0 controller ($N = 1$)

$$u_{t-1} = \frac{1}{B}w_t - \frac{S}{B}y_{t-1} = \frac{1}{B}w_t - \frac{q(1-A)}{B}y_{t-1} \tag{3}$$

$$y_t = w_t + e_t \tag{4}$$

Known Systems and Control

Alternative form: System

$$y_t = \phi_t^T \theta + e_t = b_0 u_{t-1} + \varphi_t^T \vartheta + e_t \tag{5}$$

$$\phi_t^T = (-y_{t-1}, -y_{t-2}, \ldots, u_{t-1}, u_{t-2}, \ldots), \quad \theta^T = (a_1, a_2, \ldots, b_0, b_1, \ldots) \tag{6}$$

$$\varphi_t^T = (-y_{t-1}, -y_{t-2}, \ldots, 0, u_{t-2}, \ldots), \quad \vartheta^T = (a_1, a_2, \ldots, 0, b_1, \ldots) \tag{7}$$

Alternative form: Controller

$$u_{t-1} = \frac{1}{b_0} w_t - \frac{\varphi_t^T \vartheta}{b_0} \tag{8}$$

$$y_t = w_t + e_t \tag{9}$$

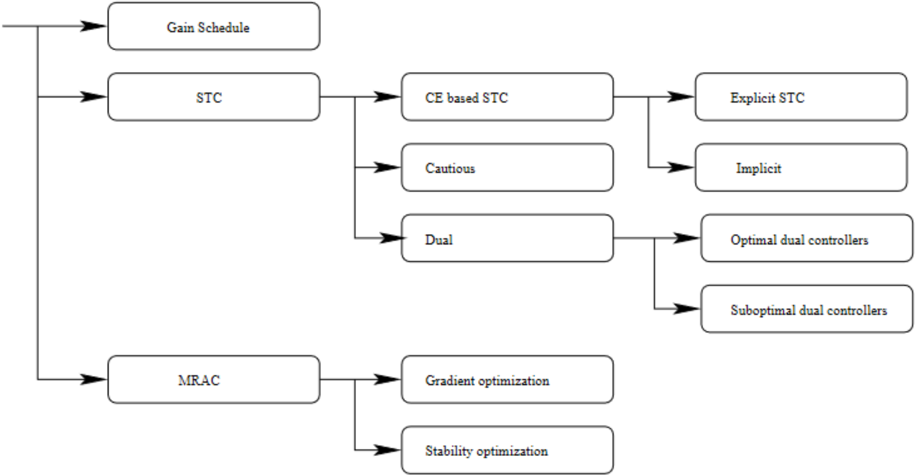
Relation between ϕ and φ

$$\varphi = \phi - \mathrm{diag}(l)\phi, \quad \vartheta = \theta - \mathrm{diag}(l)\theta \tag{10}$$

$$l^T = (0, 0, \ldots, 1, 0, \ldots) \tag{11}$$

The nonzero entry corresponds to the placement of b_0 and u_{t-1}

Adaptive control - Method Overview



Adaptive Control - CE Control (Explicit)

Certainty equivalent self-tuner: Use the parameter estimate as the true parameters

$$y_t = \phi_t^T \theta + \epsilon_t, \quad \theta \rightarrow \hat{\theta} \tag{12}$$

Update parameter estimate

$$\epsilon_t = y_t - \phi_t^T \hat{\theta}_{t-1} \tag{13}$$

$$K_t = \frac{P_{t-1}\phi_t}{1 + \phi_t^T P_{t-1}\phi_t} \tag{14}$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t \epsilon_t \tag{15}$$

$$P_t = P_{t-1} - K_t(1 + \phi_t^T P_{t-1}\phi_t)K_t^T \tag{16}$$

Redesign the control law

$$u_t = \frac{w_{t+1} - S y_t}{R} = \frac{1}{\hat{B}} w_{t+1} - \frac{q(1 - \hat{A})}{\hat{B}} y_t = \frac{1}{\hat{b}_0} w_t - \frac{\varphi_t^T \hat{\vartheta}}{\hat{b}_0} \tag{17}$$

Cautious adaptive control

Adaptive Control - Cautious Control

Cautious adaptive control: Take estimation uncertainty into account

Conditional cost function

$$J = \mathbb{E}[(y_{t+1} - w_{t+1})^2 | Y_t] \tag{18}$$

$$= (\mathbb{E}[y_{t+1} - w_{t+1} | Y_t])^2 + \mathrm{Var}(y_{t+1} - w_{t+1} | Y_t) \tag{19}$$

Uncertainty of parameter estimate

$$\hat{\theta}_t \sim N(\theta, P_t) \tag{20}$$

$$\hat{b}_{0,t} = l^T \hat{\theta}_t \tag{21}$$

$$p_{b,t} = l^T P_t l \tag{22}$$

Control law

$$u_t = \frac{\hat{b}_{0,t}^2}{\hat{b}_{0,t}^2 + p_{b,t}} \left(\frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} - \frac{\varphi_t^T P_t l}{\hat{b}_{0,t}^2} \right) \tag{23}$$

Adaptive Control - Cautious Control

If $P_t \to 0$, cautious control is equivalent to certainty equivalent control

$$u_t = \frac{\hat{b}_{0,t}^2}{\hat{b}_{0,t}^2 + p_{b,t}} \left(\frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} - \frac{\varphi_t^T P_t l}{\hat{b}_{0,t}^2} \right) \quad \to \quad u_t = \frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} \tag{24}$$

If $\hat{\theta}_t \to \theta$, certainty equivalent control is equivalent to the known control

1. $P_t \to 0:$ Cautious $=$ CE $\neq$ known
2. $\hat{\theta}_t \to \theta:$ Cautious $\neq$ CE $=$ known
3. $P_t \to 0,\ \hat{\theta}_t \to \theta:$ Cautious $=$ CE $=$ known

Adaptive Control - Cautious Control - Usage

Cautious controller

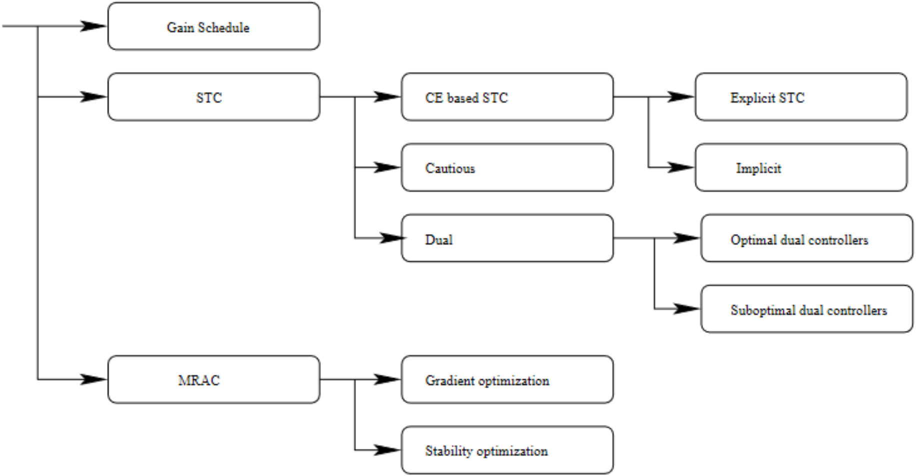
$$u_t = \frac{\hat{b}_{0,t}^2}{\hat{b}_{0,t}^2 + p_{b,t}} \left(\frac{w_{t+1} - \varphi_t^T \hat{\theta}}{\hat{b}_{0,t}} - \frac{\varphi_t^T P_t l}{\hat{b}_{0,t}^2} \right) \tag{25}$$

- *Turn-off phenomenon:* Control is dampened due to high uncertainty of b_0
- Consequence: Less information about b_0 for the next estimate, i.e., the uncertainty increases
- Turn-off usually occurs if b_0 or the control signal is small
- Conclusion: The cautious controller is useful for systems with constant or almost constant parameters, but unsuitable for general time-varying systems

Examples

Matlab example: Cautious self-tuner

Adaptive Control - Method Overview



Dual adaptive control

Adaptive Control - Optimal Dual Control

Dual control: Conditional expectation of the cost

$$J = \min_{U_t} \mathbb{E}\left[\sum_{i=1}^{N}(y_{t+i} - w_{t+i})^2\right] = \mathbb{E}_{Y_t}\left[\min_{U_t} \mathbb{E}\left[\sum_{i=1}^{N}(y_{t+i} - w_{t+i})^2 \middle| Y_t\right]\right] \tag{26}$$

If the parameter uncertainty is Gaussian, the conditional expectation is Gaussian (even if y_t is not)

$$\xi_t = [\varphi_{t-1}, \quad \hat{\theta}_t, \quad P_t] \tag{27}$$

contains the necessary information

If not Gaussian, it becomes computationally challenging to compute the hyper space and storage requirements increase

Adaptive Control - Optimal Dual Control

Bellman equation

$$V(\xi_t, t) = \min_{u_{t-1}} \mathbb{E}[(y_t - w_t)^2 + V(\xi_{t+1}, t)|Y_{t-1}] \quad (28)$$

The last (N'th) step is identical to the cautious controller

$$V(\xi_N, N) = \min_{u_{N-1}} \mathbb{E}[(y_N - w_N)^2|Y_{N-1}] \quad (29)$$

$$= (\varphi_{N-1}^T \theta_N - w_N)^2 + \sigma^2 + \varphi_{N-1}^T P_N \varphi_{N-1} - \frac{\hat{b}_{0,N} w_N - \varphi_{N-1}^T(\hat{b}_{0,N}\hat{\theta}_N + P_N l)}{\hat{b}_{0,N}^2 - p_{b,N}} \quad (30)$$

substituting into $V(\xi_{N-1}, N-1)$, the second last control can be computed, and so on

This is similar to the LQR – however, it does not have an analytical solution and must be solved numerically

Adaptive Control - Optimal Dual control

Fundamental paradox of adaptive control

1. Control objective: Small signals (control action)
2. Estimation: Large signals (probing action)

For the optimal N-step dual control problem, the solution is a compromise between these goals

1. Improved long-term estimation accuracy; sacrificing short-term loss
2. Probing adds active learning to the method

Cautious control ($N = 1$): The probing effects diminishes and any learning is "accidental"

Issue with dual control: Curse of dimensionality – the computational cost increases drastically with increasing hyperspace dimension and horizon

Adaptive Control - Sub-Optimal Dual control

As optimal dual control is impractical, sub-optimal dual controllers exist. They are based on the cautious controller and fix the issue with turn-off

Various approaches

1. Constrain the uncertainty
2. Extend the loss function
3. Serial expansion of the loss function
4. Add perturbation signals to the control

Sub-Optimal Dual Control - Constrained One-step Controller

Constrained one-step controller (minimum distance to zero control)

$$u_t = \begin{cases} u_c & \text{if } |u_c| \geq |u_l| \\ u_l \operatorname{sign}(u_c) & \text{if } |u_c| < |u_l| \end{cases} \tag{31}$$

u_c is the cautious controller input and u_l is a lower limit determined by us

The constraints do not prevent turn-off, but add extra perturbation when it happens

Sub-Optimal Dual Control - Constrained Uncertainty

Alternatively, constrain the uncertainty

$$\mathrm{Tr}(P_{t+1}^{-1}) \geq M \tag{32}$$

or constrain only p_b

$$p_{b,t+1} \leq \begin{cases} \gamma \hat{b}_{0,t+1}^2 & \text{if } p_{b,t} \leq \hat{b}_{0,t}^2 \\ \alpha p_{b,t} & \text{otherwise} \end{cases} \tag{33}$$

Sub-Optimal Dual control - extended loss function

Add uncertainty to the cost function

$$J = \mathbb{E}[(y_{t+1} - w_{t+1})^2 + \rho f(P_{t+1})] \tag{34}$$

f can be formulated in many ways

1. $f(P_{t+1}) = p_{b,t+1}$
2. $f(P_{t+1}) = R_2 \frac{p_{b,t+1}}{p_{b,t}}$
3. $f(P_{t+1}) = -\frac{\det(P_t)}{\det(P_{t+1})}$
4. $f(P_{t+1}) = -\epsilon_{t+1}^2$

This might lead to multiple local minima, and numerical optimization is required. Alternatively, use a second order serial expansion (e.g., a Taylor expansion)

Sub-Optimal Dual control - Extended Loss Function - Example

Third version of f

$$J = \mathbb{E}\left[(y_{t+1} - w_{t+1})^2 - \rho \frac{\det(P_t)}{\det(P_{t+1})} \middle| Y_t \right] \tag{35}$$

Ratio between determinants

$$\frac{\det(P_t)}{\det(P_{t+1})} = 1 + \phi_{t+1}^T P_t \phi_{t+1} \tag{36}$$

Analytical control law

$$u_t = \frac{\hat{b}_0(w_{t+1} - \varphi_{t+1}^T \hat{\vartheta}_t) + \rho (P_t l)^T \varphi_{t+1}}{\hat{b}_0^2 - \rho p_{b,t}} \tag{37}$$

Depending on ρ, we get specific controllers

1. $\rho = 0$: the CE controller
2. $\rho = -1$: the cautious controller
3. $\rho > 0$: an active learning controller

Sub-Optimal Dual control - Probing

Add probing/perturbation signal

$$u_t = u_t^c + u_t^x \tag{38}$$

Possible probing/perturbation signals include

1. PRBS
2. DOX: Design of excitation signal

They can be applied both at certain points in time (low uncertainty) or constantly

Questions

Questions?

Other interesting courses

- 34746 – Robust & fault-tolerant control
- 34791 – Topics in advanced control (PhD)
- 02619 – Model predictive control
- 02417 – Time series analysis
- 02427 – Advanced time series analysis

Subjects for special courses and MSc projects

Stochastic adaptive control of structured nonlinear systems

Bilinear systems (e.g., in heat exchangers)

$$x_{k+1} = Ax_k + Bu_k + G(x_k \otimes u_k) \tag{39}$$

Separable bilinear systems (e.g., in district heating)

$$x_{k+1} = Ax_k + Bu_k + G(x_k \otimes v_k) \tag{40}$$

Quadratic-bilinear systems (e.g., in nuclear fission)

$$x_{k+1} = Ax_k + bu_k + H(x_k \otimes x_k) + G(x_k \otimes u_k) \tag{41}$$

Develop control system examples

- TCLab device
- Electrolysis
- Tank systems
- ... or others

Control of systems with time delays

Can both be discrete- and continuous-time

- Ordinary differential equations (ODEs)

$$\dot{x}(t) = f(x(t), u(t), d(t), \theta)$$

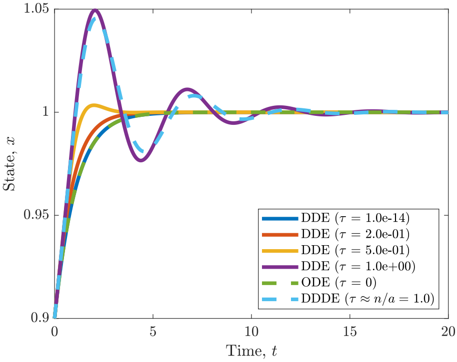
- Delay differential equations (DDEs) *with absolute delays*

$$\dot{x}(t) = f(x(t), x(t-\tau), u(t), d(t), \theta)$$

- Distributed delay differential equations (DDDEs)

$$\dot{x}(t) = f(x(t), z(t), u(t), d(t), \theta),$$

$$z(t) = \int_{-\infty}^{t} \alpha(t-s)x(s)\,\mathrm{d}s$$



Potential collaboration with Prof. John Wyller from NMBU, Norway.

Control of equilibrium/transport processes *(DAEs/PDAEs with complementarity conditions)*

DAEs

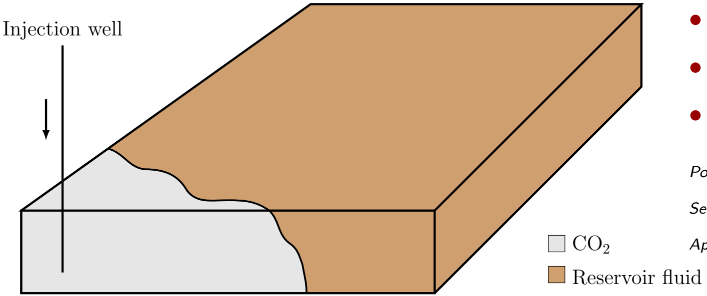
$$\dot{x}(t) = F(y(t), u(t), d(t), \theta),$$
$$0 = G(x(t), y(t), z(t), \theta)$$

Equilibrium 1/2

$$\min_{y(t)} \quad f(y(t)),$$
$$\text{s.t.} \quad g(y(t)) = x(t),$$
$$h(y(t)) = 0$$

Equilibrium 2/2

$$\min_{y(t)} \quad f(y(t)),$$
$$\text{s.t.} \quad g(y(t)) = x(t),$$
$$h(y(t)) = 0,$$
$$y(t) \geq 0$$

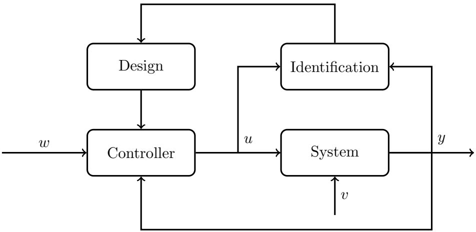


- CO_2 storage
- Geothermal energy
- Power-to-X

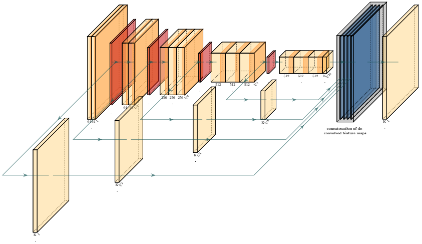
Potential collaborations with SemperCycle ApS, Nordic Hydrogen ApS, and MPI Magdeburg, Germany.

Learning-based adaptive control

Stochastic adaptive control



Reinforcement learning



Combined/hybrid approaches

Model reduction and numerical methods for optimal control

The linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

reduces to

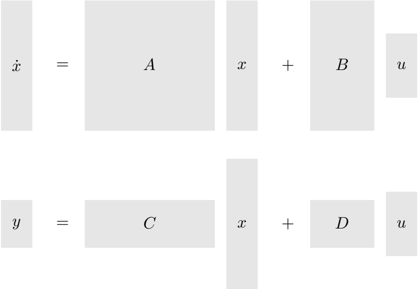
$$\dot{\hat{x}}(t) = A_r\hat{x}(t) + B_r u(t),$$
$$\hat{y}(t) = C_r\hat{x}(t) + D_r u(t).$$

What about

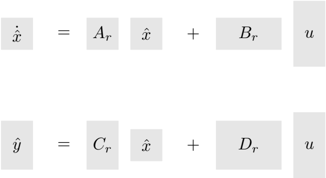
$$\dot{x}(t) = f(x(t), u(t))$$

and optimal control/dynamic optimization problems?

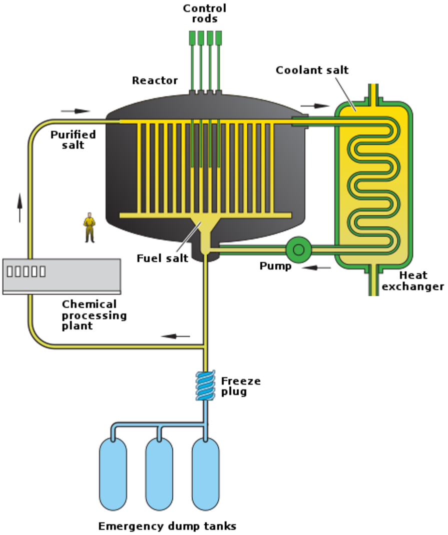
Original system



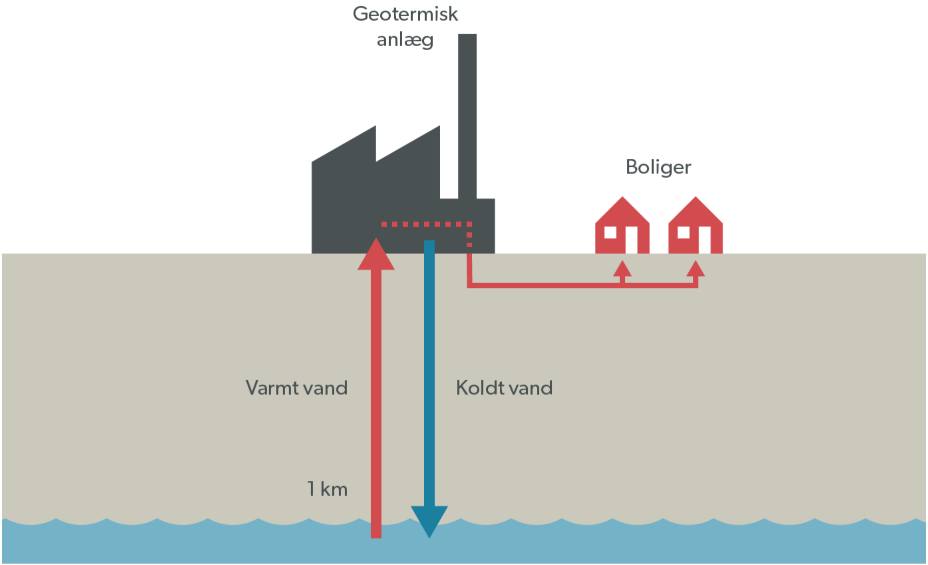
Reduced system



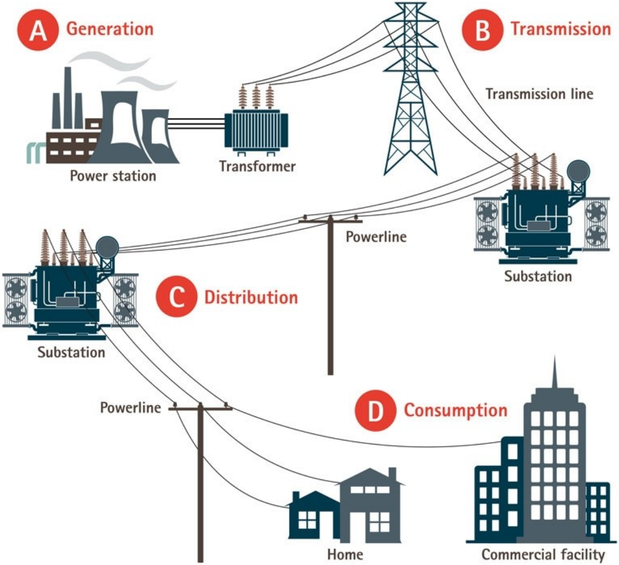
Nuclear fission



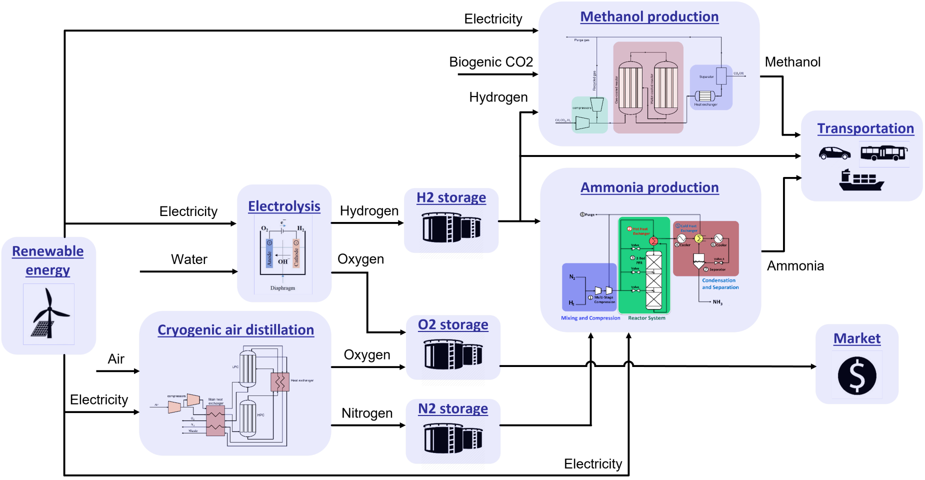
Geothermal energy



Power grids


A Generation
Power station
Transformer
B Transmission
Transmission line
Substation
Powerline
C Distribution
Substation
Powerline
D Consumption
Home
Commercial facility

Power-to-X


Electricity
Methanol production
Biogenic CO2
Hydrogen
Methanol
Transportation
Ammonia production
Electricity
Electrolysis
Hydrogen
H2 storage
Renewable energy
Water
Oxygen
Ammonia
Cryogenic air distillation
Air
Oxygen
O2 storage
Market
Electricity
Nitrogen
N2 storage
Electricity
Diaphragm
Mixing and Compression
Reactor System
Condensation and Separation