Stochastic Adaptive Control (02421)

Lecture 9

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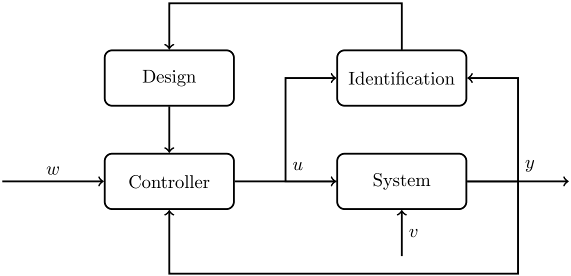
Lecture Plan

1. System theory
2. Stochastics
3. State estimation 1
4. State estimation 2
5. Optimal control 1
6. System identification 1 + adaptive control 1
7. External models + prediction
8. Optimal control 2
9. **Optimal control 3**
10. System identification 2
11. System identification 3 + model validation
12. System identification 4 + adaptive control 2
13. Adaptive control 3



Today's Agenda

- General minimum variance (GMV) control
- Generalized predictive control (GPC)
- Linear quadratic Gaussian (LQG) control

Follow-up from last time:

Questions?

Optimal External Control – Summary

So far we have considered the following controllers

A MV: $\mathbb{E}[y_{t+k}^2]$

B MV0: $\mathbb{E}[(y_{t+k} - w_t)^2]$

C MV1: $\mathbb{E}[(y_{t+k} - w_t)^2 + \rho u_t^2]$

D MV1a: $\mathbb{E}[(y_{t+k} - w_t)^2 + \rho \Delta u_t^2]$

E PZ: $\mathbb{E}[(A_m(q^{-1}) y_{t+k} - B_m(q^{-1}) w_t)^2]$

F GSP: $\mathbb{E}[(A_m(q^{-1}) y_{t+k} - B_m(q^{-1}) w_t)^2]$, $B_m = B^- \bar{B}_m$, $B = B^+ B^-$

We have discussed some of the limitations of these controllers

1. setpoints: A
2. constant disturbances: A
3. large control effort: A, B
4. undamped zeros (zeros outside of the unit circle): A, B, C, D, E

Today we will consider more methods that can deal with the 4^{th} issue

Generalized minimum variance control

Generalized Minimum Variance Strategy

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{1}$$

$\{e_t\}$ is a white-noise input with variance σ_e^2

Cost function

$$J_t = \mathbb{E}\Big[\,(\tilde{y}_t - \tilde{w}_t)^2 + \rho \tilde{u}_t^2\Big] \tag{2}$$

Filtered variables

$$\tilde{y}_t = H_y y_t = \frac{B_y(q^{-1})}{A_y(q^{-1})} y_t, \tilde{w}_t = H_w w_t = \frac{B_w(q^{-1})}{A_w(q^{-1})} w_t, \tilde{u}_t = H_u u_t = \frac{B_u(q^{-1})}{A_u(q^{-1})} u_t \tag{3}$$

$\rho > 0$ is a regularization parameter

Assume that

$$A_y(0) = A_w(0) = A_u(0) = B_u(0) = 1 \tag{4}$$

Generalized Minimum Variance Strategy

Control law

$$\Big[A_u BG + \alpha C B_u\Big] u_t = A_u \Big[C \frac{B_w}{A_w} w_t - \frac{S}{A_y} y_t - Gd\Big], \quad \alpha = \frac{\rho}{b_0} \qquad (5)$$

G and S are solutions to the Diophantine equation

$$B_y(q^{-1})C(q^{-1}) = A_y(q^{-1})A(q^{-1})G(q^{-1}) + q^{-k}S(q^{-1}), \qquad (6)$$

$G(0) = B_y(0)$, ord$[G] = k - 1$ and
ord$[S] = \max(n_a + n_{a_y} - 1, n_{b_y} + n_c - k)$

Note: The Diophantine equation is independent of the control filter $\frac{B_u}{A_u}$

Generalized Minimum Variance Strategy

Stationary closed-loop system

$$\Big[BA_uB_y + \alpha AB_uA_y\Big]y_t = q^{-k}\frac{B_w}{A_w}BA_uA_yw_t + RA_ye_t + \alpha A_yB_ud \quad (7)$$

$$\Big[BA_uB_y + \alpha AB_uA_y\Big]u_t = \frac{B_w}{A_w}AA_uA_yw_t - SA_ue_t + A_uB_yd \quad (8)$$

$$R = \Big[A_uBG + \alpha CB_u\Big] \quad (9)$$

We can affect the closed-loop poles using the control filter because it doesn't affect the Diophantine equation

GMV – Special cases

PZ-control

$$H_y(q) = A_m(q^{-1}), \quad H_w(q) = B_m(q^{-1}), \quad H_u(q) = 1, \quad \rho = 0 \tag{10}$$

Variant of MV_0 control

$$H_y(q) = 1, \quad H_w(q) = \frac{B_w(q^{-1})}{A_w(q^{-1})}, \quad H_u(q) = 1, \quad \rho = 0 \tag{11}$$

MV_{1a} control:

$$H_y(q) = 1, \quad H_w(q) = 1, \quad H_u(q) = 1 - q^{-1}, \quad \rho \neq 0 \tag{12}$$

MV_3 control:

$$H_y(q) = \frac{A_e(q^{-1})}{B_e(q^{-1})}, \quad H_w(q) = \frac{A_e(q^{-1})B_m(q^{-1})}{B_e(q^{-1})A_m(q^{-1})}, \quad H_u(q) = 1, \quad \rho = 0 \tag{13}$$

MV_3 control

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{14}$$

Cost function

$$J_t = \mathbb{E}\left[\left(\frac{A_e(q^{-1})}{B_e(q^{-1})}y_{t+k} - \frac{A_e(q^{-1})B_m(q^{-1})}{B_e(q^{-1})A_m(q^{-1})}w_t\right)^2\right] \tag{15}$$

Control law

$$BGu_t = C\frac{A_eB_m}{B_eA_m}w_t - \frac{S}{B_e}y_t - Gd \tag{16}$$

Stationary closed-loop system

$$y_t = q^{-k}\frac{B_m}{A_m}w_t + G\frac{B_e}{A_e}e_t \tag{17}$$

$$u_t = \frac{AB_m}{BA_m}w_t - \frac{SB_e}{BA_e}e_t - \frac{1}{B}d \tag{18}$$

Generalized Minimum Variance control – Example

ARMAX model

$$y_t - 1.7y_{t-1} + 0.7y_{t-2} = u_{t-1} + 0.5u_{t-2} + e_t + 1.5e_{t-1} + 0.9e_{t-2} \quad (19)$$

Cost function

$$\mathbb{E}\Big[(H_y(q)y_{t+1} - H_w(q)1)^2 + \rho\,(H_u(q)u_t)^2 \Big] \quad (20)$$

Polynomials

$$\begin{aligned} &A_y(q^{-1}) = 1, \quad A_w(q^{-1}) = 1, \quad A_u(q^{-1}) = 1 \\ &B_y(q^{-1}) = 1, \quad B_w(q^{-1}) = 1, \quad B_u(q^{-1}) = 1 - q^{-1} \end{aligned} \quad (21)$$

What is this controller also called?

Think about it for yourself for <u>one minute</u> and
then discuss with the person next to you for <u>one minute</u>.

Generalized Minimum Variance control – Example

Use that ord$[G] = 0$ and $g_0 = 1$

$$G = 1 \tag{22}$$

Diophantine equation

$$1 + 1.5q^{-1} + 0.9q^{-2} = 1 - 1.7q^{-1} + 0.7q^{-2} + s_1q^{-1} + s_2q^{-2} \tag{23}$$

Match coefficients

$$1.5 = -1.7 + s_1, \tag{24a}$$

$$0.9 = 0.7 + s_2 \tag{24b}$$

Solution

$$s_1 = 3.2, \tag{25a}$$

$$s_2 = 0.2 \tag{25b}$$

Generalized Minimum Variance control – Example

Optimal control law

$$\Big[A_u BG + \alpha C B_u\Big] u_t = A_u \Big[C \frac{B_w}{A_w} w_t - \frac{S}{A_y} y_t - Gd\Big], \quad \alpha = \frac{\rho}{b_0} \tag{26}$$

Specific control law

$$\begin{aligned} \Big[(1 + 0.5q^{-1}) + \rho(1 + 1.5q^{-1} + 0.9q^{-2})(1 - q^{-1})\Big] u_t \\ = (1 + 1.5q^{-1} + 0.9q^{-2}) w_t - (3.2 + 0.2q^{-1}) y_t \end{aligned} \tag{27}$$

Setpoint

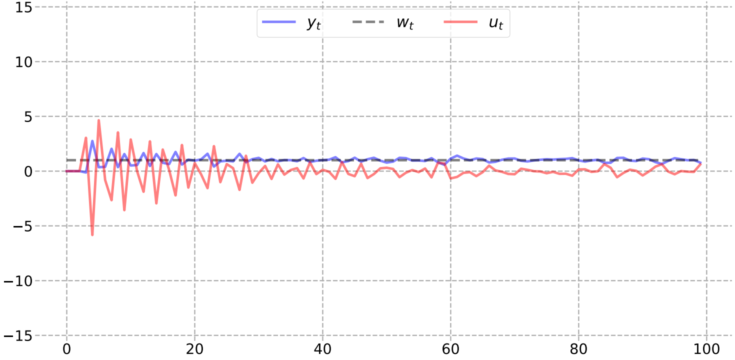
$$w_t = 1 \tag{28}$$

Control law

$$u_t = \frac{1}{1+\rho}\Big[(0.5\rho - 0.5)u_{t-1} + 0.6\rho u_{t-2} + 0.9\rho u_{t-3} + 3.4 - 3.2y_t - 0.2y_{t-1}\Big] \tag{29}$$

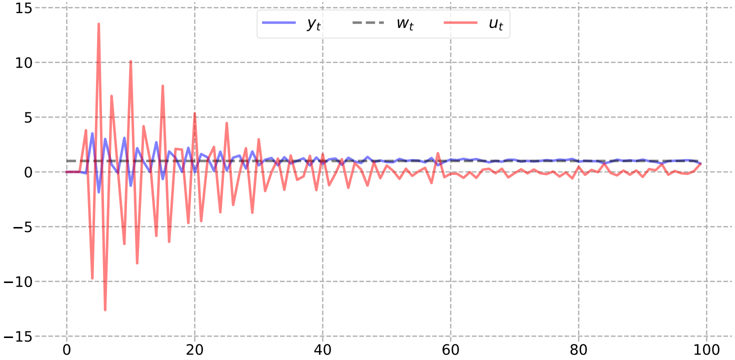
Generalized Minimum Variance Strategy – Example

Example $(\rho = 0.25)$



Generalized Minimum Variance Strategy – Example

Example $(\rho = 0.0)$



We obtain last week's example, the MV0 controller

Generalized Minimum Variance control – Exercise

ARMAX model

$$y_t + 0.5y_{t-1} + 0.3y_{t-2} = u_{t-1} + 0.1u_{t-2} + e_t + 1.1e_{t-1} - 0.3e_{t-2} \tag{30}$$

Cost function

$$\mathbb{E}\Big[\,(H_y(q)y_{t+1} - H_w(q)1)^2 + \rho\,(H_u(q)u_t)^2\,\Big] \tag{31}$$

Polynomials

$$\begin{aligned} A_y(q^{-1}) &= 1, \quad A_w(q^{-1}) = 1, \quad A_u(q^{-1}) = 1 \\ B_y(q^{-1}) &= 1, \quad B_w(q^{-1}) = 1, \quad B_u(q^{-1}) = 1 - q^{-1} \end{aligned} \tag{32}$$

Solve the exercise in 15 min.

Generalized Minimum Variance control – Exercise

Use that ord$[G] = 0$ and $g_0 = 1$

$$G = 1 \tag{33}$$

Diophantine equation

$$1 + 1.1q^{-1} - 0.3q^{-2} = 1 + 0.5q^{-1} + 0.3q^{-2} + s_1q^{-1} + s_2q^{-2} \tag{34}$$

Match coefficients

$$1.1 = 0.5 + s_1, \tag{35a}$$
$$-0.3 = 0.3 + s_2 \tag{35b}$$

Solution

$$s_1 = 0.6, \tag{36a}$$
$$s_2 = -0.6 \tag{36b}$$

Generalized Minimum Variance control – Exercise

Optimal control law

$$\Big[A_u BG + \alpha C B_u\Big]u_t = A_u\Big[C\frac{B_w}{A_w}w_t - \frac{S}{A_y}y_t - Gd\Big], \quad \alpha = \frac{\rho}{b_0} \tag{37}$$

Specific control law

$$\begin{aligned}\Big[(1+0.1q^{-1}) + \rho(1+1.1q^{-1}-0.3q^{-2})(1-q^{-1})\Big]u_t \\ = (1+1.1q^{-1}-0.3q^{-2})w_t - (0.6-0.6q^{-1})y_t\end{aligned} \tag{38}$$

Setpoint

$$w_t = 2 \tag{39}$$

Control law

$$u_t = \frac{1}{1+\rho}\Big[(0.1\rho+0.1)u_{t-1} - 1.4\rho u_{t-2} + 0.3\rho u_{t-3} + 3.6 - 0.6y_t + 0.6y_{t-1}\Big] \tag{40}$$

Generalized predictive control

Generalized Predictive Control (GPC)

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t \tag{41}$$

Cost function

$$J_t = \mathbb{E}\left[\sum_{i=1}^{N}(y_{t+i} - w_{t+i})^T q_i (y_{t+i} - w_{t+i}) + u_{t+i-1}^T \rho_i u_{t+i-1}\right] \tag{42}$$

Generalized Predictive Control – Prediction

Prediction and Diophantine equation

$$y_{t+m} = \frac{BG_m}{C} u_{t+m-k} + \frac{S_m}{C} y_t + G_m e_{t+m}, \tag{43}$$

$$C = AG_m + q^{-k} S_m, \tag{44}$$

$$ord[G_m] = m - 1, \quad ord[S_m] = \max\{n_a - 1, n_c - m\} \tag{45}$$

Rewrite future output by introducing another Diophantine equation

$$y_{t+m} = H_{m+1} u_{t+m} + \frac{F_{m+1}}{C} u_{t-1} + \frac{S_m}{C} y_t + G_m e_{t+m}, \tag{46}$$

$$q^{-k} BG_m = CH_{m+1} + q^{-m-1} F_{m+1}, \tag{47}$$

$$ord[H_{m+1}] = m, \quad ord[F_{m+1}] = \max\{n_c - 1, n_b + k - 1\} \tag{48}$$

GPC - Compact notation

Compact notation ($n = \max\{n_a - 1, n_c - 1\}$)

$$Y_t = \begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \vdots \\ y_{t+N} \end{bmatrix}, \; U_t = \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+N-1} \end{bmatrix}, \; W_t = \begin{bmatrix} w_{t+1} \\ w_{t+2} \\ \vdots \\ w_{t+N} \end{bmatrix}, \; E_t = \begin{bmatrix} e_{t+1} \\ e_{t+2} \\ \vdots \\ e_{t+N} \end{bmatrix}, \tag{49}$$

$$H = \begin{bmatrix} h_0 & 0 & \dots & 0 \\ h_1 & h_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ h_N & h_{N-1} & \dots & h_0 \end{bmatrix}, \; F = \begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix}, \; S = \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix}, \; G = \begin{bmatrix} G_1 \\ \vdots \\ G_N \end{bmatrix}, \tag{50}$$

$$Y_o = \frac{1}{C(q^{-1})} \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-(n-1)} \end{bmatrix}, \; U_o = \frac{1}{C(q^{-1})} \begin{bmatrix} u_{t-1} \\ u_{t-2} \\ \vdots \\ u_{t-n} \end{bmatrix} \tag{51}$$

GPC

Vector of future outputs

$$Y_t = SY_o + HU_t + FU_o + GE_t \tag{52}$$

Prediction and error

$$\hat{Y}_t = \mathbb{E}[Y_t|Y_o] = SY_o + HU_t + FU_0 = HU_t + f_t \tag{53}$$

$$\tilde{Y}_t = GE_t \tag{54}$$

Rewrite cost function

$$J_t = \mathbb{E}[(Y_t - W_t)^T Q_y (Y_t - W_t) + U_t^T Q_u U_t | Y_o] \tag{55}$$

$$= (\hat{Y}_t - W_t)^T Q_y (\hat{Y}_t - W_t) + U_t^T Q_u U_t + \mathrm{Var}(*) \tag{56}$$

$$Q_y(i,i) = q_i, \quad Q_u(i,i) = \rho_i \tag{57}$$

Control law

$$U_t^* = -[H^T Q_y H + Q_u]^{-1} H^T Q_y (f_t - W_t) = -L(f_t - W_t) \tag{58}$$

$$u_t = \gamma U_t^*, \quad \gamma = [1, 0, \ldots, 0] \tag{59}$$

Linear quadratic Gaussian control

LQG

ARMAX model

$$A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t \tag{60}$$

Cost function

$$J_t = \lim_{N\to\infty} \frac{1}{N} \sum_{i=t}^{N} \mathbb{E}[y_i^2 + \rho u_i^2] \tag{61}$$

Control law

$$R(q^{-1})u_t = -S(q^{-1})y_t \tag{62}$$

Diophantine equation

$$A_m(q^{-1})C(q^{-1}) = A(q^{-1})R(q^{-1}) + B(q^{-1})S(q^{-1}) \tag{63}$$

$$A_m(q^{-1})A_m(q) = B(q^{-1})B(q) + \rho A(q^{-1})A(q) \tag{64}$$

A_m is the stable solution to the second equation

What method can you use to solve (64)?
Think about it for yourself for one minute and
then discuss with the person next to you for one minute.

LQG – Setpoints and disturbances

ARMAX model

$$A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t + d \tag{65}$$

Control law (with setpoint and disturbance)

$$R(q^{-1})u_t = \frac{A_m(1)C(q^{-1})}{B(1)}w_t - S(q^{-1})y_t - \frac{R(1)}{B(1)}d \tag{66}$$

Note: The LQG for external models is equivalent to the stationary LQG for internal models

Stationary closed-loop system

$$y_t = \frac{A_m(1)}{B(1)}\frac{B(q^{-1})}{A_m(q^{-1})}w_t - \frac{R(q^{-1})}{A_m(q^{-1})}e_t \tag{67}$$

$$u_t = \frac{A_m(1)}{B(1)}\frac{A(q^{-1})}{A_m(q^{-1})}w_t - \frac{S(q^{-1})}{A_m(q^{-1})}e_t - \frac{R(1)}{B(1)}d \tag{68}$$

LQG – Example

ARMAX model

$$y_t - 1.7y_{t-1} + 0.7y_{t-2} = u_{t-1} + 0.5u_{t-2} + e_t + 1.5e_{t-1} + 0.9e_{t-2} \tag{69}$$

Setpoint and input regularization parameter

$$w_t = 1, \qquad \rho = 1 \tag{70}$$

Polynomials

$$A(q^{-1}) = 1 - 1.7q^{-1} + 0.7q^{-2}, \tag{71}$$

$$B(q^{-1}) = q^{-1} + 0.5q^{-2}, \qquad k = 1, \tag{72}$$

$$C(q^{-1}) = 1 + 1.5q^{-1} + 0.9q^{-2} \tag{73}$$

Example – Compute polynomial in Diophantine equation

Polynomial coefficients

```
>> A = [1,  -1.7,  0.7];
>> B = [0,   1,    0.5];
>> C = [1,   1.5,  0.9];
>> k = 1;
```

Right-hand side (`rho = 1; tol = 1e-14;`)

```
>> conv(B, fliplr(B))

ans =

    0     0.5000     1.2500     0.5000          0

>> conv(A, fliplr(A))

ans =

    0.7000    -2.8900     4.3800    -2.8900     0.7000

 >> rhs = conv(B, fliplr(B)) + rho*conv(A, fliplr(A))

rhs =

    0.7000    -2.3900     5.6300    -2.3900     0.7000

>> Am = spectralFactorization(rhs, tol)

Am =

    2.1414    -0.9683     0.3269
```

Example – Solve Diophantine equation

System matrix

```
>> M = [[A'; 0; 0], [0; A'; 0], [0; 0; A'], [B'; 0; 0], [0; B'; 0]]

M =

    1.0000         0         0         0         0
   -1.7000    1.0000         0    1.0000         0
    0.7000   -1.7000    1.0000    0.5000    1.0000
         0    0.7000   -1.7000         0    0.5000
         0         0    0.7000         0         0
```

Right-hand side

```
>> b = conv(Am, C)'

b =

    2.1414
    2.2438
    0.8017
   -0.3811
    0.2942
```

Solution

```
>> sol = M\b;
```

Example – Extract solution and create controller polynomials

Orders of polynomials (solution to Diophantine equation)

```
>> na = numel(A)-1;
>> nr = na + k-1;
>> ns = na - 1;
```

Solution to Diophantine equation

```
>> R = sol(1:nr+1)'

R =

    2.1414    1.3129    0.4203

>> S = sol(nr+2:nr+ns+2)'

S =

    4.5713   -1.1713
```

Remaining controller polynomial (effect of setpoint, w_t)

```
>> Q = sum(Am)/sum(B)*C

Q =

    1.0000    1.5000    0.9000
```

Example – Controller

Controller polynomials

$$R(q^{-1}) = 2.1414 + 1.3129q^{-1} + 0.4203q^{-2}, \tag{74}$$

$$Q(q^{-1}) = 1.0000 + 1.5000q^{-1} + 0.9000q^{-2}, \tag{75}$$

$$S(q^{-1}) = 4.5713 - 1.1713q^{-1} \tag{76}$$

Control law

$$R(q^{-1})u_t = Q(q^{-1})w_t - S(q^{-1})y_t \tag{77}$$

Resulting control law ($w_t = 1$)

$$u_t = \frac{1}{2.1414}\left(3.4 - 1.3129u_{t-1} - 0.4203u_{t-2} - 4.5713y_t + 1.1713y_{t-1}\right) \tag{78}$$

LQG – Exercise

ARMAX model

$$y_t + 0.5y_{t-1} + 0.3y_{t-2} = u_{t-1} + 0.1u_{t-2} + e_t + 1.1e_{t-1} - 0.3e_{t-2} \quad (79)$$

Setpoint and input regularization parameter

$$w_t = 2, \qquad \rho = 0.5 \quad (80)$$

Solve the exercise in 15 min.

Polynomials

$$A(q^{-1}) = 1 + 0.5q^{-1} + 0.3q^{-2}, \quad (81)$$
$$B(q^{-1}) = q^{-1} + 0.1q^{-2}, \qquad k = 1, \quad (82)$$
$$C(q^{-1}) = 1 + 1.1q^{-1} - 0.3q^{-2} \quad (83)$$

Exercise – Compute polynomial in Diophantine equation

Polynomial coefficients

```
>> A = [1, 0.5,  0.3];
>> B = [0, 1,    0.1];
>> C = [1, 1.1, -0.3];
>> k = 1;
```

Right-hand side (rho = 0.5; tol = 1e-14;)

```
>> conv(B, fliplr(B))

ans =

        0    0.1000    1.0100    0.1000         0

>> conv(A, fliplr(A))

ans =

    0.3000    0.6500    1.3400    0.6500    0.3000

>> rhs = conv(B, fliplr(B)) + rho*conv(A, fliplr(A))

rhs =

    0.1500    0.4250    1.6800    0.4250    0.1500

>> Am = spectralFactorization(rhs, tol)

Am =

    1.2529    0.3096    0.1197
```

Exercise – Solve Diophantine equation

System matrix

```
>> M = [[A'; 0; 0], [0; A'; 0], [0; 0; A'], [B'; 0; 0], [0; B'; 0]]

M =

    1.0000         0         0         0         0
    0.5000    1.0000         0    1.0000         0
    0.3000    0.5000    1.0000    0.1000    1.0000
         0    0.3000    0.5000         0    0.1000
         0         0    0.3000         0         0
```

Right-hand side

```
>> b = conv(Am, C)'

b =

    1.2529
    1.6878
    0.0844
    0.0388
   -0.0359
```

Solution

```
>> sol = M\b;
```

Exercise – Extract solution and create controller polynomials

Orders of polynomials (solution to Diophantine equation)

```
>> na = numel(A)-1;
>> nr = na + k-1;
>> ns = na - 1;
```

Solution to Diophantine equation

```
>> R = sol(1:nr+1)'

R =

    1.2529    0.4864   -0.1197

>> S = sol(nr+2:nr+ns+2)'

S =

    0.5750   -0.4724
```

Remaining controller polynomial (effect of setpoint, w_t)

```
>> Q = sum(Am)/sum(B)*C

Q =

    1.5293    1.6823   -0.4588
```

Exercise – Controller

Controller polynomials

$$R(q^{-1}) = 1.2529 + 0.4864q^{-1} - 0.1197q^{-2}, \tag{84}$$
$$Q(q^{-1}) = 1.5293 + 1.6823q^{-1} - 0.4588q^{-2}, \tag{85}$$
$$S(q^{-1}) = 0.5750 - 0.4724q^{-1} \tag{86}$$

Control law

$$R(q^{-1})u_t = Q(q^{-1})w_t - S(q^{-1})y_t \tag{87}$$

Resulting control law ($w_t = 2$)

$$u_t = \frac{1}{1.2529}\left(5.5056 - 0.4864u_{t-1} + 0.1197u_{t-2} - 0.5750y_t + 0.4724y_{t-1}\right) \tag{88}$$

Questions

Questions?