Stochastic Adaptive Control (02421)

Lecture 8

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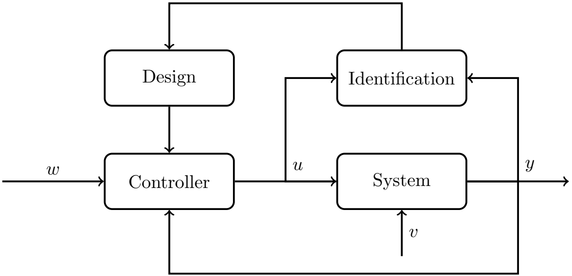
Lecture Plan

1. System theory
2. Stochastics
3. State estimation 1
4. State estimation 2
5. Optimal control 1
6. System identification 1 + adaptive control 1
7. External models + prediction
8. **Optimal control 2**
9. Optimal control 3
10. System identification 2
11. System identification 3 + model validation
12. System identification 4 + adaptive control 2
13. Adaptive control 3



Today's Agenda

- Follow-up from last lecture
- Minimum variance (MV) control
- Pole-zero (PZ) control
- General stochastic pole placement (GSP)

Follow-up from last time: Q3

System

$$x_{k+1} = \begin{bmatrix} 1 & -0.5 \\ 0.4 & -0.7 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.3 \end{bmatrix} e_k, \quad e_k \sim N(0, 3) \tag{1}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + 0.5512 e_k \tag{2}$$

DC-gain

$$K_{dc} = C(I - A)^{-1} G + F = 8.3012 \tag{3}$$

AC-gain

$$P_x = \mathrm{dlyap}(A, G\sigma_e^2 G^T), \tag{4}$$

$$\sigma_y^2 = C P_x C^T + D\sigma_e^2 D^T, \tag{5}$$

$$K_{ac} = \frac{\sigma_y^2}{\sigma_e^2} = \frac{13.1771}{3} = 4.3924 \tag{6}$$

Follow-up from last time

Questions?

External Control

External (ARMAX) model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t \tag{7}$$

Control law

$$R(q^{-1})u_t = Q(q^{-1})w_t - S(q^{-1})y_t + \gamma \tag{8}$$

Basis of optimal control laws

$$\min_{u_t} J_t(y_{t+k}, u_t) \tag{9}$$

Minimum variance control

Minimum Variance Control

System

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t \tag{10}$$

B and C are stable

Minimize the variance

$$J_t = \mathbb{E}[y_{t+k}^2] \tag{11}$$

G and S are the solution to the simple Diophantine equation

$$C(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}S(q^{-1}) \tag{12}$$

Minimum Variance Control

m-step prediction and error

$$\hat{y}_{t+m} = \frac{1}{C(q^{-1})}(B(q^{-1})G_m(q^{-1})u_{t+m-k} + S_m(q^{-1})y_t), \tag{13}$$

$$\tilde{y}_{t+m} = G_m(q^{-1})e_{t+m} \tag{14}$$

Cost function (we exploit that $\hat{y}_{t+m} \perp \tilde{y}_{t+m}$)

$$J_t = \mathbb{E}[y_{t+k}^2] = \mathbb{E}\left[\left(\frac{1}{C(q^{-1})}(B(q^{-1})G_k(q^{-1})u_t + S_k(q^{-1})y_t)\right)^2\right] \tag{15}$$

$$+\,\mathbb{E}\left[\left(G_k(q^{-1})e_{t+k}\right)^2\right] \tag{16}$$

Optimal control law

$$B(q^{-1})G_k(q^{-1})u_t = -S_k(q^{-1})y_t \tag{17}$$

Minimum Variance Control

Stationary closed-loop system

$$y_t = G_k(q^{-1})e_t, \quad u_t = -\frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})}y_t = -\frac{S_k(q^{-1})}{B(q^{-1})}e_t \tag{18}$$

The closed loop poles are determined by BC

The minimum variance controller has issues with the following

1. set-points
2. constant disturbances
3. large control effort
4. undamped zeros (zeros outside of the unit circle)

MV_0 control

ARMAX system with constant disturbance

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{19}$$

Cost function

$$J_t = \mathbb{E}[(y_{t+k} - w_t)^2] \tag{20}$$

Optimal control law

$$u_t = \frac{C(q^{-1})}{B(q^{-1})G_k(q^{-1})}w_t - \frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})}y_t - \frac{1}{B(q^{-1})}d \tag{21}$$

MV_0 control

Stationary closed-loop system

$$y_t = q^{-k} w_t + G_k(q^{-1}) e_t + \frac{G_k(q^{-1})}{C(q^{-1})}(1 - q^{-k}) d \tag{22}$$

$$= q^{-k} w_t + G_k(q^{-1}) e_t, \tag{23}$$

$$u_t = \frac{A(q^{-1})}{B(q^{-1})} w_t - \frac{S_k(q^{-1})}{B(q^{-1})} e_t - \frac{1}{B(q^{-1})} d \tag{24}$$

The poles are determined by BC

If $d = w_t = 0$, the MV_0 control becomes the minimum variance control

The MV_0 controller still has issues with the following:

1. large control effort
2. undamped zeros

Example of MV0 controller

ARMAX model

$$y_t - 1.7y_{t-1} + 0.7y_{t-2} = u_{t-1} + 0.5u_{t-2} + e_t + 1.5e_{t-1} + 0.9e_{t-2} \tag{25}$$

Criterion

$$\min J_t = \mathbb{E}[(y_{t+1} - 1)^2] \tag{26}$$

Polynomials

$$A(q^{-1}) = 1 - 1.7q^{-1} + 0.7q^{-2}, \quad B(q^{-1}) = 1 + 0.5q^{-1}, \tag{27}$$

$$C(q^{-1}) = 1 + 1.5q^{-1} + 0.9q^{-2}, \qquad d = 0, \quad w_t = 1, \quad k = 1 \tag{28}$$

Example of MV0 controller

Diophantine equation

$$1 + 1.5q^{-1} + 0.9q^{-2} = \left(1 - 1.7q^{-1} + 0.7q^{-2}\right) G(q^{-1}) + q^{-1} S(q^{-1}) \quad (29)$$

Furthermore, $G(0) = 1$, ord$[G] = k - 1$ and ord$[S] = \max(n_a - 1, n_c - k)$

$$\text{ord}[G] = 0, \qquad \text{ord}[S] = 1 \quad (30)$$

Rewrite

$$1 + 1.5q^{-1} + 0.9q^{-2} = 1 - 1.7q^{-1} + 0.7q^{-2} + s_1 q^{-1} + s_2 q^{-2} \quad (31)$$

$$1.5 = -1.7 + s_1, \qquad 0.9 = 0.7 + s_2 \quad (32)$$

Match coefficients

$$G(q^{-1}) = 1, \quad (33)$$

$$S(q^{-1}) = s_1 + s_2 q^{-1} = 3.2 + 0.2q^{-1} \quad (34)$$

Example of MV0 controller

MV0 control law

$$u_t = \frac{C(q^{-1})}{B(q^{-1})G_k(q^{-1})} w_t - \frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})} y_t - \frac{1}{B(q^{-1})} d \tag{35}$$

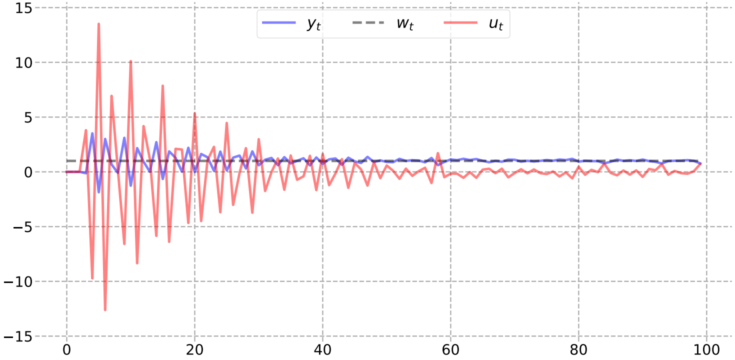
Optimal control law

$$(1 + 0.5q^{-1})u_t = (1 + 1.5q^{-1} + 0.9q^{-2})w_t - (3.2 + 0.2q^{-1})y_t \tag{36}$$

Substitute $w_t = 1$

$$u_t = 3.4 - 0.5u_{t-1} - 3.2y_t - 0.2y_{t-1} \tag{37}$$

Example of MV0 controller



The control input is large compared to the output

Exercise: MV0 controller

ARMAX model

$$y_t + 0.5y_{t-1} + 0.3y_{t-2} = u_{t-1} + 0.1u_{t-2} + e_t + 1.1e_{t-1} - 0.3e_{t-2} \quad (38)$$

Criterion

$$\min J_t = \mathbb{E}[(y_{t+1} - 2)^2] \quad (39)$$

Solve the exercise in 10 min.

Polynomials

$$A(q^{-1}) = 1 + 0.5q^{-1} + 0.3q^{-2}, \quad B(q^{-1}) = 1 + 0.1q^{-1}, \quad (40)$$

$$C(q^{-1}) = 1 + 1.1q^{-1} - 0.3q^{-2}, \quad d = 0, \quad w_t = 2, \quad k = 1 \quad (41)$$

Exercise: MV0 controller

Diophantine equation

$$1 + 1.1q^{-1} - 0.3q^{-2} = \left(1 + 0.5q^{-1} + 0.3q^{-2}\right) G(q^{-1}) + q^{-1}S(q^{-1}) \quad (42)$$

Furthermore, $G(0) = 1$, ord$[G] = k - 1$ and ord$[S] = \max(n_a - 1, n_c - k)$

$$\text{ord}[G] = 0, \qquad \text{ord}[S] = 1 \quad (43)$$

Rewrite

$$1 + 1.1q^{-1} - 0.3q^{-2} = 1 + 0.5q^{-1} + 0.3q^{-2} + s_1q^{-1} + s_2q^{-2} \quad (44)$$

$$1.1 = 0.5 + s_1, \qquad -0.3 = 0.3 + s_2 \quad (45)$$

Match coefficients

$$G(q^{-1}) = 1, \quad (46)$$

$$S(q^{-1}) = s_1 + s_2q^{-1} = 0.6 - 0.6q^{-1} \quad (47)$$

Exercise: MV0 controller

MV0 control law

$$u_t = \frac{C(q^{-1})}{B(q^{-1})G_k(q^{-1})} w_t - \frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})} y_t - \frac{1}{B(q^{-1})} d \tag{48}$$

Optimal control law

$$(1 + 0.1q^{-1})u_t = (1 + 1.1q^{-1} - 0.3q^{-2})w_t - (0.6 - 0.6q^{-1})y_t \tag{49}$$

Substitute $w_t = 2$

$$u_t = 3.6 - 0.1u_{t-1} - 0.6y_t + 0.6y_{t-1} \tag{50}$$

MV_1 control

Penalize control action

$$J_t = \mathbb{E}[(y_{t+k} - w_t)^2 + \rho u_t^2] \tag{51}$$

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{52}$$

Optimal control law

$$\left(B(q^{-1})G_k(q^{-1}) + \frac{\rho}{b_0}C(q^{-1})\right)u_t = C(q^{-1})w_t - S_k(q^{-1})y_t - G_k(q^{-1})d \tag{53}$$

MV_1 control

Stationary closed-loop system

$$y_t = q^{-k}\frac{B(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}A(q^{-1})}w_t + \frac{B(q^{-1})G_k(q^{-1}) + \frac{\rho}{b_0}C(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}A(q^{-1})}e_t \quad (54)$$

$$+ \frac{\frac{\rho}{b_0}}{B(q^{-1}) + \frac{\rho}{b_0}A(q^{-1})}d, \quad (55)$$

$$u_t = \frac{A(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}A(q^{-1})}w_t - \frac{S_k(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}A(q^{-1})}e_t \quad (56)$$

$$- \frac{1}{B(q^{-1}) + \frac{\rho}{b_0}A(q^{-1})}d \quad (57)$$

If $A(1) \neq 0$ (no pure integrator), MV_1 contains a stationary error for nonzero setpoints

MV_{1a} control

Penalize changes in the control action

$$J_t = \mathbb{E}[(y_{t+k} - w_t)^2 + \rho(u_t - u_{t-1})^2] \tag{58}$$

Optimal control law

$$\left(B(q^{-1})G_k(q^{-1}) + \frac{\rho}{b_0}C(q^{-1})\Delta\right)u_t = C(q^{-1})w_t - S_k(q^{-1})y_t - G_k(q^{-1})d \tag{59}$$

$$\Delta = 1 - q^{-1} \tag{60}$$

Stationary closed-loop system

$$y_t = q^{-k}\frac{B(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})}w_t + \frac{B(q^{-1})G_k(q^{-1}) + \frac{\rho}{b_0}\Delta C(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})}e_t, \tag{61}$$

$$u_t = \frac{A(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})}w_t - \frac{S_k(q^{-1})}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})}e_t \tag{62}$$

$$- \frac{1}{B(q^{-1}) + \frac{\rho}{b_0}\Delta A(q^{-1})}d \tag{63}$$

Pole-zero control

Pole-zero control

Relax the objective of following the setpoint

$$\tilde{w}_t = q^{-k} \frac{B_m(q^{-1})}{A_m(q^{-1})} w_t \tag{64}$$

Minimize

$$y_{t+k} - \tilde{w}_{t+k} \qquad \text{or} \qquad A_m(q^{-1}) y_{t+k} - B_m(q^{-1}) w_t \tag{65}$$

Cost function

$$J_t = \mathbb{E}\left[\left(A_m(q^{-1}) y_{t+k} - B_m(q^{-1}) w_t\right)^2\right] \tag{66}$$

ARMAX model

$$A(q^{-1}) y_t = q^{-k} B(q^{-1}) u_t + C(q^{-1}) e_t + d \tag{67}$$

Optimal control law

$$u_t = \frac{C(q^{-1}) B_m(q^{-1})}{B(q^{-1}) G_k(q^{-1})} w_t - \frac{S_k(q^{-1})}{B(q^{-1}) G_k(q^{-1})} y_t - \frac{1}{B(q^{-1})} d \tag{68}$$

Diophantine equation

$$A_m(q^{-1}) C(q^{-1}) = A(q^{-1}) G_k(q^{-1}) + q^{-k} S_k(q^{-1}) \tag{69}$$

Pole-zero control

Stationary closed-loop system

$$y_t = q^{-k}\frac{B_m(q^{-1})}{A_m(q^{-1})}w_t + \frac{G_k(q^{-1})}{A_m(q^{-1})}e_t, \tag{70}$$

$$u_t = \frac{A(q^{-1})B_m(q^{-1})}{B(q^{-1})A_m(q^{-1})}w_t - \frac{S_k(q^{-1})}{B(q^{-1})A_m(q^{-1})}e_t - \frac{1}{B(q^{-1})}d \tag{71}$$

PZ-control has an issue with

1. undamped zeros

Example: PZ control

ARMAX model

$$y_t - 1.7y_{t-1} + 0.7y_{t-2} = u_{t-1} + 0.5u_{t-2} + e_t + 1.5e_{t-1} \tag{72}$$

Setpoint model

$$(1 - 0.5q^{-1} + 0.2q^{-2})\tilde{w}_t = w_{t-k} - 0.3w_{t-k-1} \tag{73}$$

Criterion

$$\min J_t = \mathbb{E}\left[\left(A_m(q^{-1})y_{t+1} - B_m(q^{-1})w_t\right)^2\right] \tag{74}$$

Example: PZ control

Diophantine equation

$$1 + q^{-1} - 0.55q^{-2} + 0.3q^{-3} = \left(1 - 1.7q^{-1} + 0.7q^{-2}\right) G(q^{-1}) + q^{-1} S(q^{-1}) \tag{75}$$

Furthermore, $G(0) = 1$, ord$[G] = k - 1$ and ord$[S] = \max(n_a - 1, n_c - k)$

$$\text{ord}[G] = 0, \qquad \text{ord}[S] = 2 \tag{76}$$

Rewrite

$$1+q^{-1} - 0.55q^{-2} + 0.3q^{-3} = 1 - 1.7q^{-1} + 0.7q^{-2} + s_1 q^{-1} + s_2 q^{-2} + s_3 q^{-3}, \tag{77}$$

$$1 = -1.7 + s_1, \qquad -0.55 = 0.7 + s_2, \qquad 0.3 = s_3 \tag{78}$$

Match coefficients

$$G(q^{-1}) = 1, \tag{79}$$

$$S(q^{-1}) = s_1 + s_2 q^{-1} + s_3 q^{-2} = 2.7 - 1.25q^{-1} + 0.3q^{-2} \tag{80}$$

Example: PZ control

PZ control law

$$u_t = \frac{C(q^{-1})B_m(q^{-1})}{B(q^{-1})G_k(q^{-1})}w_t - \frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})}y_t - \frac{1}{B(q^{-1})}d \qquad (81)$$

Optimal control law

$$(1 + 0.5q^{-1})u_t = (1 + 1.2q^{-1} - 0.45q^{-2})w_t \qquad (82)$$

$$- (2.7 - 1.25q^{-1} + 0.3q^{-2})y_t \qquad (83)$$

Exercise: PZ control

ARMAX model

$$y_t + 0.5y_{t-1} + 0.3y_{t-2} = u_{t-1} + 0.1u_{t-2} + e_t + 1.1e_{t-1} \tag{84}$$

Setpoint model

$$(1 - 0.5q^{-1} + 0.2q^{-2})\tilde{w}_t = w_{t-k} - 0.3w_{t-k-1} \tag{85}$$

Criterion

$$\min J_t = \mathbb{E}\left[\left(A_m(q^{-1})y_{t+1} - B_m(q^{-1})w_t\right)^2\right] \tag{86}$$

Solve the exercise in 10 min.

Exercise: PZ control

Diophantine equation

$$1 + 0.6q^{-1} - 0.35q^{-2} + 0.22q^{-3} = \left(1 + 0.5q^{-1} + 0.3q^{-2}\right) G(q^{-1}) + q^{-1} S(q^{-1}) \tag{87}$$

Furthermore, $G(0) = 1$, $\text{ord}[G] = k - 1$ and $\text{ord}[S] = \max(n_a - 1, n_c - k)$

$$\text{ord}[G] = 0, \qquad \text{ord}[S] = 2 \tag{88}$$

Rewrite

$$1 + 0.6q^{-1} - 0.35q^{-2} + 0.22q^{-3} = 1 + 0.5q^{-1} + 0.3q^{-2} \tag{89}$$

$$+ s_1 q^{-1} + s_2 q^{-2} + s_3 q^{-3}, \tag{90}$$

$$0.6 = 0.5 + s_1, \qquad -0.35 = 0.3 + s_2, \qquad 0.22 = s_3 \tag{91}$$

Match coefficients

$$G(q^{-1}) = 1, \tag{92}$$

$$S(q^{-1}) = s_1 + s_2 q^{-1} + s_3 q^{-2} = 0.1 - 0.65q^{-1} + 0.22q^{-2} \tag{93}$$

Exercise: PZ control

PZ control law

$$u_t = \frac{C(q^{-1})B_m(q^{-1})}{B(q^{-1})G_k(q^{-1})}w_t - \frac{S_k(q^{-1})}{B(q^{-1})G_k(q^{-1})}y_t - \frac{1}{B(q^{-1})}d \tag{94}$$

Optimal control law

$$(1 + 0.1q^{-1})u_t = (1 + 0.8q^{-1} - 0.33q^{-2})w_t \tag{95}$$

$$- (0.1 - 0.65q^{-1} + 0.22q^{-2})y_t \tag{96}$$

General stochastic pole placement

General Stochastic Pole Placement

ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{97}$$

Relaxed setpoint/setpoint model

$$A_m(q^{-1})\tilde{w}_t = q^{-k}B_m(q^{-1})w_t \tag{98}$$

Cost function

$$\mathbb{E}\left[\left(A_m(q^{-1})y_{t+k} - B_m(q^{-1})w_t\right)^2\right] \tag{99}$$

General Stochastic Pole Placement

Assumption: $B(q^{-1})$ can be factorized

$$B(q^{-1}) = B^{+}(q^{-1})B^{-}(q^{-1}), \tag{100}$$

$B^{+}(q^{-1})$ contains the zeros that can be cancelled and $B^{-}(q^{-1})$ contains undesired zeros

Polynomial used in setpoint model

$$B_m(q^{-1}) = B^{-}(q^{-1})\bar{B}_m(q^{-1}) \tag{101}$$

General Stochastic Pole Placement

Optimal control law

$$B^{+}(q^{-1})G(q^{-1})u_t = \bar{B}_m(q^{-1})A_o(q^{-1})w_t - S(q^{-1})y_t - \frac{G(q^{-1})}{B^{-}(q^{-1})}d \quad (102)$$

Diophantine equation

$$A_o(q^{-1})A_m(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}B^{-}(q^{-1})S(q^{-1}) \quad (103)$$

$G(0) = 1$, $\text{ord}[G] = k + n_{b-} - 1$ and
$\text{ord}[S] = \max(n_a - 1, n_{a_o} + n_{a_m} - k - n_{b-})$

The polynomial $A_o(q^{-1})$ is an arbitrary stable polynomial, called the observer polynomial. Often, $A_o = C$

General Stochastic Pole Placement - Closed loop

Stationary closed-loop system

$$y_t = q^{-k}\frac{\bar{B}_m(q^{-1})B^-(q^{-1})}{A_m(q^{-1})}w_t + \frac{G(q^{-1})}{A_m(q^{-1})}\frac{C(q^{-1})}{A_o(q^{-1})}e_t, \tag{104}$$

$$u_t = \frac{A(q^{-1})}{B^+(q^{-1})}\frac{\bar{B}_m(q^{-1})}{A_m(q^{-1})}w_t - \frac{S(q^{-1})}{A_m(q^{-1})B^+(q^{-1})}\frac{C(q^{-1})}{A_o(q^{-1})}e_t - \frac{1}{B(q^{-1})}d \tag{105}$$

A_o only affects the noise terms

General Stochastic Pole Placement - special case

Special case: $B^-(q^{-1}) = 1$, $B^+(q^{-1}) = B(q^{-1})$ and $A_o(q^{-1}) = C(q^{-1})$

$$B(q^{-1})G(q^{-1})u_t = \bar{B}_m(q^{-1})C(q^{-1})w_t - S(q^{-1})y_t - G(q^{-1})d \qquad (106)$$

Stationary closed-loop system

$$y_t = q^{-k}\frac{\bar{B}_m(q^{-1})}{A_m(q^{-1})}w_t + \frac{G(q^{-1})}{A_m(q^{-1})}e_t, \qquad (107)$$

$$u_t = \frac{A(q^{-1})}{B(q^{-1})}\frac{\bar{B}_m(q^{-1})}{A_m(q^{-1})}w_t - \frac{S(q^{-1})}{A_m(q^{-1})B(q^{-1})}e_t - \frac{1}{B(q^{-1})}d \qquad (108)$$

This is exactly as for the PZ controller

GSP algorithm

- Factorize $B = B^+B^-$
- Choose A_m, $\bar{B}_m$ and A_o, such that

$$DC\left[\frac{\bar{B}_m B^-}{A_m}\right] = \frac{\bar{B}_m(1)B^-(1)}{A_m(1)} = 1 \tag{109}$$

- Find S and G, by solving

$$A_o(q^{-1})A_m(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}B^-(q^{-1})S(q^{-1}) \tag{110}$$

- Use the controller

$$B^+(q^{-1})G(q^{-1})u_t = \bar{B}_m(q^{-1})A_o(q^{-1})w_t - S(q^{-1})y_t - \frac{G(q^{-1})}{B^-(q^{-1})}d \tag{111}$$

What type of Diophantine equation is (110)?
What method could you use to solve it?

Think about it for yourself for one minute and
then discuss with the person next to you for one minute.

GSP Derivation

ARMAX model

$$Ay_t = q^{-k}Bu_t + Ce_t + d \tag{112}$$

Control law

$$Ru_t = Qw_t - Sy_t + \gamma \tag{113}$$

Relaxed setpoint

$$\tilde{w} = q^{-k}\frac{B_m}{A_m}w_t = H_{y,w}w_t \tag{114}$$

Polynomials

$$B_m = \bar{B}_m B^-, \quad B = B^+B^- \tag{115}$$

B^- contains the system zeroes which will not be cancelled

GSP Derivation

Multiply system by R

$$ARy_t = q^{-k}BRu_t + RCe_t + Rd \tag{116}$$

Substitute control law ($Ru_t = Qw_t - Sy_t + \gamma$)

$$(AR + q^{-k}BS)y_t = q^{-k}BQw_t + RCe_t + q^{-k}B\gamma + Rd \tag{117}$$

Setpoint transfer function (should equal the desired transfer function)

$$H_{y,w} = q^{-k}\frac{BQ}{AR + q^{-k}BS} = q^{-k}\frac{B_m}{A_m} = q^{-k}\frac{\bar{B}_m B^-}{A_m} \tag{118}$$

Only Q can inject new zeroes

$$Q = A_o\bar{B}_m \tag{119}$$

GSP Derivation

Substitute $B = B^+B^-$

$$H_{y,w} = q^{-k}\frac{A_o\bar{B}_m B^+B^-}{AR + q^{-k}B^+B^-S} = q^{-k}\frac{\bar{B}_m B^-}{A_m} \tag{120}$$

Cancel a subset of the zeroes

$$R = B^+G \tag{121}$$

Resulting setpoint transfer function

$$H_{y,w} = q^{-k}\frac{A_o\bar{B}_m B^-}{AG + q^{-k}B^-S} = q^{-k}\frac{\bar{B}_m B^-}{A_m} \tag{122}$$

Resulting Diophantine equation

$$A_oA_m = AG + q^{-k}B^-S \tag{123}$$

GSP Derivation

Substitute expression for R and $B = B^+B^-$ and divide through by B^+

$$(AG + q^{-k}B^-S)y_t = q^{-k}B^-Qw_t + GCe_t + q^{-k}B^-\gamma + Gd \tag{124}$$

Choose γ such that it cancels out the disturbance

$$\gamma = -\frac{G}{B^-}d \tag{125}$$

Stationary closed-loop system

$$y_t = q^{-k}\frac{B^-\bar{B}_m(1)}{A_m}w_t + \frac{G}{A_m}\frac{C}{A_o}e_t \tag{126}$$

Condition for eliminating stationary error

$$\frac{B^-(1)\bar{B}_m(1)}{A_m(1)} = 1 \tag{127}$$

QRS controller implementation

QRS Controller Implementation

Control law

$$R(q^{-1})u_t = Q(q^{-1})w_t - S(q^{-1})y_t - \gamma \quad (128)$$

Implement controller using a state-space representation

$$X_{t+1}^r = A^r X_t^r + B^r \begin{bmatrix} y_t \\ w_t \end{bmatrix}, \quad (129)$$

$$u_t = C^r X_t^r + D^r \begin{bmatrix} y_t \\ w_t \end{bmatrix} + u_0, \quad (130)$$

$$u_0 = (C_\gamma (1 - A^r)^{-1} B_\gamma + D_\gamma)\gamma \quad (131)$$

A^r, B^r, C^r, and D^r are computed using Q, R and S

QRS Controller Implementation - direct approach

Reformulate control law

$$S(q^{-1})y_t + R(q^{-1})u_t - Q(q^{-1})w_t + \gamma = 0 \tag{132}$$

Further rewrite

$$\left[u_t, \ldots, y_t, \ldots, -w_t, \ldots, 1\right] \left[r_0, \ldots, s_0, \ldots, q_0, \ldots, \gamma\right]^T = 0 \tag{133}$$

Isolate u_t

$$u_t = -\left[0, u_{t-1}, \ldots, y_t, \ldots, -w_t, \ldots, 1\right] \left[r_0, \ldots, s_0, \ldots, q_0, \ldots, \gamma\right]^T / r_0 \tag{134}$$

$$= -\phi_r^T \theta_r / r_0 \tag{135}$$

QRS Controller Implementation - direct approach

Recursively, the direct approach becomes

```
% Load system
[A, B, k, C, d, s2] = system();

% Design controller
[Q, R, S, G] = dsnmv0(A, B, k, C);

nr   = length([R, Q, S]) + 1;
phir = zeros(nr, 1);
thr  = [R, Q, S, G(1)*d]';
pil  = 1 + [0, length(R), length([R, Q])];

for it = 1:nstp,
    % Setpoint
    w = wt(it);

    % Measure output of sytem
    [y, t] = measure();

    % Ru = Q*w - S*y - G*d
    phir(2:end) =  phir(1:end-1);
    phir(pil)   =  [0, -w, y]
    u           = -phir'*thr/thr(1)
    phir(1)     = u;

    % Actuate control to the sytem
    actuate(u);
end
```

The functions system, dsnmv0, measure, and actuate are pseudocode

Example: QRS controller implementation

MV0 controller

$$(1 + 0.5q^{-1})u_t = (1 - 1.7q^{-1} + 0.7q^{-2})w_t - (3.2 + 0.2q^{-1})y_t \quad (136)$$

Reformulate

$$(1 + 0.5q^{-1})u_t - (1 - 1.7q^{-1} + 0.7q^{-2})w_t + (3.2 + 0.2q^{-1})y_t = 0 \quad (137)$$

Further rewrite (omit γ in this case)

$$\left[u_t, \ldots, y_t, \ldots, -w_t, \ldots\right] \left[1,\ 0.5,\ 3.2,\ 0.2,\ 1,\ -1.7,\ 0.7\right]^T = 0 \quad (138)$$

Isolate u_t

$$u_t = -\left[0, \ldots, y_t, \ldots, -w_t, \ldots\right] \left[1,\ 0.5,\ 3.2,\ 0.2,\ 1,\ -1.7,\ 0.7\right]^T = 0, \quad (139)$$

$$= -\phi_r^T \theta_r / r_0, \quad \theta_r = \left[1,\ 0.5,\ 3.2,\ 0.2,\ 1,\ -1.7,\ 0.7\right] \quad (140)$$

Exercise: QRS controller implementation

PZ controller

$$(1 - 0.4q^{-1} + 0.15q^{-2} + 0.02q^{-3})u_t = (1 + 0.2q^{-1} + 0.15q^{-2} - 0.09q^{-3})w_t \quad (141)$$

$$- (0.1 - 0.65q^{-1} + 0.22q^{-2})y_t \quad (142)$$

Rewrite the controller as

$$u_t = -\left[0, u_{t-1}, \ldots, y_t, \ldots, -w_t, \ldots\right]\left[r_0, \ldots, s_0, \ldots, q_0, \ldots\right]^T / r_0 \quad (143)$$

$$= -\phi_r^T \theta_r / r_0 \quad (144)$$

What is θ_r? Solve the exercise in 10 min.

$$\theta_r = \left[1,\ -0.4,\ 0.15,\ 0.02,\ 0.1,\ -0.65,\ 0.22,\ 1,\ 0.2,\ 0.15,\ -0.09\right]$$

Questions

Questions?