Stochastic Adaptive Control (02421)

Lecture 7

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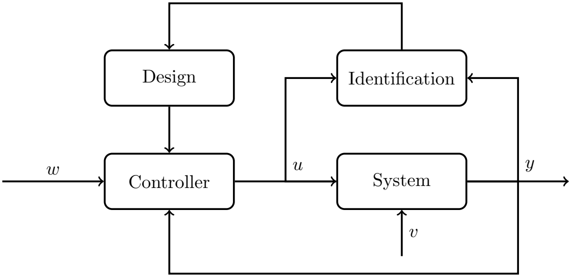
Lecture Plan

1. System theory
2. Stochastics
3. State estimation 1
4. State estimation 2
5. Optimal control 1
6. System identification 1 + adaptive control 1
7. **External models + prediction**
8. Optimal control 2
9. Optimal control 3
10. System identification 2
11. System identification 3 + model validation
12. System identification 4 + adaptive control 2
13. Adaptive control 3



Today's Agenda

- AR, MA, ARMA, and ARMAX processes
- General properties
- Spectral factorization
- Prediction

Follow-up from last time

Questions?

AR, MA, ARMA, and ARMAX processes

The Moving-Average (MA) Process

MA(n) process

$$y_t = \varepsilon_t + \sum_{k=1}^{n} c_k \varepsilon_{t-k}, \qquad c_0 = 1 \tag{1}$$

$\{\varepsilon_k\}$ is a white-noise process (independent and Gaussian with variance σ_ε^2)

The Moving-Average (MA) Process

Shift operator, q

$$q^{-1} y_t = y_{t-1}, \tag{2}$$

MA(n) process (compact notation)

$$y_t = C(q^{-1}) \varepsilon_t, \tag{3}$$

Polynomial

$$C(q^{-1}) = 1 + \sum_{k=1}^{n} c_k q^{-k} \tag{4}$$

Transfer function

$$C(z) = \frac{z^n + \sum_{k=1}^{n} c_k z^{n-k}}{z^n} \tag{5}$$

The Moving-Average (MA) Process

Properties of finite-order MA processes

- Always stationary
- Invertible if the zeros of C lie within the unit circle

Invertibility: The innovations can be represented as functions of past observations

The Moving-Average (MA) Process

Auto-covariance of MA(n) process

$$\gamma(k) = \begin{cases} \sigma_\varepsilon^2 \Big(c_k + c_1 c_{k+1} + \cdots + c_{n-k} c_n \Big), & |k| = 0, \ldots, q, \\ 0, & |k| > 0, \ldots, n \end{cases} \tag{6}$$

Variance (constant)

$$\sigma_y^2 = \gamma(0) = \sigma_\varepsilon^2 \left(1 + \sum_{k=1}^{n} c_k^2 \right) \tag{7}$$

Spectral density of MA(n) process

$$\phi(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} C\left(e^{i\omega}\right) C\left(e^{-i\omega}\right) = \frac{\sigma_\varepsilon^2}{2\pi} \left| 1 + \sum_{k=1}^{n} c_k e^{-ik\omega} \right|^2, \quad \omega \in [-\pi, \pi] \tag{8}$$

The Auto-Regressive (AR) Process

AR(m) process

$$y_t + \sum_{k=1}^{m} a_k y_{t-k} = \varepsilon_t, \qquad a_0 = 1 \tag{9}$$

$\{\varepsilon_k\}$ is a white-noise process

The Auto-Regressive (AR) Process

AR(m) process (compact notation)

$$A(q^{-1})y_t = \varepsilon_t \tag{10}$$

Polynomial

$$A(q^{-1}) = 1 + \sum_{k=1}^{m} a_k q^{-k} \tag{11}$$

Transfer function: $\dfrac{1}{A(z)}$

It is called *auto-regressive* because y_t can be viewed as a regression on past values

$$y_t = \varepsilon_t - \sum_{k=1}^{m} a_k y_{t-k} \tag{12}$$

The Auto-Regressive (AR) Process

Properties of finite-order AR processes

- Always invertible
- Stationary if the roots of A lie within the unit circle

Characteristic equation

$$A(z) = 0 \tag{13}$$

The Auto-Regressive (AR) Process

Auto-covariance function of an AR(m) process

$$\gamma(k) + \sum_{j=1}^{m} a_j \gamma(k-j) = 0, \qquad k > 0 \tag{14}$$

Initial condition

$$\gamma(0) + \sum_{j=1}^{m} a_j \gamma(j) = \sigma_\varepsilon^2 \tag{15}$$

Symmetry of auto-covariance functions: $\gamma(k) = \gamma(-k)$

Spectral density

$$\phi(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{1}{\left|1 + \sum_{k=1}^{m} a_k e^{-ik\omega}\right|^2} \tag{16}$$

The ARMA Process

ARMA(m,n) process

$$y_t + \sum_{k=1}^{m} a_k y_{t-k} = \varepsilon_t + \sum_{k=1}^{n} c_k \varepsilon_{t-k} \tag{17}$$

$\{\varepsilon_k\}$ is a white-noise process

The ARMA Process

ARMA(m,n) process (compact notation)

$$A(q^{-1})y_t = C(q^{-1})\varepsilon_t \tag{18}$$

Shift polynomials

$$A(q^{-1}) = 1 + \sum_{k=1}^{m} a_k q^{-k}, \qquad C(q^{-1}) = 1 + \sum_{k=1}^{n} c_k q^{-k} \tag{19}$$

Transfer function: $\dfrac{C(q^{-1})}{A(q^{-1})}$

Advanced External Model structures

ARMAX

$$A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t \tag{20}$$

Box-Jenkins

$$y_t = \frac{B(q^{-1})}{F(q^{-1})}u_t + \frac{C(q^{-1})}{D(q^{-1})}e_t \tag{21}$$

L-Structure

$$A(q^{-1})y_t = \frac{B(q^{-1})}{F(q^{-1})}u_t + \frac{C(q^{-1})}{D(q^{-1})}e_t \tag{22}$$

General properties

Covariance functions

Auto covariance function

$$r_x(s,t) = \mathrm{Cov}(x_s, x_t) = \mathbb{E}[x_s x_t^T] - \mathbb{E}[x_s]\mathbb{E}[x_t^T] \quad (23)$$

cross covariance function

$$r_{xy}(s,t) = \mathrm{Cov}(x_s, y_t) = \mathbb{E}[x_s y_t^T] - \mathbb{E}[x_s]\mathbb{E}[y_t^T] \quad (24)$$

Rules and notation

$$r_x(k) = r_x(t+k,t) \quad r_{xy}(k) = r_{xy}(t+k,t) \quad (25)$$

$$r_x(k) = r_x^T(-k) \quad r_{xy}(k) = r_{yx}^T(-k) \quad (26)$$

$$z_t = x_t + y_t : \qquad r_z(k) = r_x(k) + r_y(k) + r_{xy}(k) + r_{xy}^T(-k) \quad (27)$$

$$r_{zx}(k) = r_x(k) + r_{xy}^T(-k) \quad (28)$$

$$z_t = Ax_t : \qquad r_z(k) = Ar_x(k)A^T \quad r_{zx}(k) = Ar_x(k) \quad (29)$$

Variance and Spectral properties

ARMA model

$$A(q^{-1})y_t = C(q^{-1})e_t, \qquad y_t = \sum_{i=0}^{\infty} h_i q^{-i} e_t, \qquad e_t \sim N(0, \sigma_e^2) \tag{30}$$

Cross covariance

$$A(q^{-1})r_{ye}(k) = C(q^{-1})\delta_k \sigma_e^2, \qquad \delta_k = \begin{cases} 1 & k = 0, \\ 0 & \text{else,} \end{cases} \tag{31}$$

$$r_{ye}(k) = h_k \sigma_e^2 \tag{32}$$

Auto-covariance (Yule-Walker equation)

$$A(q^{-1})r_y(k) = C(q^{-1})r_{ey}(k) \tag{33}$$

$$r_y(k) = \sigma_e^2 h_k \star h_{-k} \tag{34}$$

Spectrum and Spectral density

Spectrum

$$\Psi_x(z) = \mathcal{Z}_b\{r_x(k)\} = \sum_{k=-\infty}^{\infty} r_x(k) z^{-k} \tag{35}$$

$$\Psi_{xy}(z) = \mathcal{Z}_b\{r_{xy}(k)\} = \sum_{k=-\infty}^{\infty} r_{xy}(k) z^{-k} \tag{36}$$

Spectral density (spectrum evaluated on the unit circle, $z = e^{j\omega}$)

$$\phi_x(\omega) = \Psi_x(e^{j\omega}) = \mathcal{F}(r_x(k)), \qquad \omega \in [-\pi, \pi] \tag{37}$$

Relation between spectral density and auto-covariance

$$r_x(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_x(\omega) e^{j\omega k} d\omega \tag{38}$$

$\mathcal{F}$ and $\mathcal{Z}_b$ are the Fourier and bilateral Z-transforms, respectively

Spectrum and spectral density

ARMA transfer function model

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_{n_b} z^{-n_b}}{1 + a_1 z^{-1} + \ldots + a_{n_a} z^{-n_a}} \tag{39}$$

Spectrum

$$\Psi(z) = H(z)H(z^{-1}) = \frac{\bar{b}_0 + \sum_{i=1}^{n_b} \bar{b}_i (z^i + z^{-i})}{\bar{a}_0 + \sum_{i=1}^{n_a} \bar{a}_i (z^i + z^{-i})} \tag{40}$$

$$\bar{a}_i = \sum_{j=i}^{n_a} a_j a_{j-i}, \quad \bar{b}_i = \sum_{j=i}^{n_b} b_j b_{j-i} \tag{41}$$

Spectral density

$$\phi(w) = \Psi(e^{jw}) = \frac{\bar{b}_0 + \sum_{i=1}^{n_b} 2\bar{b}_i \cos(iw)}{\bar{a}_0 + \sum_{i=1}^{n_a} 2\bar{a}_i \cos(iw)} \tag{42}$$

Spectrum and spectral density

ARMA Transfer function model

$$y_t = H(z)e_t \tag{43}$$

Spectrum and cross-spectrum

$$\Psi_y(z) = H(z)H(z^{-1})\sigma_e^2, \qquad \Psi_{ye}(z) = H(z)\sigma_e^2 \tag{44}$$

Spectral and cross-spectral density

$$\phi_y(w) = H(e^{jw})H(e^{-jw})\sigma_e^2, \qquad \phi_{ye}(w) = H(e^{-jw})\sigma_e^2 \tag{45}$$

Spectral factorization

Spectral Factorization

Problem: Assume that you know $\phi(\omega)$ from data

Can you determine $H(z)$ such that

$$\phi(\omega) = H(e^{j\omega})H(e^{-j\omega})\sigma^2? \tag{46}$$

The representation theorem: A weak stationary stochastic process with rational spectral density $\phi(\omega) \geq 0$ can be represented by

$$y_t = H(q)e_t, \qquad e_t \text{ is white} \tag{47}$$

$H(q)$ and its inverse are asymptotically stable and the spectral density of y_t is $\phi(\omega)$

Spectral factorization – iterative method

Assume that the polynomial $\Psi(z)$ is known

$$\Psi(z) = r_n z^{-n} + r_{n-1} z^{-(n-1)} + \ldots + r_{n-1} z^{n-1} + r_n z^n \tag{48}$$

Then, there exists a polynomial $P(z)$ such that

$$\Psi(z) = P(z^{-1})P(z) \tag{49}$$

$$P(z^{-1}) = p_0 + p_1 z^{-1} + \ldots + p_n z^{-n} \tag{50}$$

whose zeros are within the unit circle.

The spectrum of $H(z)$ can be considered a ratio of spectra:

$$\Psi_H(z) = H(z)H(z^{-1}) = \frac{C(z)}{A(z)} \frac{C(z^{-1})}{A(z^{-1})} = \frac{C(z)C(z^{-1})}{A(z)A(z^{-1})} = \frac{\Psi_C(z)}{\Psi_A(z)} \tag{51}$$

Spectral Factorization

We can compute the factorized polynomial using a correction polynomial, $X(z)$, and an iterative approach

1. $P_i(z^{-1})X_i(z) + P_i(z)X_i(z^{-1}) = 2\Psi(z)$
2. $P_{i+1}(z^{-1}) = \frac{1}{2}(P_i(z^{-1}) + X_i(z^{-1}))$

Each correction is obtained by solving the linear system

$$\begin{bmatrix} p_n & 0 & \dots & 0 \\ p_{n-1} & p_n & \dots & 0 \\ \vdots & \vdots & & \vdots \\ p_0 & p_1 & \dots & p_n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 & p_0 \\ 0 & \dots & p_0 & p_1 \\ \vdots & & \dots & \dots \\ p_0 & \dots & p_{n-1} & p_n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = 2 \begin{bmatrix} r_n \\ r_{n-1} \\ \vdots \\ r_0 \end{bmatrix} \tag{52}$$

Stochastic systems on external form

Stochastic systems on external form

Transfer function model

$$y_t = H_u(q)u_t + H_d(q)v_t, \qquad v_t \sim N(0,\sigma^2) \text{ is white} \tag{53}$$

Stochastic description

$$\mathbb{E}[y_t] = m_t = H_u(q)u_t, \tag{54}$$

$$A(q^{-1})r_{yv}(k) = C(q^{-1})r_v(k), \tag{55}$$

$$A(q^{-1})r_y(k) = C(q^{-1})r_{vy}(k), \tag{56}$$

$$r_{vy}(k) = r_{yv}^T(-k) \tag{57}$$

If v_t is not white, substitute $v_t = H_n e_t$ where H_n and its inverse are asymp. stable

Stochastic systems on external form

Asymptotically stable transfer function model

$$y_t = H_u(q)u_t + H_d(q)v_t, \qquad v_t \sim F(\mu_v, \sigma_v^2) \tag{58}$$

If v_t is weakly stationary, then y_t is also weakly stationary with the properties

$$\mathbb{E}[y_t] = \mu_{y,t} = H_u(1)u_0 + H_d(1)\mu_v, \tag{59}$$

$$A(q^{-1})r_{yv}(k) = C(q^{-1})r_v(k), \tag{60}$$

$$A(q^{-1})r_y(k) = C(q^{-1})r_{vy}(k), \tag{61}$$

$$r_{vy}(k) = r_{yv}^T(-k) \tag{62}$$

If v_t is Gaussian, y_t is strongly stationary

Spectra

$$\Psi_y(z) = H_d(z)\Psi_v(z)H_d^T(z^{-1}), \tag{63}$$

$$\Psi_{yv}(z) = H_d(z)\Psi_v(z) \tag{64}$$

System gains

System in internal and external form

$$x_{t+1} = Ax_t + Be_t, \tag{65}$$

$$y_t = Cx_t + De_t = (C(qI - A)^{-1}B + D)e_t = H(q)e_t \tag{66}$$

DC-Gain

$$K_{dc} = \frac{y_\infty}{e_\infty} = H(1) = C(I - A)^{-1}B + D \tag{67}$$

AC-Gain (also called variance-Gain)

$$K_{ac} = \frac{\sigma_y^2}{\sigma_e^2}, \qquad e_t \sim N(0, \sigma_e^2) \tag{68}$$

Equivalent expressions

$$P_x = AP_xA^T + B\sigma_e^2B^T, \qquad \sigma_y^2 = \int_{-\pi}^{\pi} H(e^{jw})H(e^{-jw})dw\,\sigma_e^2, \tag{69}$$

$$\sigma_y^2 = CP_xC^T + D\sigma_e^2D^T \tag{70}$$

System gains - Variance

External description of the variance

$$\sigma_y^2 = \int_{-\pi}^{\pi} H(e^{jw})H(e^{-jw})dw\,\sigma_e^2, \qquad H(z) = \frac{B(z)}{A(z)} \tag{71}$$

Variance of an nth order system

$$\sigma_y^2 = \frac{\sigma_e^2}{a_0}\sum_{i=0}^{n} b_i^i \beta_i \tag{72}$$

Parameters

$$a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k, \qquad \alpha_k = a_k^k / a_0^k, \qquad a_i^n = a_i, \tag{73}$$

$$b_i^{k-1} = b_i^k - \beta_k b_{k-i}^k, \qquad \beta_k = b_k^k / a_0^k, \qquad b_i^n = b_i \tag{74}$$

Diophantine equations

Polynomials and Transfer functions

Polynomials (time and frequency domain)

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}, \quad (75)$$

$$B(z) = b_0 + b_1 z^{-1} + \cdots + b_n z^{-n} \quad (76)$$

The polynomial is order n if $b_n \neq 0$ and $b_i = 0$ for $i > n$

If $b_0 = 1$, the polynomial is *monic*

The transfer function $H(q)$ can be written in infinitely many ways

$$H(q) = \frac{B(q^{-1})}{A(q^{-1})} = \frac{C(q^{-1})B(q^{-1})}{C(q^{-1})A(q^{-1})} \quad (77)$$

Polynomials and Transfer functions

Rewrite transfer function

$$\frac{B(q^{-1})}{A(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} \tag{78}$$

$$= b_0 + q^{-1} \frac{(b_1 - b_0 a_1) + (b_2 - b_0 a_2) q^{-1} + \cdots + (b_n - b_0 a_n) q^{-(n-1)}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} \tag{79}$$

Define the transfer function

$$H(q) = \frac{B(q^{-1})}{A(q^{-1})} = g_0 + q^{-1} \frac{S_1(q^{-1})}{A(q^{-1})}, \tag{80}$$

$$S_1(q^{-1}) = s_0 + s_1 q^{-1} + \ldots + s_{n_1} q^{-n_1}, \tag{81}$$

$$g_0 = b_0, \quad s_i = b_{i-1} - b_0 a_{i-1} \tag{82}$$

where $n_1 = n - 1$ is the order of S_1

Polynomials and Transfer functions

Repeat the rewriting for $\frac{S_1}{A}$, $\frac{S_2}{A}$, etc.

$$H(q) = g_0 + g_1 q^{-1} + \cdots + g_{m-1} q^{-(m-1)} + q^{-m} \frac{S_m(q^{-1})}{A(q^{-1})}, \tag{83}$$

$$= G_m(q^{-1}) + q^{-m} \frac{S_m(q^{-1})}{A(q^{-1})} \tag{84}$$

Diophantine equation

$$B(q^{-1}) = A(q^{-1}) G_m(q^{-1}) + q^{-m} S_m(q^{-1}) \tag{85}$$

The order of S_m is $\max(n_a - 1, n_b - m)$ and the order of G_m is $m - 1$

How to solve the Diophantine equation

This (simple) Diophantine equation can be solved iteratively

```
% Initialize
G = [];
S = [B, 0]; % Pad B with zeros to make S as long as A

for i = 1:m
    % Augment with first element of S
    G = [G, S(1)];

    % Update S
    S = [S(2:end) - S(1)*A(2:end), 0];
end

% Remove last element
S = S(1:end-1);
```

General Diophantine equation

General Diophantine equation

$$C(q^{-1}) = A(q^{-1})R(q^{-1}) + B(q^{-1})S(q^{-1}) \tag{86}$$

Polynomials

$$C(q^{-1}) = c_0 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}, \tag{87}$$

$$B(q^{-1}) = \qquad b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}, \quad b_0 = 0, \tag{88}$$

$$A(q^{-1}) = 1 + \ a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a} \tag{89}$$

The solution R and S exist if and only if all common factors of A and B are shared with C

In general, the solution is not unique

$$R(q^{-1}) = R_0(q^{-1}) + B(q^{-1})F(q^{-1}), \tag{90}$$

$$S(q^{-1}) = S_0(q^{-1}) - A(q^{-1})F(q^{-1}) \tag{91}$$

The solution is unique if $n_r = n_b - 1$ and $n_s = \max(n_a - 1, n_c - n_b)$

Diophantine - Sylvester method

Solution to the general Diophantine

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ a_1 & 1 & \ddots & \vdots & b_1 & 0 & \ddots & \vdots \\ a_2 & a_1 & & 0 & b_2 & b_1 & & 0 \\ \vdots & \vdots & & 1 & \vdots & \vdots & & 0 \\ a_{n_a} & a_{n_a-1} & \dots & a_1 & b_{n_b} & b_{n_b-1} & \dots & b_1 \\ 0 & a_{n_a} & & \vdots & 0 & b_{n_b} & & \vdots \\ \vdots & & \ddots & a_{n_a-1} & \vdots & & \ddots & b_{n_b-1} \\ 0 & 0 & & a_{n_a} & 0 & 0 & & b_{n_b} \end{array}\right] \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n_r} \\ s_0 \\ s_1 \\ \vdots \\ s_{n_s} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n_c} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (92)$$

Prediction

Prediction in the ARMA Structure

Weakly stationary process

$$A(q^{-1})y_t = C(q^{-1})e_t \tag{93}$$

e_t is a white noise signal $\mathbb{F}(0, \sigma^2)$ and A and C are monic

m-step prediction based on solution to the Diophantine equation

$$y_{t+m} = \frac{C(q^{-1})}{A(q^{-1})}e_{t+m} = G_m(q^{-1})e_{t+m} + \frac{S_m(q^{-1})}{A(q^{-1})}e_t \tag{94}$$

Prediction and error

$$\hat{y}_{t+m|t} = \frac{S_m(q^{-1})}{A(q^{-1})}e_t = \frac{S_m(q^{-1})}{A(q^{-1})}\left(\frac{A(q^{-1})}{C(q^{-1})}y_t\right) = \frac{S_m(q^{-1})}{C(q^{-1})}y_t, \tag{95}$$

$$\tilde{y}_{t+m|t} = G_m(q^{-1})e_{t+m} \tag{96}$$

$\hat{y}_t$ and $\tilde{y}_t$ are independent

This approach requires that $C(q^{-1})$ is inversely stable

Prediction in the ARMAX structure

System

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t \tag{97}$$

k is the control delay

m-step prediction

$$\hat{y}_{t+m|t} = \frac{1}{C(q^{-1})}(B(q^{-1})G_m(q^{-1})u_{t+m-k} + S_m(q^{-1})y_t), \tag{98}$$

$$\tilde{y}_{t+m|t} = G_m(q^{-1})e_{t+m} \tag{99}$$

Diophantine equation

$$C(q^{-1}) = A(q^{-1})G_m(q^{-1}) + q^{-m}S_m(q^{-1}) \tag{100}$$

The order of G and S are $m-1$ and $\max(n_a-1, n_c-m)$ and $G(0)=1$

Proof of ARMAX prediction

Rewrite future output using the Diophantine equation

$$y_{t+m} = \frac{C(q^{-1})}{C(q^{-1})} y_{t+m} \tag{101}$$

$$= \frac{A(q^{-1})G_m(q^{-1}) + q^{-m}S_m(q^{-1})}{C(q^{-1})} y_{t+m} \tag{102}$$

$$= \frac{G_m(q^{-1})}{C(q^{-1})} A(q^{-1}) y_{t+m} + \frac{S_m(q^{-1})}{C(q^{-1})} y_t \tag{103}$$

Substitute system description

$$y_{t+m} = \frac{G_m(q^{-1})}{C(q^{-1})} (B(q^{-1}) u_{t+m-k} + C(q^{-1}) e_{t+m}) + \frac{S_m(q^{-1})}{C(q^{-1})} y_t \tag{104}$$

$$= \frac{G_m(q^{-1})B(q^{-1})}{C(q^{-1})} u_{t+m-k} + \frac{S_m(q^{-1})}{C(q^{-1})} y_t + G_m(q^{-1}) e_{t+m} \tag{105}$$

$$= \hat{y}_{t+m|t} + \tilde{y}_{t+m|t} \tag{106}$$

Questions

Questions?

Examples

Today's Matlab example topics:

- Spectrum/Spectral density: back and forth
- Spectral factorization
- Addition of Spectra
- Plotting Spectra
- Matlab functions