Stochastic Adaptive Control (02421)

Lecture 8

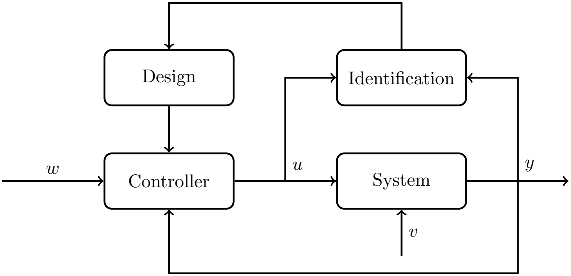
Tobias K. S. Ritschel

Section for Dynamical Systems, DTU Compute

DTU Compute
Department of Applied Mathematics and Computer Science

Lecture Plan

1. Systems theory
2. Stochastics
3. State estimation - Kalman filter 1
4. State estimation - Kalman filter 2
5. Optimal control 1 - internal models
6. External models
7. Prediction + optimal control 1 - external models
8. **Optimal control 2 - external models**
9. System identification 1
10. System identification 2
11. System identification 3 + model validation
12. Adaptive control 1
13. Adaptive control 2



Today's Agenda

- Follow-up from last lecture
- General Stochastic Pole Placement
- General Minimum Variance
- General Predictive Control
- LQG in external models

Follow-up from last time: polynomials

Let us consider the 2 polynomials

$$A = a_0 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3} \quad (1)$$

$$B = b_0 + b_1 q^{-1} + b_2 q^{-2} \quad (2)$$

How to add 2 polynomials:

First we pad the short polynomial:

$$A = a_0 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3} \quad (3)$$

$$B = b_0 + b_1 q^{-1} + b_2 q^{-2} + 0 q^{-3} \quad (4)$$

we then simply add each term:

$$A + B = (a_0 + b_0) + (a_1 + b_1) q^{-1} + (a_2 + b_2) q^{-2} + (a_3 + 0) q^{-3} \quad (5)$$

Follow-up from last time: polynomials

Consider the polynomials

$$A = a_0 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3} \tag{6}$$
$$B = b_0 + b_1 q^{-1} + b_2 q^{-2} \tag{7}$$

How to multiply 2 polynomials:

We multiply each term of A with all terms of B:

$$AB = a_0(b_0 + b_1 q^{-1} + b_2 q^{-2}) + a_1 q^{-1}(b_0 + b_1 q^{-1} + b_2 q^{-2}) \tag{8}$$
$$+ a_2 q^{-2}(b_0 + b_1 q^{-1} + b_2 q^{-2}) + a_3 q^{-3}(b_0 + b_1 q^{-1} + b_2 q^{-2}) \tag{9}$$
$$= a_0 b_0 + (a_1 b_0 + a_0 b_1) q^{-1} + (a_0 b_2 + b_1 a_1 + b_0 a_2) q^{-2} \tag{10}$$
$$+ (b_1 a_2 + b_2 a_1 + a_3 b_0) q^{-3} + (b_2 a_2 + b_1 a_3) q^{-4} + b_2 a_3 q^{-5} \tag{11}$$

In Matlab, if we use the parameter notation, we can use

$$AB = conv(A, B) \tag{12}$$
$$A = [a0, a1, a2, a3] \quad B = [b0, b1, b2] \tag{13}$$

Follow-up from last time: Q5.2

We are given the system

$$A(q^{-1})y_t = q^{-2}B(q^{-1})u_t + C(q^{-1})e_t, \quad e_t \in N(0, 0.1)$$
$$A(q^{-1}) = 1 - 1.5q^{-1} + 0.7q^{-2}$$
$$B(q^{-1}) = 1 - 0.5q^{-1}$$
$$C(q^{-1}) = 1 - 0.2q^{-1} + 0.5q^{-2}$$

To get the MV0-controller, we solve the Diophantine for G and S

$$C = AG + q^{-2}S, \quad [G] = [1, 1.3] \quad [S] = [1.75, -0.91] \tag{14}$$

and we then get:

$$Q = C \tag{15}$$
$$R = BG = 1 + 0.8q^{-1} - 0.65q^{-2} \tag{16}$$
$$[R] = conv(B, G) = [1, 0.8, -0.65] \tag{17}$$

and the control law:

$$Ru_t = Qw_t - Sy_t \tag{18}$$

Follow-up from last time:

Questions?

Optimal External Control - Resume

So far we have considered the following controllers

A MV: $E\{y_{t+k}^2\}$

B MV0: $E\{(y_{t+k} - w_t)^2\}$

C MV1: $E\{(y_{t+k} - w_t)^2 + \rho u_t^2\}$

D MV1a: $E\{(y_{t+k} - w_t)^2 + \rho(\Delta u_t)^2\}$

E PZ: $E\{(A_m(q^{-1})y_{t+k} - B_m(q^{-1})w_t)^2\}$

And we have discussed some of the issues of these controllers:

1. set-points: A
2. constant disturbances: A
3. large control effort : A, B
4. non damped zeros (zeros outside to the unit circle): A, B, C, D, E

Today we will consider methods that can deal with the 4th issue.

General Stochastic Pole Placement

The Pole-Zero (PZ) method can be generalized by accepting the presence of the unstable zeros.
We will consider the stochastic system given by the ARMAX model

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})\varepsilon_t + d \tag{19}$$

where $\{\varepsilon_t\}$ is a white-noise input with variance σ_ε^2.

The goal is still to construct a feedback strategy such that $\{y_t\}$ tracks a set-point model, $\{\tilde{w}_t\}$, given by

$$A_m(q^{-1})\tilde{w}_t = q^{-k}B_m(q^{-1})w_t, \tag{20}$$

where $\{w_t\}$ is some reference sequence (e.g. a set-point). The goal is to design the feedback strategy which minimizes the objective

$$\mathrm{E}\Big[\Big(A_m(q^{-1})y_{t+k} - B_m(q^{-1})w_t\Big)^2\Big]. \tag{21}$$

General Stochastic Pole Placement

We will now assume that $B(q^{-1})$ can be factorized according to

$$B(q^{-1}) = B^+(q^{-1})B^-(q^{-1}), \tag{22}$$

where $B^+(q^{-1})$ contains the zeros of the system which are well-behaved, and can be cancelled, while $B^-(q^{-1})$ contains unwanted zeros. Note that this factorization is subject to a design choice of the end-user; you need to specify which zeros to be contained in $B^+(q^{-1})$ and $B^-(q^{-1})$, respectively.

With this in-mind, the set-point model polynomial, $B_m(q^{-1})$, will be built upon this design choice according to

$$B_m(q^{-1}) = B^-(q^{-1})\bar{B}_m(q^{-1}) \tag{23}$$

where $\bar{B}_m(q^{-1})$ contains additional zeros in the resulting closed-loop transfer function.

General Stochastic Pole Placement

Using this factorization, the stochastic pole placement feedback strategy is then given by

$$B^{+}(q^{-1})G(q^{-1})u_t = \bar{B}_m(q^{-1})A_o(q^{-1})w_t - S(q^{-1})y_t - \frac{G(q^{-1})}{B^{-}(q^{-1})}d \quad (24)$$

where the polynomials, G and S, are solutions to the Diophantine equation given by

$$A_o(q^{-1})A_m(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}B^{-}(q^{-1})S(q^{-1}) \quad (25)$$

where $G(0) = 1$, ord$[G] = k + n_{b-} - 1$ and
ord$[S] = \max(n_a - 1, n_{a_o} + n_{a_m} - k - n_{b-})$. The polynomial $A_o(q^{-1})$ is an arbitrary stable polynomial, called the observer polynomial. Often, $A_o = C$.

From (24), we have the relation to the reference term

$$Q(q^{-1}) = \bar{B}_m(q^{-1})A_o(q^{-1}) \quad (26)$$

where we introduce the desired new zeroes ($\bar{B}_m(q^{-1})$).

General Stochastic Pole Placement - Closed loop

The closed-loop system takes the form

$$y_t = q^{-k}\frac{\bar{B}_m(q^{-1})B^-(q^{-1})}{A_m(q^{-1})}w_t + \frac{G(q^{-1})}{A_m(q^{-1})}\frac{C(q^{-1})}{A_o(q^{-1})}\varepsilon_t \tag{27}$$

$$u_t = \frac{A(q^{-1})}{B^+(q^{-1})}\frac{\bar{B}_m(q^{-1})}{A_m(q^{-1})}w_t - \frac{S(q^{-1})}{A_m(q^{-1})B^+(q^{-1})}\frac{C(q^{-1})}{A_o(q^{-1})}\varepsilon_t - \frac{1}{B(q^{-1})}d \tag{28}$$

Here we can see that choice of A_o only affects the noise terms in the close-loop

General Stochastic Pole Placement - special

If we consider the special case $B^-(q^{-1}) = 1$, $B^+(q^{-1}) = B(q^{-1})$ and $A_o(q^{-1}) = C(q^{-1})$,

$$B(q^{-1})G(q^{-1})u_t = \bar{B}_m(q^{-1})C(q^{-1})w_t - S(q^{-1})y_t - G(q^{-1})d \tag{29}$$

The closed-loop system takes the form

$$y_t = q^{-k}\frac{\bar{B}_m(q^{-1})}{A_m(q^{-1})}w_t + \frac{G(q^{-1})}{A_m(q^{-1})}\varepsilon_t \tag{30}$$

$$u_t = \frac{A(q^{-1})}{B(q^{-1})}\frac{\bar{B}_m(q^{-1})}{A_m(q^{-1})}w_t - \frac{S(q^{-1})}{A_m(q^{-1})B(q^{-1})}\varepsilon_t - \frac{1}{B(q^{-1})}d \tag{31}$$

Here we can see that we obtain the PZ-controller, this is even the case for some constant p such that $B^-(q^{-1}) = p$.

GSP algorithm

- Factorize $B = B^+ B^-$
- Choose A_m, $\bar{B}_m$ and A_o, such that

$$DC\left[\frac{\bar{B}_m B^-}{A_m}\right] = \frac{\bar{B}_m(1)B^-(1)}{A_m(1)} = 1 \tag{32}$$

- Find S and G, by solving

$$A_o(q^{-1})A_m(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}B^-(q^{-1})S(q^{-1}) \tag{33}$$

- Use the controller

$$B^+(q^{-1})G(q^{-1})u_t = \bar{B}_m(q^{-1})A_o(q^{-1})w_t - S(q^{-1})y_t - \frac{G(q^{-1})}{B^-(q^{-1})}d \tag{34}$$

What type of Diophantine equation is (33)?
What method could you use to solve it?

Think about it for yourself for <u>one minute</u> and
then discuss with the person next to you for <u>one minute</u>.

GSP Derivation

Consider the system:

$$Ay_t = q^{-k}Bu_t + Ce_t + d \tag{35}$$

We seek a controller in the form

$$Ru_t = Qw_t - Sy_t + \gamma \tag{36}$$

which minimizes the deviation from

$$\tilde{w} = q^{-k}\frac{B_m}{A_m}w_t = H_{y,w}w_t \tag{37}$$

where

$$B_m = \bar{B}_m B^-, \quad B = B^+B^- \tag{38}$$

and B^- contains the system zeroes which will not be cancelled.

GSP Derivation

We start by multiplying the system by R

$$ARy_t = q^{-k}BRu_t + RCe_t + Rd \tag{39}$$

We can the substitute (36) into the system

$$(AR + q^{-k}BS)y_t = q^{-k}BQw_t + RCe_t + q^{-k}B\gamma + Rd \tag{40}$$

The reference transfer function is then given by

$$H_{y,w} = q^{-k}\frac{BQ}{AR + q^{-k}BS} = q^{-k}\frac{B_m}{A_m} = q^{-k}\frac{\bar{B}_m B^-}{A_m} \tag{41}$$

We can then see that only Q can inject the new zeroes:

$$Q = A_o\bar{B}_m \tag{42}$$

GSP Derivation

$$H_{y,w} = q^{-k} \frac{A_o \bar{B}_m B^+ B^-}{AR + q^{-k} B^+ B^- S} = q^{-k} \frac{\bar{B}_m B^-}{A_m} \tag{43}$$

Since we only cancel a subset of the zeroes:

$$R = B^+ G \tag{44}$$

we have

$$H_{y,w} = q^{-k} \frac{A_o \bar{B}_m B^-}{AG + q^{-k} B^- S} = q^{-k} \frac{\bar{B}_m B^-}{A_m} \tag{45}$$

Leaving the following general Diophantine equation as a requirement

$$A_o A_m = AG + q^{-k} B^- S \tag{46}$$

GSP Derivation

Applying our found R

$$(AG + q^{-k}B^{-}S)y_t = q^{-k}B^{-}Qw_t + GCe_t + q^{-k}B^{-}\gamma + Gd \tag{47}$$

we can determine γ

$$\gamma = -\frac{G}{B^{-}}d \tag{48}$$

applying γ, Diophantine and Q, we get the closed-loop:

$$y_t = q^{-k}\frac{B^{-}\bar{B}_m(1)}{A_m}w_t + \frac{G}{A_m}\frac{C}{A_o}e_t \tag{49}$$

therefore if the reference filter is chosen as

$$\frac{B^{-}(1)\bar{B}_m(1)}{A_m(1)} = 1 \tag{50}$$

we get no stationary error in our reference tracking

Generalized Minimum Variance Strategy

We will now consider the generalized minimum variance strategy. We will again consider an ARMAX model on the form

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})\varepsilon_t + d, \tag{51}$$

where $\{\varepsilon_t\}$ is a white-noise input with variance σ_ε^2. Our goal is to define the feedback strategy which minimizes the objective

$$\mathrm{E}\Big[\,(\tilde{y}_t - \tilde{w}_t)^2 + \rho\tilde{u}_t^2\Big], \tag{52}$$

where we have defined the filtered versions as

$$\tilde{y}_t = H_y y_t = \frac{B_y(q^{-1})}{A_y(q^{-1})}y_t, \tilde{w}_t = H_w w_t = \frac{B_w(q^{-1})}{A_w(q^{-1})}w_t, \tilde{u}_t = H_u u_t = \frac{B_u(q^{-1})}{A_u(q^{-1})}u_t, \tag{53}$$

and $\rho > 0$ is a regularization parameter. We will assume that

$$A_y(0) = A_w(0) = A_u(0) = B_u(0) = 1. \tag{54}$$

Generalized Minimum Variance Strategy

The Generalized MV control strategy is then given by

$$\Big[A_u B G + \alpha C B_u\Big] u_t = A_u \Big[C \frac{B_w}{A_w} w_t - \frac{S}{A_y} y_t - G d\Big], \quad \alpha = \frac{\rho}{b_0} \tag{55}$$

for which the polynomials, G and S, are solutions to the Diophantine equation given by

$$B_y(q^{-1}) C(q^{-1}) = A_y(q^{-1}) A(q^{-1}) G(q^{-1}) + q^{-k} S(q^{-1}), \tag{56}$$

where $G(0) = B_y(0)$, ord$[G] = k - 1$ and
ord$[S] = \max(n_a + n_{a_y} - 1, n_{b_y} + n_c - k)$.

Note that the Diophantine equation is independent of the control filter $\frac{B_u}{A_u}$.

Generalized Minimum Variance Strategy

The closed-form of the system, then becomes

$$\Big[BA_uB_y + \alpha AB_uA_y\Big]y_t = q^{-k}\frac{B_w}{A_w}BA_uA_yw_t + RA_ye_t + \alpha A_yB_ud \quad (57)$$

$$\Big[BA_uB_y + \alpha AB_uA_y\Big]u_t = \frac{B_w}{A_w}AA_uA_yw_t - SA_ue_t + A_uB_yd \quad (58)$$

$$R = \Big[A_uBG + \alpha CB_u\Big] \quad (59)$$

Given the independence between the Diophantine equation and the controller filter, we can design the control filter to affect the closed-loop poles.

GMV - special versions

PZ-control

$$H_y(q^{-1}) = A_m(q^{-1}), \quad H_w(q^{-1}) = B_m(q^{-1}), \quad H_u(q^{-1}) = 1, \quad \rho = 0 \tag{60}$$

Variant of MV_0 control

$$H_y(q^{-1}) = 1, \quad H_w(q^{-1}) = \frac{B_w(q^{-1})}{A_w(q^{-1})}, \quad H_u(q^{-1}) = 1, \quad \rho = 0 \tag{61}$$

MV_{1a} control:

$$H_y(q^{-1}) = 1, \quad H_w(q^{-1}) = 1, \quad H_u(q^{-1}) = 1 - q^{-1}, \quad \rho \neq 0 \tag{62}$$

MV_3 control:

$$H_y(q^{-1}) = \frac{A_e(q^{-1})}{B_e(q^{-1})}, \quad H_w(q^{-1}) = \frac{A_e(q^{-1})B_m(q^{-1})}{B_e(q^{-1})A_m(q^{-1})}, \quad H_u(q^{-1}) = 1, \quad \rho = 0 \tag{63}$$

MV_3 control

Let us consider MV_3 control method:

$$J_t = E\left\{\left(\frac{A_e(q^{-1})}{B_e(q^{-1})}y_{t+k} - \frac{A_e(q^{-1})B_m(q^{-1})}{B_e(q^{-1})A_m(q^{-1})}w_t\right)^2\right\} \tag{64}$$

and let us consider the ARMAX system:

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t + d \tag{65}$$

The minimizing control is then given by the equation:

$$BGu_t = C\frac{A_eB_m}{B_eA_m}w_t - \frac{S}{B_e}y_t - Gd \tag{66}$$

Resulting in the closed-loop form:

$$y_t = q^{-k}\frac{B_m}{A_m}w_t - G\frac{B_e}{A_e}e_t \tag{67}$$

$$u_t = \frac{AB_m}{BA_m}w_t - \frac{SB_e}{BA_e}e_t - \frac{1}{B}d \tag{68}$$

Generalized Minimum Variance Strategy

Example

Consider the ARMAX model

$$y_t - 1.7y_{t-1} + 0.7y_{t-2} = u_{t-1} + 0.5u_{t-2} + \varepsilon_t + 1.5\varepsilon_{t-1} + 0.9\varepsilon_{t-2}. \quad (69)$$

We want to design a feedback strategy such that

$$\mathtt{E}\Big[\Big(H_y(q^{-1})y_{t+1} - H_w(q^{-1})1\Big)^2 + \rho\Big(H_u(q^{-1})u_t\Big)^2\Big], \quad (70)$$

is minimal. We have the following polynomials

$$\begin{aligned} A_y(q^{-1}) &= 1, \quad A_w(q^{-1}) = 1, \quad A_u(q^{-1}) = 1 \\ B_y(q^{-1}) &= 1, \quad B_w(q^{-1}) = 1, \quad B_u(q^{-1}) = 1 - q^{-1}. \end{aligned} \quad (71)$$

What is this controller also called?

Think about it for yourself for <u>one minute</u> and
then discuss with the person next to you for <u>one minute</u>.

Generalized Minimum Variance Strategy

Example

Based on the orders, we have

$$G = 1 \tag{72}$$

Using the Diophantine equation we can find the unknown coefficients, s_1 and s_2,

$$1 + 1.5q^{-1} + 0.9q^{-2} = 1 - 1.7q^{-1} + 0.7q^{-2} + s_1 q^{-1} + s_2 q^{-2}. \tag{73}$$

From this relation, we get the equations that

$$1.5 = -1.7 + s_1, \tag{74a}$$

$$0.9 = 0.7 + s_2, \tag{74b}$$

and thus

$$s_1 = 3.2, \tag{75a}$$

$$s_2 = 0.2. \tag{75b}$$

Generalized Minimum Variance Strategy

Example

The optimal controller is therefore given by

$$\Big[(1+0.5q^{-1})+\rho(1+1.5q^{-1}+0.9q^{-2})(1-q^{-1})\Big]u_t \\ = (1+1.5q^{-1}+0.9q^{-2})w_t-(3.2+0.2q^{-1})y_t, \tag{76}$$

where

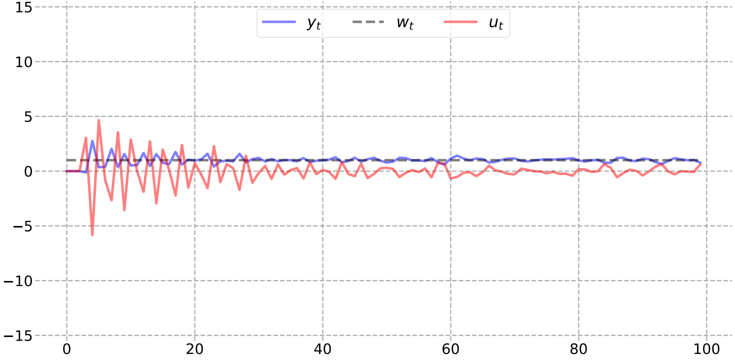
$$w_t = 1, \quad \text{for any} \quad t. \tag{77}$$

Inserting this, we find that

$$u_t = \frac{1}{1+\rho}\Big[(0.5\rho-0.5)u_{t-1}+0.6\rho u_{t-2}+0.9\rho u_{t-3}+3.4-3.2y_t-0.2y_{t-1}\Big]. \tag{78}$$

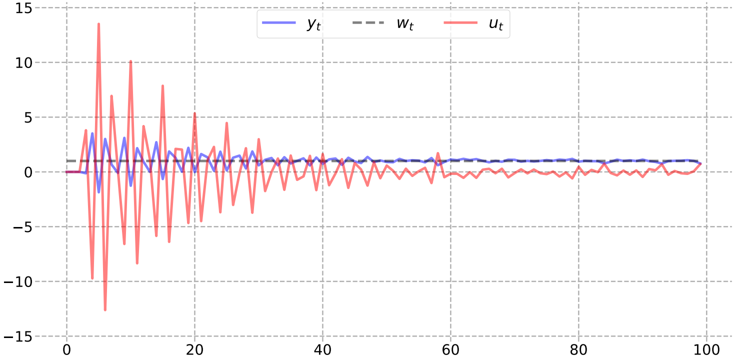
Generalized Minimum Variance Strategy

Example $(\rho = 0.25)$.



Generalized Minimum Variance Strategy

Example $(\rho = 0.0)$.



We obtain last weeks example, the MV0 controller.

GPC - Generalized Predictive Control

Let us consider the ARMAX system

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t \tag{79}$$

And a controller based on the cost function

$$J_t = E\left\{\sum_{i=1}^{N}(y_{t+i} - w_{t+i})^T q_i (y_{t+i} - w_{t+i}) + u_{t+i-1}^T \rho_i u_{t+i-1}\right\} \tag{80}$$

We remember the equation for external predictions, with corresponding Diophantine

$$y_{t+m} = \frac{BG_m}{C}u_{t+m-k} + \frac{S_m}{C}y_t + G_m e_{t+m} \tag{81}$$

$$C(q^{-1}) = AG_m + q^{-k}S_m \tag{82}$$

GPC - Vector definition

Let us define some vector variables:

$$Y_t = \begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \vdots \\ y_{t+N} \end{bmatrix}, \quad U_t = \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+N-1} \end{bmatrix}, \quad W_t = \begin{bmatrix} w_{t+1} \\ w_{t+2} \\ \vdots \\ w_{t+N} \end{bmatrix} \tag{83}$$

$$Y_o = \frac{1}{C(q^{-1})} \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-(n-1)} \end{bmatrix}, \quad U_o = \frac{1}{C(q^{-1})} \begin{bmatrix} u_{t-1} \\ u_{t-2} \\ \vdots \\ u_{t-n} \end{bmatrix}, \quad E_t = \begin{bmatrix} e_{t+1} \\ e_{t+2} \\ \vdots \\ e_{t+N} \end{bmatrix} \tag{84}$$

$$n = \max\{n_a - 1, n_c - 1\} \tag{85}$$

GPC - the control formulation

We can rewrite the future output by using an extra Diophantine equation:

$$y_{t+m} = H_{m+1}(q^{-1})u_{t+m} + \frac{F_{m+1}(q^{-1})}{C(q^{-1})}u_{t-1} + \frac{S_m(q^{-1})}{C(q^{-1})}y_t + G_m(q^{-1})e_{t+m} \quad (86)$$

$$q^{-k}B(q^{-1})G_m(q^{-1}) = C(q^{-1})H_{m+1}(q^{-1}) + q^{-m-1}F_{m+1}(q^{-1}) \quad (87)$$

We can then formulate the full prediction vectors as

$$\hat{Y}_t = SY_o + HU_t + FU_o = HU_t + f_t \quad (88)$$

$$\tilde{Y}_t = GE_t \quad (89)$$

where the matrices are given by

$$H = \begin{bmatrix} h_0 & 0 & \dots & 0 \\ h_1 & h_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ h_N & h_{N-1} & \dots & h_0 \end{bmatrix} \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix} \quad S = \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix} \quad G = \begin{bmatrix} G_1 \\ \vdots \\ G_N \end{bmatrix}$$

GPC

We can then rewrite our cost function:

$$Y_t = SY_0 + HU_t + FU_0 + GE_t \tag{90}$$

$$J_t = E\{(Y_t - W_t)^T Q_y (Y_t - W_t) + U_t^T Q_u U_t | Y_o\} \tag{91}$$

$$= (\hat{Y}_t - W_t)^T Q_y (\hat{Y}_t - W_t) + U_t^T Q_u U_t + Var(*) \tag{92}$$

$$Q_y(i,i) = q_i, \quad Q_u(i,i) = \rho_i \tag{93}$$

Where our prediction and error is given by

$$\hat{Y}_t = E\{Y_t | Y_o\} = SY_0 + HU_t + FU_0 = HU_t + f_t \tag{94}$$

$$\tilde{Y}_t = GE_t \tag{95}$$

GPC - the control formulation

Finding the optimum, then gives the receding horizon control law

$$U_t^* = -[H^T Q_y H + Q_u]^{-1} H^T Q_y (f_t - W_t) = -L(f_t - W_t) \tag{96}$$

$$u_t = \gamma U_t^*, \quad \gamma = [1, 0, \ldots, 0] \tag{97}$$

GPC - QRS form

The matrix form is not the only formulation of GPC, we can also use transfer function form or QRS control form. If our reference is constant we have $W_t = \mathbf{1}w_t$
Let us define a vector of delays:

$$\mathbf{q} = \begin{bmatrix} 1 & q^{-1} & \dots & q^{1-n} \end{bmatrix}^T \tag{98}$$

By rewriting Y_o and U_o, the control law becomes

$$u_t = \gamma L \mathbf{1} w_t - \frac{\gamma L S \mathbf{q}}{C(q^{-1})} y_t - \frac{\gamma L F \mathbf{q}}{C(q^{-1})} u_{t-1} \tag{99}$$

By isolating u_t we then get the QRS form

$$(C(q^{-1}) + q^{-1}\gamma L F \mathbf{q}) u_t = C(q^{-1}) \gamma L \mathbf{1} w_t - \gamma L S \mathbf{q} y_t \tag{100}$$

$$\Rightarrow R(q^{-1}) u_t = Q(q^{-1}) w_t - S(q^{-1}) y_t \tag{101}$$

LQG

We will consider the ARMAX strucutre

$$A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t \tag{102}$$

with the objective function

$$J_t = \lim_{N\to\infty} \frac{1}{N}\sum_{i=t}^{N} E\{y_i^2 + \rho u_i^2\} \tag{103}$$

The external LQG controller then takes the form

$$R(q^{-1})u_t = -S(q^{-1})y_t \tag{104}$$

with the Diophantine given as

$$A_m(q^{-1})C(q^{-1}) = A(q^{-1})R(q^{-1}) + B(q^{-1})S(q^{-1}) \tag{105}$$

$$A_m(q^{-1})A_m(q) = B(q^{-1})B(q) + \rho A(q^{-1})A(q) \tag{106}$$

Note that A_m is the stable solution to the second equation.

What method can you use to solve (106)?
Think about it for yourself for one minute and
then discuss with the person next to you for one minute.

LQG - setpoints and disturbances

Consider the ARMAX structure with disturbances

$$A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t + d \tag{107}$$

Then the controller with considering a reference w_t, is given by

$$R(q^{-1})u_t = \frac{A_m(1)C(q^{-1})}{B(1)}w_t - S(q^{-1})y_t - \frac{R(1)}{B(1)}d \tag{108}$$

This LQG for the external model, is equivalent to the stationary solution of the LQG for the internal models.

LQG

If we consider the closed-loop

$$y_t = \frac{A_m(1)}{B(1)} \frac{B(q^{-1})}{A_m(q^{-1})} w_t - \frac{R(q^{-1})}{A_m(q^{-1})} e_t \quad (109)$$

$$u_t = \frac{A_m(1)}{B(1)} \frac{A(q^{-1})}{A_m(q^{-1})} w_t - \frac{S(q^{-1})}{A_m(q^{-1})} e_t - \frac{R(1)}{B(1)} d \quad (110)$$

Special case 1:

$\rho = 0$, $B(q^{-1}) = q^{-k} B_1(q^{-1})$, where $B_1(q^{-1})$ is stable
results in $A_m(q^{-1}) = B_1(q^{-1})$

Special case 2:

$\rho = 0$, $B(q^{-1}) = q^{-k} B_1(q^{-1})$, where $B_1(q^{-1})$ is unstable
results in $A_m(q^{-1}) = B_{1+}(q^{-1}) B^*_{1-}(q^{-1})$

where $B_1(q^{-1}) = B_{1+}(q^{-1}) B_{1-}(q^{-1})$ and $B^*_{1-}(q^{-1})$ is $B_{1-}(q^{-1})$ with mirrored zeros:

$$B^*_{1-}(q^{-1}) B^*_{1-}(q) = B_{1-}(q^{-1}) B_{1-}(q) \quad (111)$$

Mirrored zeros

How do we find the mirrored zeros? Let us consider a single zero:

$$H(q) = (q - a), \quad H(q^{-1}) = \frac{1 - aq}{q} \tag{112}$$

$$H(q)H(q^{-1}) = \frac{(q - a)(1 - aq)}{q} \tag{113}$$

If $|a| > 1$, then we can simply define the mirrored zero as

$$H^*(q) = (1 - aq), \quad H^*(q^{-1}) = \frac{q - a}{q} \tag{114}$$

$$H^*(q)H^*(q^{-1}) = \frac{(q - a)(1 - aq)}{q} \tag{115}$$

Summary

LQG

1. $R(q^{-1})u_t = \frac{A_m(1)C(q^{-1})}{B(1)}w_t - S(q^{-1})u_t - \frac{R(1)}{B(1)}d$
2. equivalent to stationary LQG for internal models

GPC

1. $U_t^* = -[H^T Q_y H + Q_u]^{-1} H^T Q_y (f_t - W_t)$
2. uses receding horizon approach

Questions

Questions?

Examples

Today's Matlab example topics:

- simulation using transfer functions
- simulation of transfer function: internal method