Stochastic Adaptive Control (02421)

Lecture 5

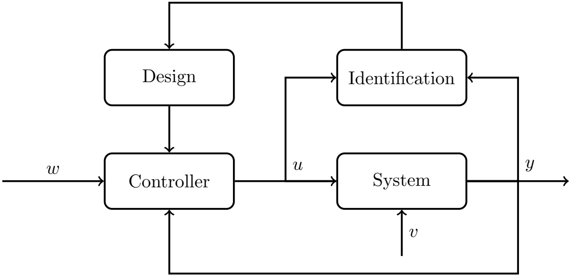
Tobias K. S. Ritschel

Section for Dynamical Systems, DTU Compute

DTU Compute
Department of Applied Mathematics and Computer Science

Lecture Plan

1. Systems theory
2. Stochastics
3. State estimation - Kalman filter 1
4. State estimation - Kalman filter 2
5. **Optimal control 1 - internal models**
6. External models
7. Prediction + optimal control 1- external models
8. Optimal control 2 - external models
9. System identification 1
10. System identification 2
11. System identification 3 + model validation
12. Adaptive control 1
13. Adaptive control 2



Today's Agenda

- Info: Project 1
- Follow-up from last lecture
- Linear Control Theory - Pole Placement
- Optimal Control Theory - LQR
- Optimal Control Theory - GPC
- LQR with incomplete state information
- observer-based control: LQG

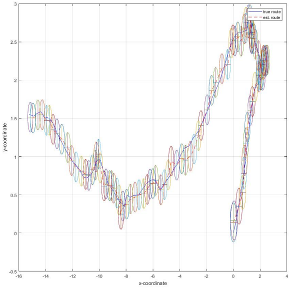
Info: Project 1

1. Released: March, the 7th at 10:00
2. Deadline: April, the 4th at 23:59
3. Page limit: 20 pages
4. Format: Individual Reports

Follow-up from last time:

How to make confidence interval for the ship trajectory

```
plot(xcord, ycord) % true path
plot(xcord_est,ycord_est) % est. path
for i = 1:N
    c = [xcord_est(i), ycord_est(i)]
    P = Pcord(:,:,i) % variance
    Niveau(c, inv(P), ...
        sqrt(chi2inv(0.95, 2)))
end
```



Follow-up from last time

Questions?

General Linear Control Theory - introduction

In deterministic state space control, we consider the system:

$$x_{t+1} = Ax_t + Bu_t + d \tag{1}$$

Based on this system, and our desired goals of the system; a control law/strategy is designed:

$$u = -Lx + w \tag{2}$$

where the control gain L in general assures stability, and changes the system properties, the input w is then used for directing the system according to some reference.

This gives the closed-loop system:

$$x_{t+1} = (A - BL)x_t + Bw_t + d \tag{3}$$

General Linear Control Theory - Pole Placement

One approach to stabilize a system, is by changing its poles. This also allow for changing the dynamic properties of the system. As define by the relation between poles, eigenvalues and time constants τ:

$$\text{discrete-time Poles } \lambda_d = eig(A_d) = e^{-\frac{T_s}{\tau}} \tag{4}$$

$$\text{continues-time Poles } \lambda_c = eig(A_c) = -\frac{1}{\tau} \tag{5}$$

If one uses external models then pole placement can be done simply by using $u = H_{pole}(q)w$:

$$y = H(q)u = \frac{B(q)}{A(q)}u = \frac{B(q)}{A(q)}\frac{A(q)}{A_{pole}(q)}w = \frac{B(q)}{A_{pole}(q)}w \tag{6}$$

General Linear Control Theory - Pole Placement

In internal models, pole placement can easiest be explained by bringing the system on Controller Canonical form:

$$\Phi_c = \begin{bmatrix} -a_1 & \dots & -a_{n-1} & -a_n \\ 1 & \dots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} \qquad \Gamma_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

$$\phi_c^T = (b_1 - b_0 a_1, b_2 - b_0 a_2, ..., b_n - b_0 a_n) \qquad \Delta_c = b_0 \tag{8}$$

where a_i is the n coefficients in $A(q)$. Then if α_i represent the coefficients in $A_{poles}(q)$:

$$L = [\alpha_1 - a_1, \dots, \alpha_n - a_n] \tag{9}$$

defines the control gain, that gives the exact pole placement. Remember the polynomials relation to the poles:

$$A(q) = \prod_{i=1}^{n} (q - \lambda_{d,i}) \tag{10}$$

Example on pole placement

Consider the system

$$X_{k+1} = \begin{bmatrix} 6 & -8 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k, \qquad eig(A_x) = [2, 4] \tag{11}$$

$$A(q) = q^2 - 6q + 8 \tag{12}$$

if we want a system with the poles $0.5, -0.5$, the polynomial should be

$$A_{new}(q) = q^2 + 0q - 0.25 \tag{13}$$

designing the controller we gain the following results

$$L = \begin{bmatrix} 0 - -6 & -0.25 - 8 \end{bmatrix} = \begin{bmatrix} 6 & -8.25 \end{bmatrix} \tag{14}$$

$$A_{new} = A - BL = \begin{bmatrix} 0 & 0.25 \\ 1 & 0 \end{bmatrix}, \qquad eig(A_x) = [0.5, -0.5] \tag{15}$$

What is the feedback gain, L, if we wanted the poles to be 0.25 and -0.25? What is the corresponding closed-loop system matrix, $A - BL$?

Think about it for yourself for one minute and then discuss with the person next to you for one minute.

Optimal Control Theory

In Optimal control theory, one consider how to operate a system (17) best or optimal, according to some scalar cost criteria (16)

$$J = \min_{u} l(x, u) \tag{16}$$

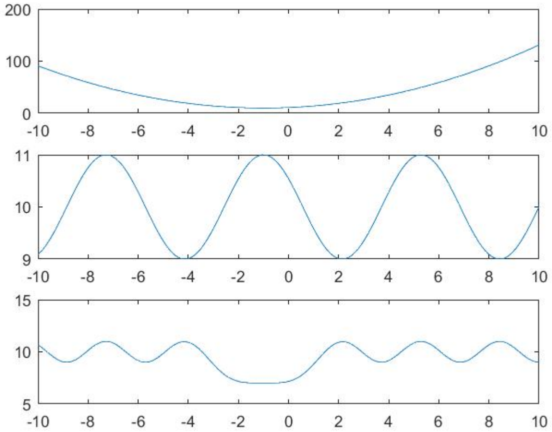
$$x_{k+1} = Ax_k + Bu_k \tag{17}$$

The optimal control minimizing the cost is then found by substituting (17) into a new cost $\bar{V}$, and finding the minimum:

$$\frac{\mathrm{d}\bar{V}(x_0, u)}{\mathrm{d}u} = 0 \tag{18}$$

The resulting control trajectory is then computed by isolating u

Optimal Control Theory - minimum of cost function



Optimal Control Theory - Example

Let us go through a simple example, consider the optimal control problem:

$$J_k = \min_{u_k} x_{k+1}^T Q x_{k+1} + u_k^T R u_k + 2u_k^T H x_{k+1} \tag{19}$$

$$x_{k+1} = Ax_k + Bu_k \tag{20}$$

The optimal cost can then be written as:

$$\bar{V}(x_k, u) = (Ax_k + Bu_k)^T Q(Ax_k + Bu_k) + u_k^T R u_k + 2u_k^T H(Ax_k + Bu_k) \tag{21}$$

$$= x_k^T A^T Q A x_k + u_k^T (B^T Q B + R + 2HB)u_k + u_k^T (2B^T Q A + 2HA)x_k \tag{22}$$

We can the find the minimizing control law to be results from:

$$\frac{\mathrm{d}\bar{V}(x_k, u)}{\mathrm{d}u_k} = 2(B^T Q B + R + 2HB)u_k + 2(B^T Q A + HA)x_k \tag{23}$$

$$u_k = -(B^T Q B + R + 2HB)^{-1}(B^T Q A + HA)x_k \tag{24}$$

How do Q, R, and H affect the solution, u_k? Think about it for yourself for one minute and then discuss with the person next to you for one minute.

Optimal Control - Linear Quadratic Regulator

We will now consider the quadratic case over a longer period, covering a finite state trajectory $\{x_k\}_{k=0}^{N}$ and an input trajectory $\{u_k\}_{k=0}^{N-1}$:

$$l = x_N' Q_0 x_N + \sum_{k=0}^{N-1} \left[x_k' Q_1 x_k + u_k' Q_2 u_k \right]. \tag{25}$$

The matrices Q_0 and Q_1 are symmetric positive semi-definite and the matrix Q_2 is positive definite. In quadratic optimization one considers costs of the form $x^T S x$, where S is a positive semi-definite matrix

$$x^T S x \geq 0 \textit{ and convex} \tag{26}$$

$$S_s = \frac{1}{2}(S + S^T) = S_s^T, \qquad S_a = \frac{1}{2}(S - S^T) = -S_a^T \tag{27}$$

$$x^T S_s x = x^T S x, \qquad x^T S_a x = -x^T S_a^T x = 0 \tag{28}$$

$$x^T A^T S A x \geq 0 \tag{29}$$

Optimal Control - Linear Quadratic Regulator

For the rest of our discussion let us consider the system:

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad v_k \sim N(0, R_1) \tag{30a}$$

$$y_k = Cx_k + e_k, \quad e_k \sim N(0, R_2) \tag{30b}$$

$$v_k \perp e_k, \qquad v_k, e_k \perp x_k, \qquad x_0 \sim N(m_0, P_0) \tag{30c}$$

In the stochastic case, we use expectation in our optimization:

$$J = \min_u E\{l(x,u)|\mathcal{F}\} = \min_u E\{x^T Qx + u^T Ru^T|\mathcal{F}\} \tag{31}$$

$$= \min_u E\{x|\mathcal{F}\}^T QE\{x|\mathcal{F}\} + u^T Ru^T + tr\{QV\{x|\mathcal{F}\}\} \tag{32}$$

Where $\mathcal{F}$ is the information available

Optimal Control - Quadratic cost versions

There exist multiple ways to formulate quadratic costs. Some of the most common is given below.

The cost on deviation from references, and control usages:

$$J = E\left\{ \sum_{i=t}^{t+N} (y_i - r_i)^T Q_y (y_i - r_i) + u_i^T Q_u u_i | \mathcal{F} \right\} \tag{33}$$

The cost on deviation from references, and control deviation:

$$J = E\left\{ \sum_{i=t}^{t+N} (y_i - r_i)^T Q_y (y_i - r_i) + (u_i - u_0)^T Q_u (u_i - u_0) | \mathcal{F} \right\} \tag{34}$$

The cost on deviation from references, and control changes:

$$J = E\left\{ \sum_{i=t}^{t+N} (y_i - r_i)^T Q_y (y_i - r_i) + (u_i - u_{i-1})^T Q_u (u_i - u_{i-1}) | \mathcal{F} \right\} \tag{35}$$

Optimal Control - Linear Quadratic Regulator

For a stochastic control problem it is very important to specify which data (or information) that is available for determining the optimal control trajectory $\{u_k\}_{k=0}^{N-1}$. In this discussion, we will assume that we have perfect state information $y_k = x_k$.

In the stochastic case, our cost function is also a stochastic function, and as stated we will consider the expected value of l instead:

$$\mathrm{E}\Big[l\Big] = \mathrm{E}\left[x_N' Q_0 x_N + \sum_{k=0}^{N-1} x_k' Q_1 x_k + u_k' Q_2 u_k\right] \tag{36}$$

$$= \mathrm{E}\left[\sum_{k=0}^{t-1}\Big[x_k' Q_1 x_k + u_k' Q_2 u_k\Big]\right] + \mathrm{E}\left[x_N' Q_0 x_N + \sum_{k=t}^{N-1}\Big[x_k' Q_1 x_k + u_k' Q_2 u_k\Big]\right] \tag{37}$$

The goal is now, to find a control strategy for the stochastic system (30a) such that the cost function (36) is minimal.
Notice that we can split the equation at time t, with the first term being independent of $u_t, ..., u_{N-1}$

Optimal Control - minimum and definition

Assume that $l(x,u)$ has a unique minimum with respect to u for all x, and let $u^0(x)$ denote the value of u where this minimum is attained. Then,

$$\min_{u(x)} \mathtt{E}\Big[l(x,u)\Big] = \mathtt{E}\Big[l(x,u^0(x))\Big] = \mathtt{E}\Big[\min_{u} l(x,u)\Big] \tag{38}$$

We now apply the result (38) and find that

$$\min_{u_t,\ldots,u_{N-1}} \mathtt{E}\left[x_N' Q_0 x_N + \sum_{k=t}^{N-1}\left[x_k' Q_1 x_k + u_k' Q_2 u_k\right]\right] = \mathtt{E}\Big[V(x_t,t)\Big], \tag{39}$$

where we define

$$V(x_t,t) = \min_{u_t,\ldots,u_{N-1}} \mathtt{E}\left[x_N' Q_0 x_N + \sum_{k=t}^{N-1}\left[x_k' Q_1 x_k + u_k' Q_2 u_k\right] \;\Big|\; x_t\right] \tag{40}$$

Optimal Control - Dynamic Programming

We can do this repeatedly to achieve the recursion, known as the Bellman equation:

$$V(x,t) = \min_{u_t} \Big\{ \mathtt{E}\Big[x_t' Q_1 x_t + u_t' Q_2 u_t + V(x_{t+1}, t+1) \mid x \Big] \Big\} \tag{41a}$$

$$= \min_{u_t} \Big\{ x' Q_1 x + u_t' Q_2 u_t + \mathtt{E}\Big[V(x_{t+1}, t+1) \mid x \Big] \Big\}. \tag{41b}$$

where we for $t = N$ has the end-point condition that

$$V(x,N) = \min_{u} \Big\{ \mathtt{E}\Big[x_N' Q_0 x_N \mid x \Big] \Big\} = x' Q_0 x. \tag{42}$$

being the initial condition.

Optimal Control - Dynamic Programming

We will now show that the solution to (41) with the initial conditions given by (42) is a quadratic function

$$V(x,t) = x' S_t x + s_t, \tag{43}$$

where S_t is a non-negative definite matrix.

Optimal Control - Dynamic Programming

It is true for $t = N$ since we by definition have that

$$V(x, N) = x' Q_0 x, \tag{44}$$

by our initial condition. We will now make a proof-by-induction. Assume that it holds for $t + 1$, we will then have to show that it also holds for t.

By our assumption we have that

$$V(x_{t+1}, t+1) = x'_{t+1} S_{t+1} x_{t+1} + s_{t+1}. \tag{45}$$

We also find that

$$\begin{aligned} \mathtt{E}\Big[V(x_{t+1}, t+1) \mid x\Big] = {} & [Ax + Bu_t]' \, S_{t+1} \, [Ax + Bu_t] \\ & + \mathtt{trace}\left[S_{t+1} R_1\right] + s_{t+1}. \end{aligned} \tag{46}$$

Optimal Control - Dynamic Programming

Using this result, we find that

$$V(x,t) = \min_{u_t} \Big\{ x' Q_1 x + u_t' Q_2 u_t + [Ax + Bu_t]' S_{t+1} [Ax + Bu_t] \\ + \texttt{trace}\,[S_{t+1} R_1] + s_{t+1} \Big\}. \tag{47}$$

Collecting terms and defining

$$L_t = \left[Q_2 + B' S_{t+1} B\right]^{-1} B' S_{t+1} A, \tag{48}$$

we can formulate $V(x,t)$ according to

$$V(x,t) = x' \left[A' S_{t+1} A + Q_1 - L_t'(Q_2 + B' S_{t+1} B) L_t\right] x \\ + \texttt{trace}\,[S_{t+1} R_1] + s_{t+1}, \tag{49}$$

where this minimum value is attained for

$$u_t = -L_t x_t, \tag{50}$$

which is a linear feedback controller (or strategy).

Optimal Control - Dynamic Programming

We have now shown that the function $V(x,t)$ indeed is a quadratic function with

$$S_t = A' S_{t+1} A + Q_1 - L_t' \left[Q_2 + B' S_{t+1} B \right] L_t \tag{51a}$$

$$s_t = \texttt{trace} \left[S_{t+1} R_1 \right] + s_{t+1}. \tag{51b}$$

However, we still need to show that S_t is a non-negative definite matrix. Re-arranging terms in (51a) yields

$$S_t = \left[A - BL_t \right]' S_{t+1} \left[A - BL_t \right] + L_t' Q_2 L_t + Q_1, \tag{52}$$

from which (given the properties of Q_1 and Q_2) it is seen that if S_{t+1} is a non-negative definite matrix, then S_t is also a non-negative definite matrix.

What is the trick you need to get from (51a) to (52)?

Think about it for yourself for one minute and then discuss with the person next to you for one minute.

Optimal Control - Dynamic Programming

To summarize an optimal control strategy for the stochastic system (30a) such that cost function (36) is minimal is given by

$$u_k = -L_k x_k, \tag{53}$$

where

$$L_k = \left[Q_2 + B' S_{k+1} B\right]^{-1} B' S_{k+1} A, \tag{54}$$

and S_k is given by

$$S_k = \left[A - B L_k\right]' S_{k+1} \left[A - B L_k\right] + L_k' Q_2 L_k + Q_1, \tag{55}$$

with initial condition

$$S_N = Q_0. \tag{56}$$

Optimal Control - General Predictive Control

Let us consider the ith prediction of a linear system:

$$x_i = A^i x_0 + [A^{i-1}B, \ldots, AB, B] \begin{bmatrix} u_0 \\ \vdots \\ u_{i-2} \\ u_{i-1} \end{bmatrix} + [A^{i-1}G, \ldots, AG, G] \begin{bmatrix} v_0 \\ \vdots \\ v_{i-2} \\ v_{i-1} \end{bmatrix} \tag{57}$$

$$= A^i x_0 + \mathcal{B}_{i-1} U_{i-1} + \mathcal{G}_{i-1} V_{i-1} \tag{58}$$

$$y_i = C x_i + D u_i + e_i \tag{59}$$

$$= C A^i x_0 + [C\mathcal{B}_{i-1}, D] U_i + C\mathcal{G}_{i-1} V_{i-1} + e_i \tag{60}$$

Optimal Control - GPC: General Predictive Control

We can then define all predictions as:

$$Y_i = \mathcal{W}_{o,i} x_0 + \mathbb{G}_I U_i + \mathbb{H}_i V_{i-1} + E_i \tag{61}$$

$$\mathcal{W}_{o,i} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^i \end{bmatrix} \quad \mathbb{M}_i(B,D) = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & \vdots & \ddots & \ddots & \\ CA^{i-1}B & CA^{i-2}B & \dots & CB & D \end{bmatrix} \tag{62}$$

$$\mathbb{G}_i = \mathbb{M}_i(B,D) \quad \mathbb{H}_i = \mathbb{M}_i(G,0) \tag{63}$$

Using optimization the predictive control is then given as:

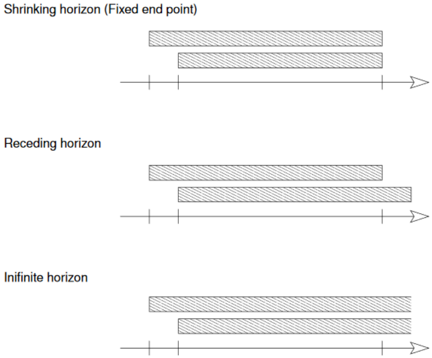
$$J_i = \min_{U_i} E\{(Y_i - W_i)^T Q_y (Y_i - W_i) + U_i^T Q_u U_i\} \tag{64}$$

$$U_i = [\mathbb{G}_i^T Q_y \mathbb{G}_i + Q_u]^{-1} \mathbb{G}_i^T Q_y (W_i - \mathcal{W}_{o,i} E\{x_0\} - \mathbb{H}_i E\{V\}_{i-1} - E\{E_i\}) \tag{65}$$

GPC: Horizons

Predictive control, is commonly used in a iterative manner as:

$u_t = [I, 0, \ldots, 0]U_i$



where i in U_i is the length of the Horizon

LQR - Linear Quadratic Regulator w. complete state info

The result of the dynamic programming earlier is also known as finite horizon LQR and defined by

$$J_t = E\left\{ \sum_{i=t}^{t+N} \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} \right\}, \quad x_t \in F(x_0, \Sigma_0) \tag{66}$$

$$x_{i+1} = Ax_i + Bu_i + v_i, \quad v_i \in F(0, R_1) \tag{67}$$

With an optimal control law on the form:

$$u_t = -L_t x_t = -[B^T S_{t+1} B + Q_2]^{-1} [B^T S_{t+1} A + Q_{12}^T] x_t \tag{68}$$

Where the optimal state weight at time t is S_t, given by

$$S_t = A^T S_{t+1} A + Q_1 - A^T S_{t+1} B (B^T S_{t+1} B + Q_2)^{-1} B^T S_{t+1} A \tag{69}$$

$$S_{t+N+1} = 0 \tag{70}$$

LQR - closed-loop w. complete state info

For the closed-loop analysis, let us consider the system:

$$x_{t+1} = Ax_t + Bu_t + v_t \tag{71}$$

$$z_t = C_z x_t + D_z u_t \tag{72}$$

with the control $u_t = -L_t x_t$

The closed-loop description the becomes:

$$x_{t+1} = (A - BL)x_t + v_t = A_{cl}x_t + v_t \tag{73}$$

$$z_t = (C_z - D_z L_t)x_t = C_{zcl}x_t \tag{74}$$

With the state mean/variance evolution:

$$E\{x_t\} = A_{cl}E\{x_{t-1}\}, \quad E\{x_0\} = m_0 \tag{75}$$

$$Var\{x_t\} = A_{cl}Var\{x_{t-1}\}A_{cl}^T + R_1, \quad Var\{x_0\} = \Sigma_0 \tag{76}$$

and the output mean/variance evolution:

$$E\{z_t\} = C_{zcl}E\{x_t\} \tag{77}$$

$$Var\{z_t\} = C_{zcl}Var\{x_t\}C_{zcl}^T \tag{78}$$

LQR - stationary control

If we consider the infinite horizon LQR, $N = \infty$, then we have a stationary controller.
We can see that $S_t = S_{t+1}$ results in a constant control gain L_t.

The Discrete Algebraic Ricatti Equation (DARE):

$$S_\infty = A^T S_\infty A + Q_1 - A^T S_\infty B (B^T S_\infty B + Q_2)^{-1} B^T S_\infty A \tag{79}$$

$$L_\infty = -[B^T S_\infty B + Q_2]^{-1} [B^T S_\infty A + Q_{12}] \tag{80}$$

This applicable iif (A,B) is at least stabilizable (controllable,reachable).

If (A,Q_1) is observable, then DARE have a unique positive semi-definite solution, and (A-BL) is asymptotically stable.

LQR - complete/incomplete state information

From the deduction of LQR, we remember the Bellman equation, and how it described the optimal cost to go. In the more general form it is given by:

$$V_t(\mathcal{F}_t) = \min_{u_t,\dots,u_{t+N}} E\left\{ \sum_{i=t}^{t+N} I_i(x_i,u_i)|\mathcal{F}_t \right\} = \min_{u_t} E\{I_t(x_t,u_t) + V_{t+1}(\mathcal{F}_{t+1})|\mathcal{F}_t\} \tag{81}$$

$$\mathcal{F}_t = \{x_t, Y_t, Y_{t-1}\} \tag{82}$$

If we follow the same deduction, the control law of LQR becomes:

$$u_t = -L_t E\{x_t|\mathcal{F}_t\} \tag{83}$$

$$L_t = [B^T S_{t+1} B + Q_2]^{-1}[B^T S_{t+1} A + Q_{12}] \tag{84}$$

$$S_t = A^T S_{t+1} A + Q_1 - L_t^T (B^T S_{t+1} B + Q_2) L_t \tag{85}$$

$$S_{t+N+1} = 0 \tag{86}$$

For case of the incomplete state information, the control becomes:

$$u_t = -L_t E\{x_t|Y_t\} = -L_t \hat{x}_{t|t}, \quad u_t = -L_t E\{x_t|Y_{t-1}\} = -L_t \hat{x}_{t|t-1} \tag{87}$$

LQG: the optimal Linear Quadratic Gaussian observer-based controller

For linear systems, we have discussed both controllers and observers:

1. LQ control: Optimal state control based on perfect state & system knowledge.
2. Kalman filter: Optimal state estimation based on perfect system knowledge.

When the assumption of full state knowledge is unattainable, we can combine the controller with an observer.

The optimal observer-based controller in the linear case, is the Linear Quadratic Gaussian controller or LQG:

$$J_k = \min_{u_t,\ldots,u_{t+N}} E\left\{ \sum_{i=t}^{t+N} \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12} & Q_2 \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} |\mathcal{F} \right\} \quad (88)$$

$$x_{i+1} = Ax_i + Bu_i + v_i, \quad v_i \sim N(0, R_1) \quad (89)$$

$$y_i = Cx_i + e_i, \quad e_i \sim N(0, R_2), cov(v_i, e_i) = R_{12} \quad (90)$$

Both the controller and observe can be designed independently (separation Principle)

LQG - Duality and Stationarity

Controlling vs Observation - two sides of the same coin.

Consider quadratic optimal control (LQ) and quadratic optimal observers (Kalman filter):

Optimal Gain:

$$L_t^T = [A^T S_{t+1} B + Q_{12}][B^T S_{t+1} B + Q_2]^{-1} \tag{91}$$

$$K_t = [AP_t C^T + R_{12}][CP_t C^T + R_2]^{-1} \tag{92}$$

Riccati Equations:

$$S_t = A^T S_{t+1} A + Q_1 - L_t^T [B^T S_{t+1} B + Q_2] L_t, \quad S_{N+1} = 0 \tag{93}$$

$$P_{t+1} = AP_t A^T + R_1 - K_t [CP_t C^T + R_2] K_t^T, \quad P_{t_0} = P_0 \tag{94}$$

Algebraic Riccati Equations: (Stationary case)

$$S = A^T SA + Q_1 - [A^T SB + Q_{12}][B^T SB + Q_2]^{-1}[B^T SA + Q_{12}^T] \tag{95}$$

$$P = APA^T + R_1 - [APC^T + R_{12}][CPC^T + R_2]^{-1}[CPA^T + R_{12}^T] \tag{96}$$

Brief Proof of Separation principle

It can be shown, that independently designed optimal controller and observer design results in the optimal controller/observer of the combined system, this is known as the separation principle.

Let us consider the system:

$$x_{t+1} = Ax_t + Bu_t + v_t \tag{97}$$

$$y_t = Cx_t + e_t \tag{98}$$

and let us have designed both an LQR and a Kalman filter independently, with the gains L_t and K_t respectively. Our state and estimation systems can then be written as:

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix}_{t+1|t} = \begin{bmatrix} A & -BL_t \\ K_tC & A - K_tC - BL_t \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}_{t|t-1} + \begin{bmatrix} I & 0 \\ 0 & K_t \end{bmatrix} \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{99}$$

Brief Proof of Separation principle

As we saw earlier, an LQR based on estimation provides the same control equation as one based on complete state information. Meaning that we only have to prove the individually designed Kalman filter still is optimal besides the control.

If we consider the systems estimation error $\tilde{x}_t = x_t - \hat{x}_{t+1}$:

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t+1|t} = \begin{bmatrix} A - BL_t & BL_t \\ 0 & A - K_t C \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t-1} + \begin{bmatrix} I & 0 \\ I & -K_t \end{bmatrix} \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{100}$$

We can observe the description for the estimation error is independent of the control and true state. Additionally given the expanded state matrix is triangular, its eigenvalues only dependend on $(A - BL_t)$ and $A - K_t C$, meaning stability is designed individually.

Conclusion: separation principle holds, given the iterative properties of x_t and $\tilde{x}_t$ is independent of the design of the other.

Closed loop LQG - Predictive

Using a LQG controller based on a predictive Kalman filter, gives the closed-loop description:

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t+1|t} = \begin{bmatrix} A - BL_t & BL_t \\ 0 & A - K_t C \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t-1} + \begin{bmatrix} I & 0 \\ I & -K_t \end{bmatrix} \begin{bmatrix} v_t \\ e_t \end{bmatrix} \quad (101)$$

$$= A_{cl} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t-1} + G \begin{bmatrix} v_t \\ e_t \end{bmatrix} \quad (102)$$

LQG properties:

$$m_{t+1} = A_{cl} m_t \to 0 \qquad \text{(iff asym. stable)} \quad (103)$$

$$\Sigma_{t+1} = A_{cl} \Sigma_t A_{cl}^T + G \bar{R}_1 G^T \to \begin{bmatrix} P_x & P_\infty \\ P_\infty & P_\infty \end{bmatrix} \qquad \text{(iff asym. stable)} \quad (104)$$

$$\bar{R}_1 = diag(R_v, R_e) \quad (105)$$

where P_∞ comes from the ricatti equation for the ordinary Kalman filter:

$$P_{t+1} = AP_t A^T + R_1 - K_t (CP_t C^T + R_2) K_t^T \quad (106)$$

Closed loop LQG - Predictive

The Predictive LQG closed-loop form:

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t+1|t} = \begin{bmatrix} A - BL_t & BL_t \\ 0 & A - K_t C \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t-1} + \begin{bmatrix} I & 0 \\ I & -K_t \end{bmatrix} \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{107}$$

$$= A_{cl} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t-1} + G \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{108}$$

The control and output properties are

$$u_t = -L_t \hat{x}_{t|t-1}$$

$$= -L_t(x_t - \tilde{x}_{t|t-1}) \in N\left(\begin{bmatrix} L_t & -L_t \end{bmatrix} m_t, \begin{bmatrix} L_t & -L_t \end{bmatrix} \Sigma_t \begin{bmatrix} L_t^T \\ -L_t^T \end{bmatrix} \right) \tag{109}$$

$$y_t = Cx_t \in N\left(\begin{bmatrix} C & 0 \end{bmatrix} m_t, \begin{bmatrix} C & 0 \end{bmatrix} \Sigma_t \begin{bmatrix} C \\ 0 \end{bmatrix} \right) = N(Cm_{x,t}, CP_{x,t}C^T) \tag{110}$$

Stationary: $\tilde{x} \in N(0, P_\infty)$

Closed loop LQG - Ordinary

Using a LQG controller based on an ordinary Kalman filter, gives the closed-loop description:

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t+1|t+1} = \begin{bmatrix} A - BL_t & BL_t \\ 0 & A - \kappa_t CA \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t} + \begin{bmatrix} I & 0 \\ I - \kappa C & -\kappa_t \end{bmatrix} \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{111}$$

$$= A_{cl} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t} + G \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{112}$$

LQG properties:

$$m_{t+1} = A_{cl} m_t \to 0 \qquad \text{(iff asym. stable)} \tag{113}$$

$$\Sigma_{t+1} = A_{cl} \Sigma_t A_{cl}^T + G \bar{R}_1 G^T \to \begin{bmatrix} P_x & \bar{P}_\infty \\ \bar{P}_\infty & \bar{P}_\infty \end{bmatrix} \qquad \text{(iff asym. stable)} \tag{114}$$

$$\bar{R}_1 = diag(R_v, R_e) \tag{115}$$

where $\bar{P}_\infty$ comes from the Riccati equation for the ordinary Kalman filter:

$$\bar{P}_{t+1} = (I - \kappa_{t+1} C)(A \bar{P}_t A^T + R_1)(I - \kappa_{t+1} C)^T + \kappa_{t+1} R_2 \kappa_{t+1} \tag{116}$$

Closed loop LQG - Ordinary

The ordinary LQG closed-loop form:

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t+1|t+1} = \begin{bmatrix} A - BL_t & BL_t \\ 0 & A - \kappa_t CA \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t} + \begin{bmatrix} I & 0 \\ I - \kappa C & -\kappa_t \end{bmatrix} \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{117}$$

$$= A_{cl} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}_{t|t} + G \begin{bmatrix} v_t \\ e_t \end{bmatrix} \tag{118}$$

The control and output properties:

$$u_t = -L_t \hat{x}_{t|t}$$

$$= -L_t(x_t - \tilde{x}_{t|t}) \in N\left(\begin{bmatrix} L_t & -L_t \end{bmatrix} m_t, \begin{bmatrix} L_t & -L_t \end{bmatrix} \Sigma_t \begin{bmatrix} L_t^T \\ -L_t^T \end{bmatrix} \right) \tag{119}$$

$$y_t = Cx_t \in N\left(\begin{bmatrix} C & 0 \end{bmatrix} m_t, \begin{bmatrix} C & 0 \end{bmatrix} \Sigma_t \begin{bmatrix} C \\ 0 \end{bmatrix} \right) = N(Cm_{x,t}, CP_{x,t}C^T) \tag{120}$$

Note the similarity with the predictive LQG. Stationary: $\tilde{x} \in N(0, \bar{P}_\infty)$.

Questions

Questions?

Kalman Filter Cheat-Sheet

Given the process model

$$x_{k+1} = Ax_k + Bu_k + G\xi_k \tag{121a}$$

$$y_k = Cx_k + Du_k + Fe_k. \tag{121b}$$

Given $\hat{x}_{k-1|k-1}$ (filtered mean) and $\hat{P}_{k-1|k-1}$ (filtered covariance), we predict (1-step) using

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1} \tag{122a}$$

$$\hat{P}_{k|k-1} = A\hat{P}_{k-1|k-1}A' + GR_\xi G'. \tag{122b}$$

Kalman Filter Cheat-Sheet

Given y_k, we reconstruct (or update) $\hat{x}_{k|k}$ and $\hat{P}_{k|k}$ using

$$\hat{e}_k = y_k - C\hat{x}_{k|k-1} - Du_k \tag{123a}$$

$$R_k = C\hat{P}_{k|k-1}C' + FR_eF' \tag{123b}$$

$$K_k = \hat{P}_{k|k-1}C'R_k^{-1} \tag{123c}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\hat{e}_k \tag{123d}$$

$$\hat{P}_{k|k} = (I - K_kC)\hat{P}_{k|k-1}. \tag{123e}$$

REPEAT!