Stochastic Adaptive Control (02421)

Lecture 1

Tobias K. S. Ritschel

Assistant Professor

Section for Dynamical Systems, DTU Compute

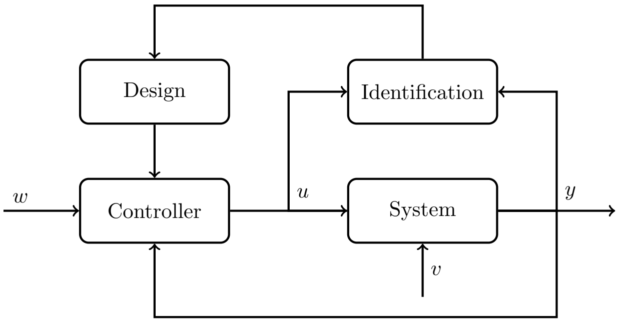
Course Content

Course details

- Time: Tuesday 08:00 - 12:00
  (2 hours lecture, 2 hours exercises)
- 5 ECTS points
- Evaluation: 2 individual reports
- Software: MATLAB (free choice)

Course plan

- Stochastic process and systems
- Filter and control design
  (state space and transfer function models)
- System identification
- Adaptive control



Teachers and teaching assistant

Tobias K. S. Ritschel
Assistant Professor
Course responsible
Contact: tobk@dtu.dk

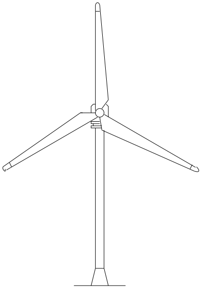
Henrik Madsen
Professor
Course co-responsible
Contact: hmad@dtu.dk

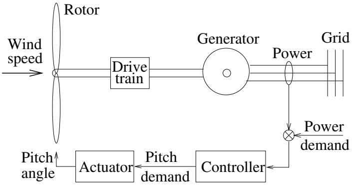
Pietro Aldo Refosco
MSc student
Teaching assistant
Contact: s212976@student.dtu.dk

You

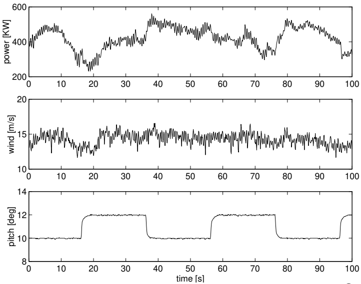
- Most of you are MSc students.
- A few BSc students, guest students, and single-course students.
- Most of you are from electrical or mathematical engineering (incl. autonomous systems).
- A few of you are from chemical engineering.
- A few from sustainable energy.

Stochastic systems – Wind Turbine





Stochastic/uncertain weather conditions



Lecture Plan

1. System theory
2. Stochastics
3. State estimation - Kalman filter 1
4. State estimation - Kalman filter 2
5. Optimal control 1 - internal models
6. External models
7. External prediction + external optimal control 1
8. Optimal control 2 - external models
9. System identification 1
10. System identification 2
11. System identification 3 + model validation
12. Adaptive control 1
13. Adaptive control 2

Structure of Lectures

Each lecture will contain 2 parts and a break.

Part 1:

- Outline of the day + practical information
- Resumé of previous lecture: e.g., an example of a difficult topic
- Topics of the day

Part 2:

- Continue the topics of the day
- Questions

Terminology

In the course, we will use the following terminology and notations.

- $x : \mathbb{R} \mapsto \mathbb{R}^{n_x}$ is the state vector of dimension n_x.
- $u : \mathbb{R} \mapsto \mathbb{R}^{n_u}$ is the input vector of dimension n_u.
- $y : \mathbb{R} \mapsto \mathbb{R}^{n_y}$ is the output vector of dimension n_y.
- $x_0 \in \mathbb{R}^{n_x}$ is the initial state vector of dimension n_x.
- $p \in \mathbb{R}^{n_p}$ is a parameter vector of dimension n_p.
- q^{-1} is the unit delay operator

Sometimes u is split into controllable and non-controllable inputs.

Common abbreviations

- LTI: Linear time-invariant
- ODE: Ordinary differential equation
- exp/e: exponential function
- iff: if and only if

Today's Topics

Systems Theory

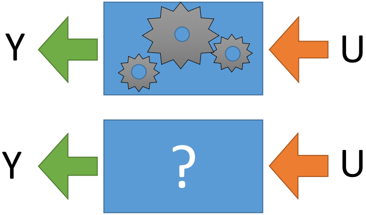
- Dynamical systems
- Domains - time/frequency
- Linearization and discretization
- System properties

Dynamical Systems: External and Internal Models

We describe dynamical systems in two ways:

Internal Models

- States of the system
- Differential equations



External Models

- Transfer functions
- Zeros and poles

Dynamical Systems: ODE and Algebraic Equation (Internal)

We consider dynamical systems in the form

$$\dot{x}(t) = \frac{\partial x}{\partial t}(t) = f(x(t), u(t); p) = A(p)x(t) + B(p)u(t) \tag{1a}$$

$$x(t_0) = x_0, \tag{1b}$$

or

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f(x(\tau), u(\tau); p)\, \mathrm{d}\tau \\ &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(p)(\tau - t_0)} B(p)u(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{2}$$

Furthermore, we consider output equations in the form

$$y(t) = g(x(t), u(t); p) = C(p)x(t) + D(p)u(t) \tag{3}$$

Later, we will also consider stationary points, (x^*, u^*), which are defined by

$$f(x^*, u^*; p) = 0. \tag{4}$$

Dynamical systems: ODE (External)

For external models, the general LTI N-th order inhomogeneous (1D) ODE is given by

$$\sum_{k=0}^{N} \alpha_k \frac{\partial^k y}{\partial t^k} = \sum_{l=0}^{M} \beta_l \frac{\partial^l u}{\partial t^l}, \tag{5}$$

where $\alpha_k, \beta_l \in \mathbb{R}$. The solution can be formulated as

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(s) u(t-s) \, \mathrm{d}s, \tag{6}$$

where $h(t)$ is the impulse response. The Laplace transformed variables are given by

$$Y(s) = H(s)U(s), \qquad U(s) = \mathcal{L}(u(t)), \tag{7}$$

where

$$\begin{aligned} H(s) = \mathcal{L}(h(t)) &= \int_{-\infty}^{\infty} h(s) e^{-st} \, \mathrm{d}s \\ &= C(p) \left(sI - A(p)\right)^{-1} B(p) + D(p). \end{aligned} \tag{8}$$

Time and Frequency Domain

Continuous-time time-domain

$$t$$

$$y(t) = h(t) * u(t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = sY(s)$$

Continuous-time frequency-domain

$$s = a + iw$$

$$Y(s) = H(s)U(s)$$

$$H(s) = \frac{\sum_{l=0}^{M} \beta_l s^l}{\sum_{k=0}^{N} \alpha_k s^k}$$

Discrete-time time-domain

$$t_k = kT_s$$

$$y_k = H_d(q^{-1})u_k$$

$$u_k = u(t_k) = u(kT_s)$$

$$u_{k-1} = q^{-1}u_k$$

Discrete-time frequency-domain

$$z = e^{T_s s}$$

$$Y(z) = H_z(z)U(z)$$

$$H_d(q^{-1}) = H_z(q)$$

where T_s is the sampling time

Impulse response

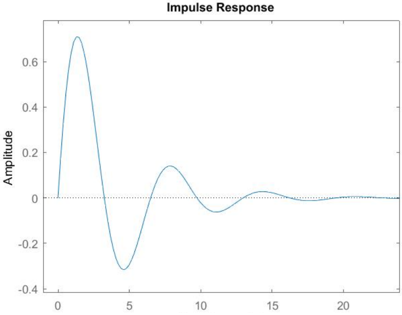


Figure: An impulse response

Consider $x_0 = 0$. Then the response from u to y is given by

$$h(t) = \begin{cases} C^T e^{At} B + D\delta(0), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Causal system:
No reaction before an impact.

Step Response

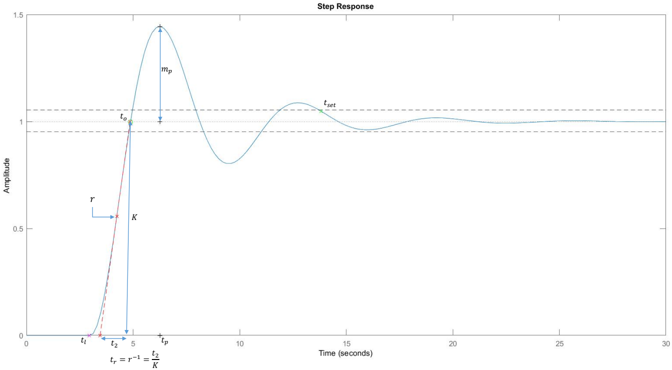


Figure: Step response

t_r is the rise time, r is the reaction rate, t_{set} is the setling time, t_p is the peak time, t_o is the growth time, t_l is the reaction time m_p is the overswing.

Linearization

We will now briefly return to the general ODE system given by (1) with output equation given by (3). Usually, we linearize in a stationary point (or steady state); i.e. a point (x^*, u^*) that satisfies

$$f(x^*, u^*; p) = 0. \tag{9}$$

Doing a Taylor expansion (truncated after first-order) on (1a) and (3) in (x^*, u^*) yields

$$\dot{x} = f(x^*, u^*; p) + \frac{\partial f}{\partial x}(x^*, u^*; p)(x - x^*) + \frac{\partial f}{\partial u}(x^*, u^*; p)(u - u^*) \tag{10a}$$

$$y = g(x^*, u^*; p) + \frac{\partial g}{\partial x}(x^*, u^*; p)(x - x^*) + \frac{\partial g}{\partial u}(x^*, u^*; p)(u - u^*). \tag{10b}$$

Linearization

Often, we will re-define the dynamical variables according to deviations from the stationary point; i.e. according to

$$\tilde{x} = x - x^* \tag{11a}$$

$$\tilde{u} = u - u^* \tag{11b}$$

$$\tilde{y} = y - g(x^*, u^*; p), \tag{11c}$$

and define the system matrices

$$A(p, x^*, u^*) = \frac{\partial f}{\partial x}(x^*, u^*; p), \qquad B(p, x^*, u^*) = \frac{\partial f}{\partial u}(x^*, u^*; p) \tag{12a}$$

$$C(p, x^*, u^*) = \frac{\partial g}{\partial x}(x^*, u^*; p), \qquad D(p, x^*, u^*) = \frac{\partial g}{\partial u}(x^*, u^*; p). \tag{12b}$$

This leads to the linear time invariant (LTI) system given by

$$\dot{\tilde{x}} = A(p, x^*, u^*)\tilde{x} + B(p, x^*, u^*)\tilde{u} \tag{13a}$$

$$\tilde{y} = C(p, x^*, u^*)\tilde{x} + D(p, x^*, u^*)\tilde{u}, \tag{13b}$$

Discretization: Sampling of Continuous Systems

A major focus in this course is on discrete-time linear systems.
Discrete sampling at fixed intervals:

$$x_k = x(t_0 + T_s k) \quad \text{and} \quad y_k = y(t_0 + T_s k) \tag{14}$$

Zero-order Hold: ZOH is the assumption/choice of input being constant between samples.

$$u(t_c) = u_k, \quad \text{for } kT_s \leq t_c < (k+1)T_s \tag{15}$$

Shannon's Sampling Theorem: if the highest frequency of the system is w_0, then a sampling frequency of at least the double is needed for reconstruction:

$$w_s \geq 2w_0, \quad w_s = \frac{2\pi}{T_s} \tag{16}$$

Choosing based on desired samples per rise time:

$$T_s = t_r / N_r, \quad N_r \in [2; 4] \tag{17}$$

Discretization: Internal Model

In state-space models, we consider discretization of the continuous-time solutions:

$$x(t) = e^{A(p)(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(p)(t-s)}B(p)u(s)\,\mathrm{d}s, \tag{18a}$$

$$y(t) = C(p)x(t) + D(p)u(t). \tag{18b}$$

Using a sampling period T_s, the discrete-time system is given by

$$x_{k+1} = A_d(p,T_s)x_k + B_d(p,T_s)u_k \quad A_d(p,T_s) = e^{A(p)T_s} \tag{19a}$$

$$y_k = C(p)x_k + D(p)u_k \quad B_d(p,T_s) = \int_0^{T_s} e^{A(p)s}B(p)\,\mathrm{d}s \tag{19b}$$

These discrete-time system matrices, can be computed using the matrix exponential:

$$\begin{bmatrix} A_d(p,T_s) & B_d(p,T_s) \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A(p) & B(p) \\ 0 & 0 \end{bmatrix} T_s\right). \tag{20}$$

Discretization: External Model

For the external model, we consider the frequency domain:

$$y(s) = H(s)u(s), \quad H(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \tag{21}$$

The discretization is then done with a Z-transform as:

$$H_z(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{H(s)}{s}\right), \quad z \in \mathbb{C} \tag{22}$$

providing a new transfer function

$$y(z) = H_z(z)u(z) = \frac{\bar{b}_0 z^n + \bar{b}_1 z^{n-1} + \cdots + \bar{b}_n}{z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n} u(z) \tag{23}$$

for the time-domain we can utilize $H_d(q^{-1}) = H_z(q)$

$$y_t = H_d(q^{-1})u_t = \frac{\bar{b}_0 + \bar{b}_1 q^{-1} + \cdots + \bar{b}_n q^{-n}}{1 + \bar{a}_1 q^{-1} + \cdots + \bar{a}_n q^{-n}} u_t \tag{24}$$

also given as a difference model:

$$y_t + \bar{a}_1 y_{t-1} + \cdots + \bar{a}_n y_{t-n} = \bar{b}_0 u_t + \bar{b}_1 u_{t-1} + \cdots + \bar{b}_m u_{t-n} \tag{25}$$

Poles and Zeros

Consider the factor terms of transfer functions:

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = K_0 \frac{\Pi_i (s - z_i)}{\Pi_i (s - p_i)} \tag{26}$$

$$H_d(q^{-1}) = \frac{B_d(q^{-1})}{A_d(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}} = K_{d,0} \frac{\Pi_i (q - z_{d,i})}{\Pi_i (q - p_{d,i})}$$

Transfer functions has following properties

Zeros: $H(z_i) = 0$ (27)

Poles: $|H(p_i)| = \infty$ (28)

DC-gain: $H(s = 0), H_z(z = 1) = H_d(q^{-1} = 1)$ (29)

Poles

The Poles of the external models are the Eigenvalues of the internal model.

$$\mathcal{C}(A) = A(s) \tag{30}$$

Poles are related to the stability of LTI systems. An LTI system is unstable if

$$\textbf{Continuous:} \;\; 0 < \text{Re}(p_c) \tag{31a}$$

$$\textbf{Discrete:} \;\; 1 < |p_d| \tag{31b}$$

The relation between the poles of discrete (p_d) and continuous (p_c) systems is

$$p_d = e^{p_c T_s} \tag{32}$$

Zeros

The number of zeros m and poles n of a system:

$$\textbf{Continuous:}\ m \leq n \tag{33}$$

$$\textbf{Discrete:}\ \begin{cases} m = n - 1 & \text{(for } D = 0) \\ m = n & otherwise \end{cases} \tag{34}$$

with the additional zeros being from sampling with ZOH.
The m Continuous system zeros relates to a subset of the discrete zeros:

$$z_d = e^{z_c T_s} \tag{35}$$

Zero-Pole Cancellation,

$$z_i = p_i : \quad H(s) = \frac{s - z_i}{(s - p_i)(s - p_1)} = \frac{1}{(s - p_1)} \tag{36}$$

Transforms - Similarity Transform and Diagonal Transform

For some reasons (computation), we might want to change the states of the internal model by

$$z_t = \Upsilon x_t \tag{37}$$

$$z_{t+1} = \Upsilon A \Upsilon^{-1} z_t + \Upsilon B u_t \tag{38}$$

$$y_t = C \Upsilon^{-1} z_t + D u_t \tag{39}$$

The external model is unchanged by the transformation:

$$H(q) = C\Upsilon^{-1}(qI - \Upsilon A \Upsilon^{-1})^{-1}\Upsilon B + D = C(qI - A)^{-1}B + D \tag{40}$$

A simple transformation is the **diagonal transform**:

$$A_{diag} = \Upsilon A \Upsilon^{-1} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{41}$$

with Υ being constructed by the right eigenvectors of A.

Transforms - External ⇒ Internal

Consider the external system:

$$y_t + a_1 y_{t-1} + \cdots + a_n y_{t-n} = b_0 u_t + b_1 u_{t-1} + \cdots + b_n u_{t-n} \tag{42}$$

we then have the transfer function:

$$H(q) = \frac{B(q^{-1})}{A(q^{-1})} = \frac{b_0 + b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} = \sum_{i=0}^{\infty} h_i q^{-1} \tag{43}$$

Minimal representation: An internal model with minimum number of states.

Examples of forms with minimal representation is the 4 canonical forms

Transforms - Canonical Forms

Controller canonical form:

$$A_c = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{44}$$

$$C_c = [b_1 - b_0 a_1, b_2 - b_0 a_2, \ldots, b_n - b_0 a_n] \qquad D_c = b_0 \tag{45}$$

Observer canonical form:

$$A_o = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} \qquad B_o = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ \vdots \\ b_n - b_0 a_n \end{bmatrix} \tag{46}$$

$$C_o = [1, 0, \cdots, 0] \qquad D_o = b_0 \tag{47}$$

Transforms - Canonical Forms

Controllability canonical form:

$$A_{co} = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & \cdots & 0 & -a_{n-1} \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & -a_1 \end{bmatrix} \qquad B_{co} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{48}$$

$$C_{co} = (h_1, h_2, \cdots, h_n) \qquad D_{co} = h_0 \tag{49}$$

Observability canonical form:

$$A_{ob} = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{ob} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \tag{50}$$

$$C_{ob} = (1, 0, \ldots, 0) \qquad D_{ob} = h_0 \tag{51}$$

Transforms - Canonical Forms

relation between canonical forms:

$$A_c = A_o^T, \quad A_{co} = A_{ob}^T \tag{52}$$

$$B_c = C_o^T, \quad B_{co} = C_{ob}^T \tag{53}$$

$$B_o = C_c^T, \quad B_{ob} = C_{co}^T \tag{54}$$

$$D_c = D_o = D_{co} = D_{ob} = b_0 = h_0 \tag{55}$$

Transforms - Direct realization

consider the more general external model:

$$y_t + a_1 y_{t-1} + \cdots + a_{n_a} y_{t-n_a} = b_0 u_t + b_1 u_{t-1} + \cdots + b_{n_b} u_{t-n_b} \tag{56}$$

A non-minimal internal model can be constructed as:

$$\Phi_d = \left[\begin{array}{cccc|c|cccc}
-a_1 & \cdots & -a_{n_a-1} & -a_{n_a} & -b_1 & \cdots & -b_{n_b-1} & -b_{n_b} \\
1 & & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & \ddots & & \vdots & \vdots & & \vdots & \vdots \\
0 & & 1 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & \cdots & 0 & 0 & 1 & & 0 & 0 \\
\vdots & & \vdots & \vdots & & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & & 1 & 0
\end{array}\right] \qquad \Gamma_d = \left[\begin{array}{c}
-b_0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ \hline 0 \\ \vdots \\ 0
\end{array}\right] \tag{57}$$

$$\phi_d^T = (a_1, \ldots, a_{n_a}, b_1, \ldots, b_{n_b}) \qquad \Delta_d = b_0 \tag{58}$$

Controllability and Reachability

Definition:
A system is said to be controllable, if it is possible to move the system from an arbitrary state value to the origin in finite time.

Definition:
A system is said to be reachable, if it is possible to move the system from one arbitrary state value to another arbitrary state in finite time.

Reachable $\Rightarrow$ Controllable, not the reverse

An n-state system is reachable if and only if the reachability matrix W_c has full rank $(k > n)$.

$$W_c(k) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{k-1}B \end{bmatrix} \tag{59}$$

with the reachability Gramian given by $\Sigma_k^c = W_c(k) W_c(k)^T$

Controllability and Reachability - Control

For discrete-time LTI systems, it is possible to give explicit results on how to construct a k-step input sequence

$$x_k = A^k x_0 + W_c(k) U_{k-1} \tag{60}$$

$$U_{k-1}^T = \begin{bmatrix} u_{k-1} & u_{k-2} & \cdots & u_0 \end{bmatrix} \tag{61}$$

which brings the system from any initial condition, x_0, to a desired state, $\hat{x}$.

Though no unique sequence exist, the sequence minimizing the control usage is given by

$$U_{k-1}^* = W_c(k)^T (\Sigma_k^c)^{-1} \Big[\hat{x} - A^k x_0\Big]. \tag{62}$$

minimizing

$$\min_{u_{k-1},\ldots,u_0} \sum_{j=0}^{k-1} u_j^T u_j \tag{63a}$$

Controllability and Reachability - Continuous-Time

For a general continuous-time LTI system

$$\dot{x} = Ax + Bu, \tag{64}$$

the reachability question is related to properties of the continuous-time reachability Gramian, Σ^c. This matrix function satisfies the dynamical condition

$$\dot{\Sigma}^c = A\Sigma^c + \Sigma^c A^T + BB^T \tag{65a}$$

$$\Sigma^c(t_0) = 0. \tag{65b}$$

The LTI system (64) is said to be reachable if $W(t)$ is symmetric and positive semi-definite for any $t \geq t_0$.

For continuous-time, reachability is equivalent to controlability

Observability and constructability

Definition:
A system is said to be observable if, any initial state can be estimated using only the information from the following outputs and inputs.

Definition:
A system is said to be constructable if, for any possible evolution of state and control vectors, the current state can be estimated using only the information from outputs.

Observable $\Rightarrow$ constructable, but the reverse is not true

An n-state system is reachable if and only if the observability matrix W_o has full rank ($k > n$).

$$W_o(k)^T = \left[C^T \ \ (CA)^T \ \ (CA^2)^T \ \ \cdots \ \ (CA^{k-1})^T\right] \tag{66}$$

with the observability Gramian given by $\Sigma_k^o = W_o(k)W_o(k)^T$

Observability and constructability - Continuous-Time

For a general continuous-time LTI system,

$$\dot{x} = Ax + Bu, \tag{67a}$$

$$y = Cx + Du, \tag{67b}$$

the observability question is related to properties of the continuous-time observability Gramian, Σ^o. This matrix function satisfies the dynamical condition

$$\dot{\Sigma}^o = A\Sigma^o + \Sigma^o A^T + C^T C \tag{68a}$$

$$\Sigma^o(t_0) = 0. \tag{68b}$$

The LTI system (67) is said to be observable if $\Sigma^o(t)$ is symmetric and positive semi-definite for any $t \geq t_0$.

Stability

For operations of systems, an important aspect is the system's stability near a stationary point x_s.

Several definitions of stability exist; e.g. marginally stable systems and asymptotically stable systems.

- **Marginally stable**: x_s is said to be (marginally) stable if any solution trajectory $\{x(t),\ t \in [t_0, \infty]\}$ is bounded.
- **Asymptotically stable**: x_s is said to be asymptotically stable if any solution trajectory converges to x_s $(x(t) \to x_s)$ as time progresses $(t \to \infty)$.

A system which is not stable (i.e. not marginally stable) is said to be unstable.

Additionally we say a system is BIBO stable, if for any bounded input, the output is also bounded (Asymptotic $\Rightarrow$ BIBO)

Stability of LTI Systems

For LTI systems, the requirements of the different definitions of stability is given below, with a system being that type of stable if and only if all of the requirements is fulfilled

Continuous-time

Marginally stable:

- $\mathrm{Re}\{eig(A)\} \leq 0$
- $\forall\, \mathrm{Re}\{eig(A)_i\} = 0$, the AM=GM

Asymptotically stable:

- $\mathrm{Re}\{eig(A)\} < 0$

Discrete-time

Marginally stable:

- $|eig(A)| \leq 1$
- $\forall |eig(A)_i| = 1$, the AM=GM

Asymptotically stable:

- $|eig(A)| < 1$

* AM = Algebraic multiplicity (# of identical eigenvalues)
** GM = geometric multiplicity (# of associated eigenvectors)

Stability of General Systems: Lyapunov's 2nd Method

For general systems, the conditions for stability is more complex, but one method is Lyapunov's second method

A Lyapunov function $V(x)$ for a system is defined as

Continuous-time

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0$$

Scalar function $V(x)$:

- $V(0) = 0$
- $V(x)$ is C^1 (differential)
- $\dot{V}(x(t)) = \left(\frac{\partial V}{\partial x}\right)^T f(x(t)) \leq 0$

Discrete-time

$$x_{t+1} = f(x_t), \quad f(0) = 0$$

- $V(0) = 0$
- $V(x)$ is C^1 (differential)
- $V(x_{t+1}) - V(x_t) \leq 0$

A stationary point x_s is stable if a Lyapunov function exists in the neighbourhood of x_s. If the inequalities are satisfied strictly, x_s is asymptotically stable

Local Stability of Nonlinear Systems

Alternatively, if we only consider stability of a specific section of the system space around a stationary point x_s,

$$\dot{x} = 0 = f(x_s, u_s) \text{ or } x_s = f(x_s, u_s), \tag{69}$$

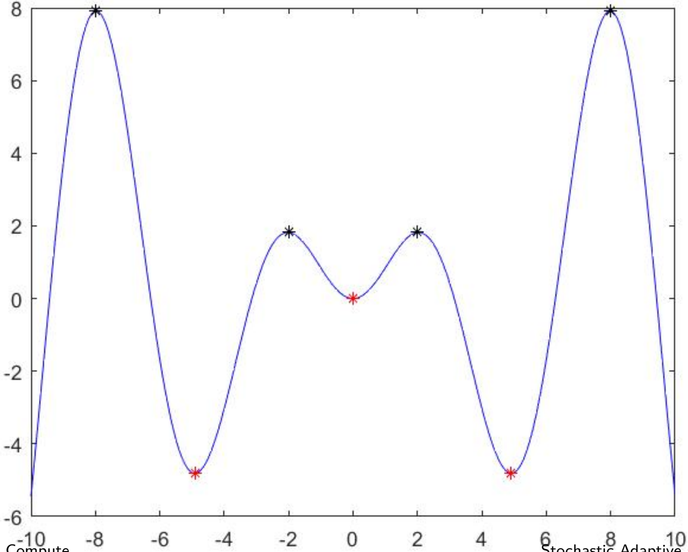
a linear approximation can be used if f is differentiable at the stationary point:

$$A = \frac{\partial f}{\partial x}(x^*, u^*). \tag{70}$$

The system is locally stable (marginal or asymptotic) around the stationary point if the LTI requirements are fulfilled.

Example of stationary points

Consider a ball lying on the curve. Which points are stationary, and which are stable?



Questions?