Time Series Analysis

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Outline of the lecture

Spectral Analysis (Chapter 7)

- The periodogram

- Consistent estimates of the spectrum

Spectrum analysis

- Describes the variations in the frequency domain.

- Useful if the time series contains more frequencies.

- A parametric approach is obtained by estimating a model and then find the 'theoretical' spectrum for the estimated model.

- Here we shall focus on the classical non-parametric approaches.

The periodogram

Based on the known theoretical relationship, it seems obvious to apply the following estimate for the spectrum

$$I_N(\omega) = \frac{1}{2\pi} \sum_{k=-(N-1)}^{N-1} C(k)e^{-i\omega k} \qquad (1)$$

($|\omega| \leq \pi$),where $C(k)$ is the estimate of the autocovariance function based on N observations: $Y_1, \ldots, Y_N$.

The periodogram

If we assume that $\{Y_t\}$ has the mean 0, then we can write $I_N(\omega)$ as

$$I_N(\omega) = \frac{1}{2\pi} \sum_{k=-(N-1)}^{N-1} C(k)e^{-i\omega k} \quad |\omega| \leq \pi$$

$$= \frac{1}{2\pi} \sum_{k=-(N-1)}^{N-1} \frac{1}{N} \sum_{t=1}^{N-|k|} Y_t Y_{t+|k|} e^{-i\omega k}$$

$$= \frac{1}{2\pi N} * \sum_{t=1}^{N} Y_t e^{-i\omega t} * \sum_{t=1}^{N} Y_t e^{i\omega t}$$

$$= \frac{1}{2\pi N} \left| \sum_{t=1}^{N} Y_t e^{-i\omega t} \right|^2 , \tag{2}$$

which we can formulate as

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Periodogram

The periodogram is defined for all ω in $[-\pi, \pi]$, but in order to achieve independence between $I_N(\omega)$ at different values of ω (more about this later) it is advisable only to calculate the periodogram at the so-called *fundamental frequencies*,

$$\omega_p = 2\pi p/N \ p = 0, 1, \ldots, \lfloor N/2 \rfloor. \tag{4}$$

It is seen that the sample spectrum is proportional to the squared amplitude of the Fourier transform of the time series: $Y_1, \ldots, Y_N$.

Properties of the periodogram

Let $\{Y_t\}$ be normally distributed white noise having variance σ_Y^2. Then the following holds

1. $\{I(\omega_p)\}$ $p = 0, 1, \ldots, [N/2]$ are stochastic independent

2. $\frac{I(\omega_p)4\pi}{\sigma_Y^2} \in \chi^2(2)$ $\quad p \neq 0, N/2$ $\quad$ for N even.

3. $\frac{I(\omega_p)2\pi}{\sigma_Y^2} \in \chi^2(1)$ $\quad p = 0, N/2$.

If the assumption of normality does not hold then the theorem is only an approximation.

Consistent estimates of the spectrum

The problem with the periodogram, is that it contains too many values of the estimated autocovariance function. Thus, it is obvious to apply the *truncated periodogram*

$$\widehat{f}(\omega) = \frac{1}{2\pi} \sum_{k=-M}^{M} C(k) e^{-i\omega k} \ M < (N-1), \tag{5}$$

where M is the *truncation point*. The truncated periodogram is a linear combination of $M + 1$ values of $C(k)$, and thus

$$V[\widehat{f}(\omega)] = O(M/N). \tag{6}$$

Consisteny estimates

- A lag-window is identified with a sequence of $\{\lambda_k\}$, which fulfills

  1. $\lambda_0 = 1$
  2. $\lambda_k = \lambda_{-k}$
  3. $\lambda_k = 0 \ |k| > M$,

  where M is the *truncation point*.

- Corresponding to a lag-window $\{\lambda_k\}$ we have the *smoothed spectrum*

$$\widehat{f}(\omega) = \frac{1}{2\pi} \sum_{k=-(N-1)}^{N-1} \lambda_k C(k) e^{-i\omega k}. \tag{7}$$

Examples using S-PLUS

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## Estimates using a parametric approach


## Estimates of periodogram (raw spectrum)

## Estimates of smoothed spectrum using a Daniell windo
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