Some results on Dynamic Gossip

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August 26, 2015

Abstract

The gossip problem consists in a network of agents trying to learn all of their respective secrets using phone calls, in which they exchange all the secrets they know. A dynamic version of this problem does not assume that the agents initially know all the phone numbers, but still they can learn them by exchanging the known secrets and phone numbers in a call. Here we study the class of graphs that are always successful for a variety of calling protocols.

1 Introduction

Gossip protocols are procedures for spreading secrets among a group of agents (nodes) in a network. After a call from agent x to agent y, denoted xy, these agents exchange all the information they have, e.g. all the secrets they know at the time.

A **solution** for a gossip problem is a sequence of phone calls such that after its execution, all the agents know all the secrets.

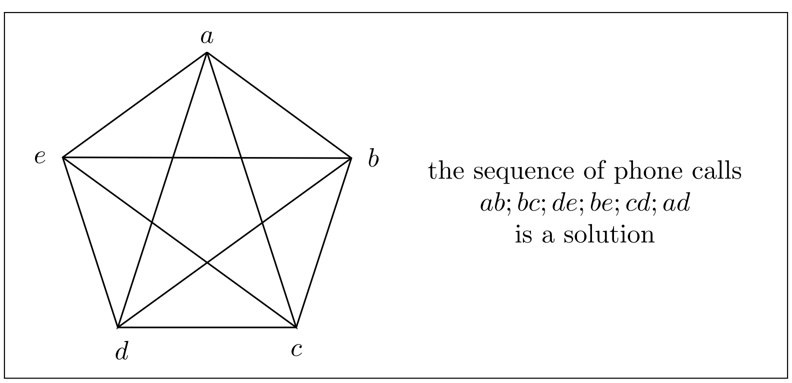

Figure 1: An illustration of the complete (undirected) graph with $n = 5$ agents (nodes) representing the classical gossip problem. This problem can be solved with $2n - 4 = 6$ phone calls. Moreover, this number of phone calls are needed [3, 2], for any $n > 3$.

One might consider deleting the assumption that the agents initially know all the phone numbers of the other agents. In this case, more phone calls might be needed before all the secrets are known by every agent; see also [1].

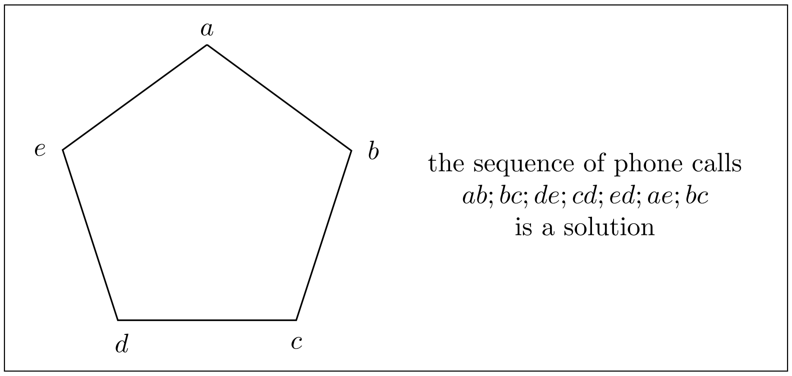

Figure 2: An illustration of a ring (as an undirected graph) for the gossip problem with $n = 5$ agents; i.e. agents related by an edge can call each other. For the class of such gossip problems that can be solved, $2n - 3 = 7$ phone calls might be needed for any $n > 4$.

In these incomplete graphs where not all agents have all the phone numbers (at least initially), we might consider that in phone calls the agents exchange the secrets as well as their contact lists. This way, if an agent x learns a new phone number, say that of agent y, then the call xy may take place afterwards. Indeed, in the following all the protocols considered will assume as a condition for a call xy that the agent x knows y's number. We call this problem the **dynamic gossip problem**.

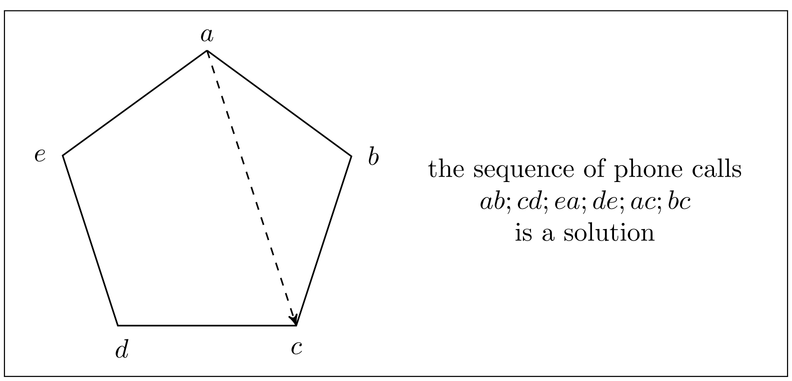

Figure 3: An illustration of the dynamic gossip problem with $n = 5$ agents. After the call ab, agent a learns c's number and so the call ac (dashed arrow) is enabled. For this particular case, the fact that agents can learn new phone numbers enables to recover the lower bound of $2n - 4 = 6$ phone calls.

From here on, we represent a dynamic gossip problem as a triple $G = (A, N, S)$, where:

- A is a finite set of agents $A = \{a, b, \ldots\}$

- N is the "number" relation $N \subseteq A \times A$.
- S is the "secrets" relation $S \subseteq A \times A$ encoding the secrets initially known by agents: namely, their own secrets $S = \{(x, x) \mid x \in A\}$.

Let us denote by $S_x \subseteq \{x\} \times A$ the sets of agents y such that Sxy in a given graph G. And similarly for N_x. Then, after a call xy in such a graph, the resulting state can again be represented as such a sequence, denoted $G^{xy} = (A, N^{xy}, S^{xy})$, whose components are defined as

$$S_z^{xy} = \begin{cases} S_z & \text{if } z \notin \{x, y\} \\ S_x \cup S_y & \text{otherwise} \end{cases}$$

$$N_z^{xy} = \begin{cases} N_z & \text{if } z \notin \{x, y\} \\ N_x \cup N_y & \text{otherwise} \end{cases}$$

For a given sequence of calls $\sigma = xy; \ldots ; x'y'$, let us denote the resulting gossip state as follows

$$G^\sigma = ((G^{xy}) \cdots)^{x'y'}$$

Fact 1 *Given an arbitrary dynamic gossip graph $G = (A, N, S)$ and any sequence of executable calls σ, the resulting state $G^\sigma = (A, N^\sigma, S^\sigma)$ has the following property: $S^\sigma \subseteq N^\sigma$.*

Some properties of graphs that will be interesting for our purpose are the following:

(WC) graph $G = (A, N, S)$ is weakly connected iff $(N \cup N^{-1})^* = A \times A$

(SC) a graph $G = (A, N, S)$ is strongly connected iff $N^* = A \times A$

(skinning) the skinning of a graph G, denoted $s(G)$ is the graph resulting from deleting N-terminal points, i.e. $s(G) = (A', N', S')$ where $A' = \{z \in A \mid N_z \neq \varnothing\}$, and $N' = N \cap (A' \times A')$, $S' = S \cap (A' \times A')$.

(Sun) a graph $G = (A, N, S)$ is a sun iff $s(G)$ is strongly connected; (in other words, a sun graph consists of a strongly connected component with outgoing edges)

Call an agent **non-expert** iff she does not know all the secrets. We can list some protocols that are definable for the dynamic gossip problem. A protocol is defined by some condition $\varphi(x, y)$ upon pairs of agents $x, y \in A$, and has the following general form (where we use the following notation: non-deterministic choice is "$\bigcup$", test is "?" and concatenation is ";")

$$\texttt{while} \text{ not all agents are expert} \quad \texttt{do} \left(\textstyle\bigcup_{x,y \in A} ?\varphi(x, y); \texttt{execute the call } xy\right)$$

We define in particular the following protocols:

- Random Protocol: defined by the (minimal) condition Nxy that agent x has the phone number of y, for a call xy;

- Learn New Secrets (LNS): defined by the condition that $Nxy \wedge \neg Sxy$, i.e. the agent x knows the number but not the secret of y;
- Call Me Once (CO): any two agents x, y can only call each other once;
- Fresh Calls (FRE): an agent x can call agent y iff it is the first call of x and x knows y's number, or x heard y's number in the last call x made/received.

We say that a protocol Π is **successful** for a graph $G = (A, N, S)$ iff each possible execution σ (i.e. sequence of calls) leads to a graph of the form $G = (A, A \times A, A \times A)$. Recall that if σ is (possibly infinite) execution of a protocol Π in a graph G, we say that σ is a **fair execution** iff for any call $ij \in A \times A$ that can be made infinitely-many times during σ (w.r.t. Π), is actually made infinitely-many times during σ. We say that a protocol Π is **fairly successful** for a graph G iff it is successful in each fair execution σ of the protocol Π.

We also say that a protocol is (fairly) successful in a class of graphs $\mathbb{G}$ iff for each $G \in \mathbb{G}$ the protocol is (fairly) successful in G. The following table summarizes some of the results obtained so far.

protocol		class of graphs
Random	fairly successful	WC
LNS	successful	Sun
CO	successful	WC
FRE	fairly successful	WC

References

[1] S. M. Hedetniemi, S. T. Hedetniemi, and A. L. Liestman. A survey of gossiping and broadcasting in communication networks. *Networks*, 18(4):319–349, 1988.

[2] C. A. J. Hurkens. Spreading gossip efficiently. *NAW*, 5(1):208–210, 2000.

[3] R. Tijdeman. On a telephone problem. *Nieuw Archief voor Wiskunde*, 3(19):188–192, 1971.