01622 Advanced Dynamical Systems: Applications in Science and Engineering

Week 9: Distributed time delays

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What are distributed time delays?

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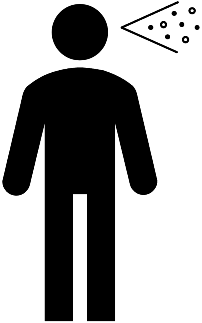
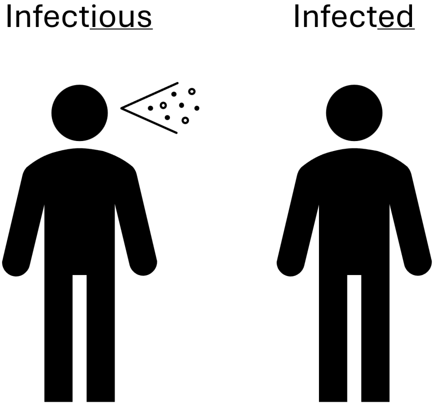
Infectious Susceptible



Shortly after . . .

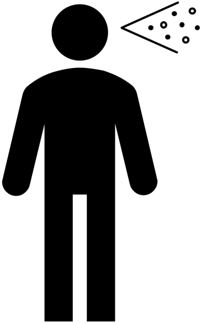


Infectious

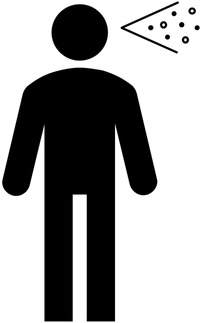
Infected

A while later (incubation period)

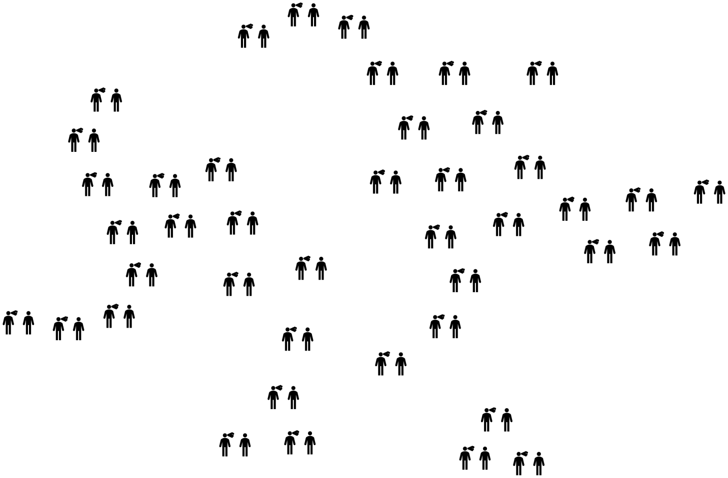
Infectious



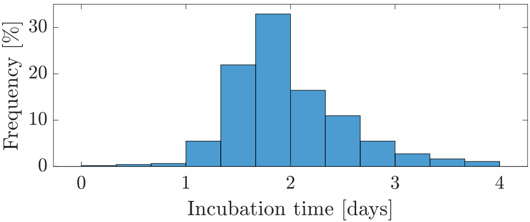
Infectious



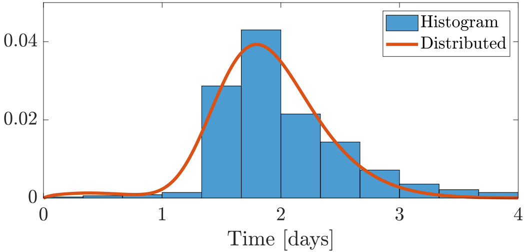
The incubation period varies across the population



Distribution of incubation periods



Distribution of incubation periods



Susceptible-infectious-recovered model

SIR model without delays

$$\begin{aligned}\dot{S}(t) &= -\beta S(t) I(t),\\ \dot{I}(t) &= \beta S(t) I(t) - \eta I(t),\\ \dot{R}(t) &= \eta I(t)\end{aligned}$$

SIR model with distributed time delay

$$\begin{aligned}\dot{S}(t) &= -\beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\, \mathrm{d}s,\\ \dot{I}(t) &= \beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\, \mathrm{d}s - \eta I(t),\\ \dot{R}(t) &= \eta I(t)\end{aligned}$$

Population dynamics (modified logistic equation)

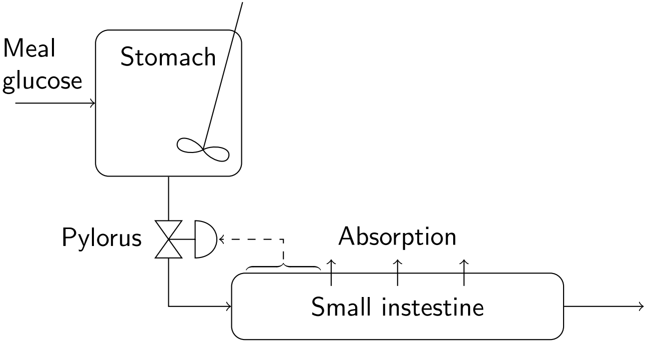
Logistic equation

$$\dot{N}(t) = \kappa N(t) \left(1 - \frac{N(t)}{K} \right)$$

Logistic equation w. distributed time delay

$$\dot{x}(t) = \kappa N(t) \left(1 - \frac{1}{K} \int_{-\infty}^{t} \alpha(t-s) N(s) \, \mathrm{d}s \right)$$

Diabetes (or physiology in general)



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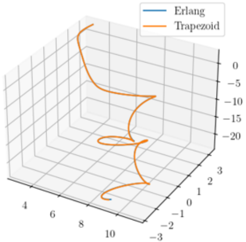
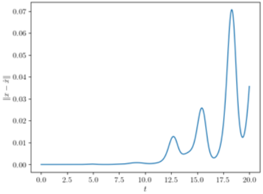
Particle flow in velocity field

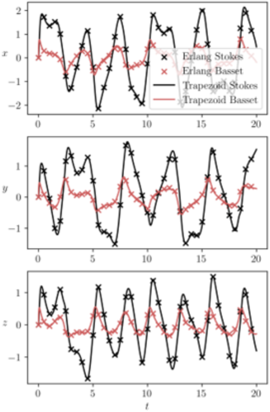
Particle subject to Stoke's drag force and Basset history force

$$\dot{x}_p = u_p,$$

$$\dot{u}_p = \underbrace{\frac{1}{\mathrm{St}}\mathcal{F}(a)(u_p - u_f)}_{\text{Nonlinear drag}} + \underbrace{C\int_0^t \frac{1}{\sqrt{t-s}}(\dot{u}_p - \dot{u}_f)\,\mathrm{d}s}_{\text{Basset history force}}$$







Collaboration with PhD candidate Zejian You, Asst. Prof. Qi Wang, and Prof. Gustaaf Jacobs from San Diego State University.

Why are distributed time delays important?

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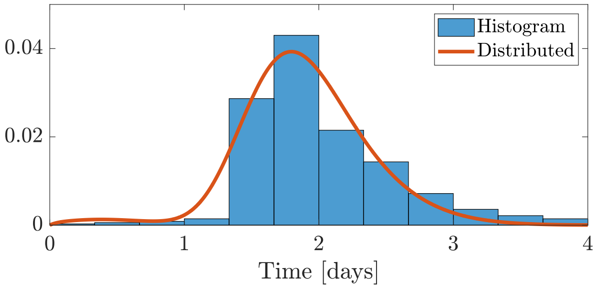
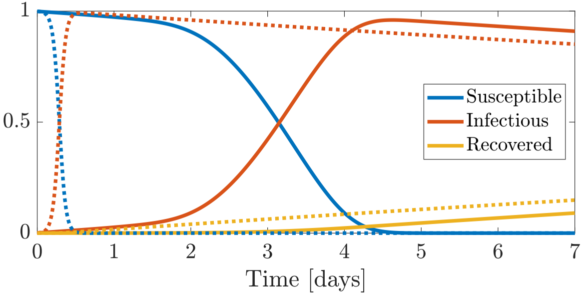


Why are distributed time delays important?

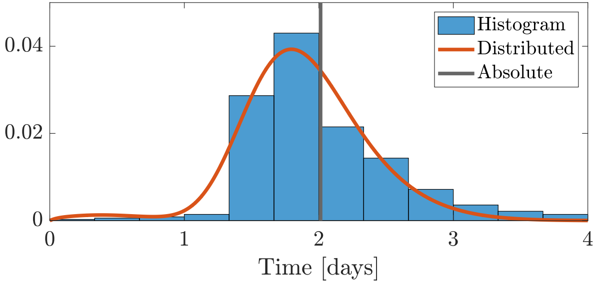
$$\dot{S}(t) = -\beta S(t) I(t), \qquad \dot{S}(t) = -\beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s,$$

$$\dot{I}(t) = \beta S(t) I(t) - \eta I(t), \qquad \dot{I}(t) = \beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s - \eta I(t),$$

$$\dot{R}(t) = \eta I(t), \qquad \dot{R}(t) = \eta I(t)$$

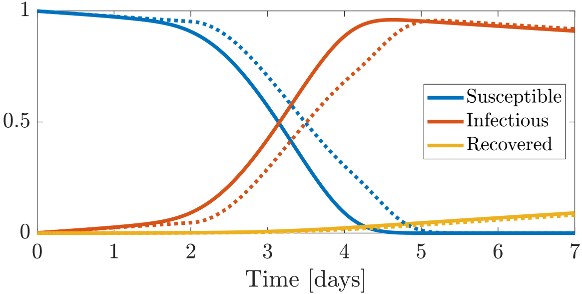


Why are distributed time delays important?



Why are distributed time delays important?

$$\begin{aligned}
\dot{S}(t) &= -\beta S(t) I(t-\tau), & \dot{S}(t) &= -\beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s, \\
\dot{I}(t) &= \beta S(t) I(t-\tau) - \eta I(t), & \dot{I}(t) &= \beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s - \eta I(t), \\
\dot{R}(t) &= \eta I(t), & \dot{R}(t) &= \eta I(t)
\end{aligned}$$



Stability analysis

Stability – Linear systems

For linear system systems in the form

$$\dot{x}(t) = Ax(t) + \int_{-\infty}^{t} \alpha(t-s)x(s)\,\mathrm{d}s + Bu(t) + Ed(t) \tag{1}$$

the stability is determined by A and α (which is a matrix)

Characteristic equation

$$P(\lambda) = \det\left(A + \int_{0}^{\infty} e^{-\lambda s}\alpha(s)\,\mathrm{d}s - \lambda I\right) = 0 \tag{2}$$

In general, infinitely many solutions

Stability – Nonlinear systems

For nonlinear systems in the general form

$$\begin{aligned} \dot{x}(t) &= f(x(t), z_1(t), \ldots, z_m(t), u(t), d(t), p), && \text{(3a)} \\ z_i(t) &= \int_{-\infty}^{t} \alpha_i(t-s) x(s) \, \mathrm{d}s && \text{(3b)} \end{aligned}$$

the stability is determined by A and α_i for $i = 1, \ldots, m$

Steady state ($x(t) = \bar{x}$ for all t)

$$\begin{aligned} 0 &= f(\bar{x}, \bar{z}_1, \ldots, \bar{z}_m, \bar{u}, \bar{d}, p), && \text{(4a)} \\ \bar{z}_i &= \int_{-\infty}^{t} \alpha_i(t-s) \bar{x} \, \mathrm{d}s \\ &= \int_{-\infty}^{t} \alpha_i(t-s) \, \mathrm{d}s \, \bar{x} \\ &= \int_{0}^{\infty} \alpha_i(t) \, \mathrm{d}t \, \bar{x} && \text{(4b)} \end{aligned}$$

Stability – Nonlinear systems

Characteristic equation

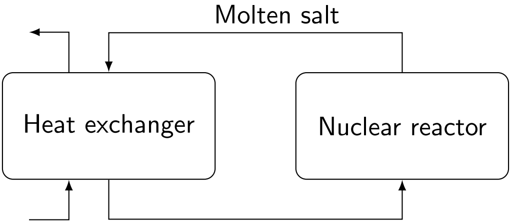
$$P(\lambda) = \det\left(A + \sum_{i=1}^{m} G_i \int_0^{\infty} e^{-\lambda s}\alpha_i(s)\,\mathrm{d}s - \lambda I\right) = 0 \tag{5}$$

Matrices

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad G_i = \frac{\partial f}{\partial z_i} = \begin{bmatrix} \frac{\partial f_1}{\partial z_{i,1}} & \cdots & \frac{\partial f_1}{\partial z_{i,k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_{i,1}} & \cdots & \frac{\partial f_1}{\partial z_{i,k}} \end{bmatrix} \tag{6}$$

Nuclear reactor models

Molten salt nuclear reactor and nonuniform flow in pipes

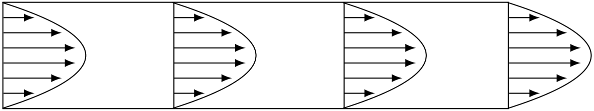


Molten salt

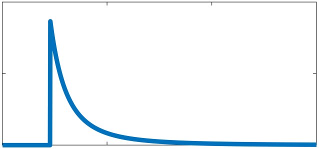
Heat exchanger

Nuclear reactor

Non-uniform velocity profile



Kernel for Hagen-Poiseuille flow (quadratic velocity profile)



Ritschel, T.K.S., 2025. Numerical Optimal Control for Distributed Delay Differential Equations: A Simultaneous Approach based on Linearization of the Delayed Variables. In: Proceedings of the 2025 European Control Conference (ECC), June 24-27, Thessaloniki, Greece. DOI: 10.23919/ECC65951.2025.11187183.

Nuclear reactor model 8 – Model 7 revisited

Reactivity and thermal reactivity

$$\rho(t) = \rho_{th}(t) + \rho_{ext}(t), \qquad \dot{\rho}_{th}(t) = -\kappa \dot{T}_r \tag{7}$$

Mass balance equations (ρ_s is the salt density and v_a is the average velocity)

$$\dot{C}_n(t) = \frac{\rho(t) - \beta}{\Lambda} C_n(t) + \sum_{i=1}^{m} \lambda_i C_i(t), \tag{8a}$$

$$\dot{C}_i(t) = \frac{\beta_i}{\Lambda} C_n(t) - \lambda_i C_i(t) + (C_{i,in}(t) - C_i(t)) D \tag{8b}$$

$$D = \frac{F}{V}, \quad f = \rho_s F, \quad F = A v_a \tag{8c}$$

Energy balance equations

$$\dot{T}_r(t) = \frac{f}{n_r}(T_{r,in}(t) - T_r(t)) + \frac{Q_g(t)}{n_r c_P}, \tag{9a}$$

$$\dot{T}_{hx}(t) = \frac{f}{n_{hx}}(T_{hx,in}(t) - T_{hx}(t)) - \frac{k_{hx}}{n_{hx} c_P}(T_{hx}(t) - T_c) \tag{9b}$$

Memory states

Memory states

$$C_{i,in}(t) = e^{-\lambda_i \tau} \int_{-\infty}^{t} \alpha_f(t-s) C_i(s) \, \mathrm{d}s, \tag{10a}$$

$$T_{r,in}(t) = \int_{-\infty}^{t} \alpha_h(t-s) T_{hx}(s) \, \mathrm{d}s, \tag{10b}$$

$$T_{hx,in}(t) = \int_{-\infty}^{t} \alpha_h(t-s) T_r(s) \, \mathrm{d}s \tag{10c}$$

Decay time

$$\tau = 2\tau_0 \tag{11}$$

Kernels

$$\alpha_f(t) = \begin{cases} 2\frac{\tau_0^2}{t^3}, & t \geq \tau_0, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_0 = \frac{L}{2v_a}, \tag{12a}$$

$$\alpha_h(t) = \begin{cases} \frac{1}{2}\frac{\tau_0^2}{t^3}, & t \geq \tau_0/2, \\ 0, & \text{otherwise} \end{cases} \tag{12b}$$

ODE approximation

ODE approximation through kernel approximation

DDE with distributed time delay

$$\dot{x}(t) = f(x(t), z(t)), \qquad z(t) = \int_{-\infty}^{t} \alpha(t-s)x(s)\,\mathrm{d}s \tag{13}$$

Approximate kernel

$$\alpha(t) \approx \hat{\alpha}(t) = \sum_{m=0}^{M} c_m \ell_m(t), \qquad \ell_m(t) = \frac{a^{m+1}}{m!} t^m e^{-at} \tag{14}$$

Derivatives of basis functions

$$\dot{\ell}_m(t) = a(\ell_{m-1}(t) - \ell_m(t)), \qquad \dot{\ell}_0 = -a\ell_0(t) \tag{15}$$

Substitute into integral

$$z(t) \approx \hat{z}(t) = \sum_{m=0}^{M} c_m \int_{-\infty}^{t} \ell_m(t-s)x(s)\,\mathrm{d}s = \sum_{m=0}^{M} c_m z_m(t) \tag{16}$$

Differentiate z_m

$$\dot{z}_m(t) = \ell_m(0)x(t) + \int_{-\infty}^{t} \dot{\ell}_m(t-s)x(s)\,\mathrm{d}s = \begin{cases} a(x(t) - z_0(t)), & m = 0, \\ a(z_{m-1}(t) - z_m(t)), & m \geq 1 \end{cases} \tag{17}$$

Approximate ODEs

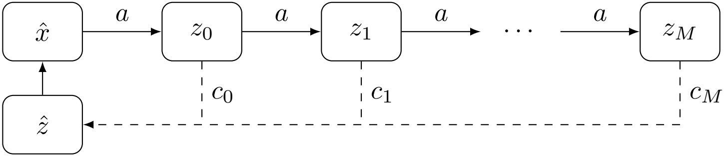
Approximate system

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{z}(t)), \qquad \hat{z}(t) = \sum_{m=0}^{M} c_m z_m(t) \tag{18}$$

Auxiliary memory states

$$\dot{z}_0(t) = a(\hat{x}(t) - z_0(t)), \tag{19}$$

$$\dot{z}_m(t) = a(z_{m-1}(t) - z_m(t)), \qquad m = 1, \dots, M \tag{20}$$



Approximate ODEs and stability analysis

Approximate ODEs

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{z}(t)), \tag{21}$$
$$\dot{Z}(t) = AZ(t) + B\hat{x}(t), \tag{22}$$
$$\hat{z}(t) = CZ(t) \tag{23}$$

Matrices

$$A = a \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}, \qquad B = a \begin{bmatrix} 1 \\ \\ \\ \\ \end{bmatrix}, \tag{24}$$
$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_M \end{bmatrix} \tag{25}$$

Approximate steady state and stability analysis

Steady state ($\hat{x}(t) = \bar{x}$ for all t)

$$
\begin{aligned}
0 &= f(\bar{x}, \bar{z}), && \text{(26a)}\\
0 &= A\bar{Z} + B\bar{x}, && \text{(26b)}\\
\bar{z} &= C\bar{Z} && \text{(26c)}\\
&= -CA^{-1}B\bar{x} = \left(\sum_{m=0}^{M} c_m\right)\bar{x} && \text{(26d)}
\end{aligned}
$$

The steady state state is the same as for the original system if $\sum_{m=0}^{M} c_M = \int_0^\infty \alpha(t)\,\mathrm{d}t$

Characteristic equation

$$P(\lambda) = \det\left(\begin{bmatrix} F & GC \\ B & A \end{bmatrix}\right) = 0 \tag{27}$$

Jacobian matrices (evaluated in steady state)

$$F = \frac{\partial f}{\partial x}, \qquad G = \frac{\partial f}{\partial z} \tag{28}$$

Does it work?

Choose the coefficients

$$c_m = \int_{s_m}^{s_{m+1}} \alpha(s)\,\mathrm{d}s, \qquad s_m = m\Delta s, \qquad \Delta s = 1/a \tag{29}$$

As $a \to \infty$ and $M/a \to \infty$, the kernel approximation converges

- uniformly

$$\hat{\alpha}(t) \to \alpha(t), \qquad t \in [0, \infty) \tag{30}$$

- weakly

$$\int_0^t \hat{\alpha}(s)\,\mathrm{d}s \to \int_0^t \alpha(s)\,\mathrm{d}s, \tag{31}$$

- in L_1-norm

$$\int_0^\infty |\hat{\alpha}(s) - \alpha(s)|\,\mathrm{d}s \to 0, \tag{32}$$

if α is continuous and bounded

Does it work?

If α is also exponentially bounded,

- The steady state converges

$$\hat{\bar{x}} \to \bar{x} \tag{33}$$

- The state and memory state converge uniformly

$$\hat{x}(t) \to x(t), \qquad \hat{z}(t) \to z(t), \qquad t \in [t_0, t_f] \tag{34}$$

- For every isolated root, there are roots of the characteristic equation that converge*

Numerical example: Bifurcation analysis

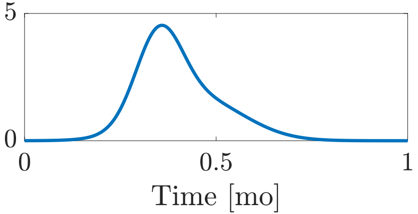
Numerical example

- System

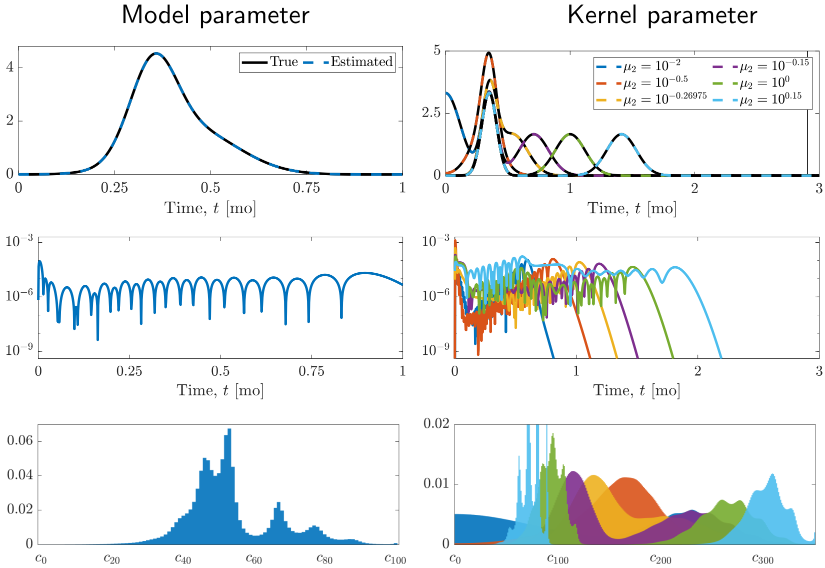
$$\dot{N}(t) = \kappa N(t) \left(1 - \frac{1}{K} \int_{-\infty}^{t} \alpha(t-s) N(s) \, \mathrm{d}s\right)$$

- Kernel

$$\alpha(t) = \gamma_1 F(t; \mu_1, \sigma_1) + \gamma_2 F(t; \mu_2, \sigma_2),$$
$$F(t; \mu, \sigma) = \frac{\exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) + \exp\left(-\frac{1}{2}\left(\frac{t+\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\sigma}$$

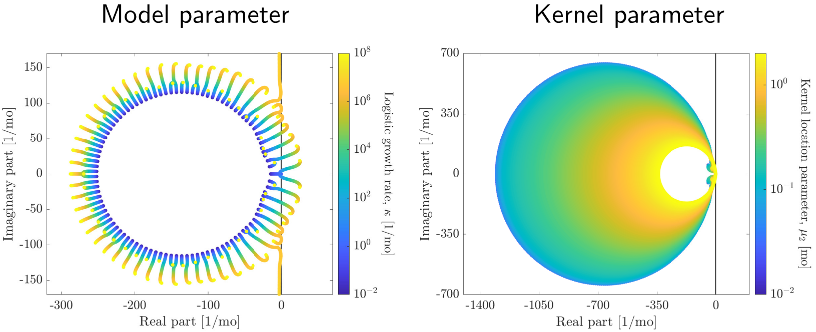


Numerical example: Bifurcation analysis


Model parameter

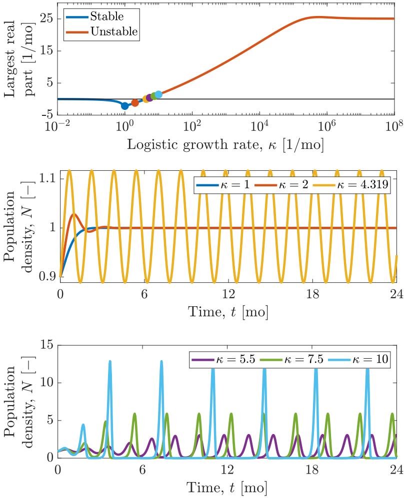
Kernel parameter

Numerical example: Bifurcation analysis

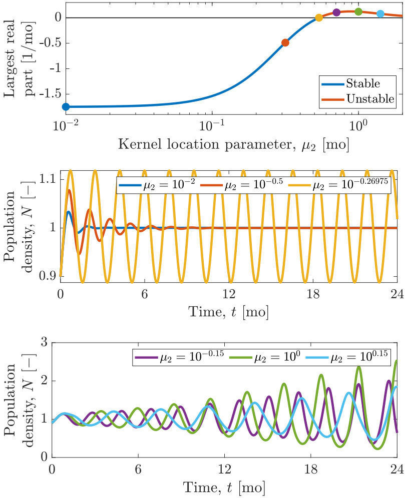


Numerical example: Bifurcation analysis

Model parameter



Kernel parameter



The numerical simulations are obtained with Euler's implicit method and a right rectangle rule for approximating the integral.

Questions?

Bibliography I

[1] T. K. S. Ritschel, "Numerical optimal control for distributed delay differential equations: A simultaneous approach based on linearization of the delayed variables," in *Proceedings of the 23rd European Control Conference (ECC'25)*, (Thessaloniki, Greece), pp. 759–764, June 2025. Preprint.

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