Compact Multi-frame Blind Deconvolution

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Motivation

Big Data Imaging Problems

- Multi-Frame Blind Deconvolution (MFBD) where multi = very large
- MFBD combined with 3D (shape) or 4D (shape and color) reconstructions.

Requirements

- Powerful computers
- More efficient algorithms
  - More efficient use of current algorithms
  - Smarter ways to process massive data sets
- Collaborative/synergistic teams
  - e.g., physics, math, computer science, engineering
Outline

1. Introduction
2. Compact Multi-Frame Blind Deconvolution
3. Global Variable Consensus
4. Higher Dimensional Image Reconstruction
Consider the convolution image formation model:
Deconvolution

Deconvolution: Given

- Blurred image, and
- Point spread function (convolution kernel).

⇒ compute ⇒

- Compute estimate of true image.
Blind Deconvolution: Given

- Blurred image.

⇒ compute ⇒

- Compute estimate of true image, and
- Compute estimate of PSF.
Multi-Frame Blind Deconvolution (MFBD):

- Given multiple frames of blurred images:

- Reconstruct PSFs and object:
Single Frame Blind Deconvolution (SFBD) Model

Parameterize point spread function

- Using convolution model: \( b = \text{psf}(y) \ast x + \eta \)
- Or, using matrix notation: \( b = A(y)x + \eta \)

Example parameterizations:

- PSF pixels: \( \text{psf} \)

- Wavefront phase: \( \text{psf} = |F^{-1}(Pe^{iy})|^2 \)

- Wavefront phase with Zernikes:
  \[
  \text{psf} = |F^{-1}(Pe^{i(y_1z_1 + \cdots + y_mz_m})|^2
  \]
General Mathematical Model

General mathematical model for image formation:

\[ b = A(y)x + \eta \]

where

- **b** = vector representing observed image
- **x** = vector representing true image
- **A(y)** = matrix defining blurring operation
  
  For example,
  
  - Convolution with imposed boundary conditions
  - Spatially variant blurs

- **y** = vector of parameters defining blurring operation

Goal: Given **b**, jointly compute approximations of **y** and **x**.
Multi-Frame Blind Deconvolution (MFBD)

The MFBD problem is:

\[ b_1 = A(y_1)x + \eta_1 \]
\[ b_2 = A(y_2)x + \eta_2 \]
\[ \vdots \]
\[ b_m = A(y_m)x + \eta_m \]

To solve, could consider least squares best fit objective:

\[ \| \begin{bmatrix} b_1 - A(y_1)x \\ \vdots \\ b_m - A(y_m)x \end{bmatrix} \|^2 = \| \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} - \begin{bmatrix} A(y_1) \\ \vdots \\ A(y_m) \end{bmatrix} x \|^2 = \| b - A(y)x \|^2 \]

Also need regularization, but we omit that complication for now.
Processing a large number of frames is computationally intensive.

**Compact MFBD (CMFBD)**


- Identify a small set of *control frames* that contain most independent information.
- Reduce full set of data to small set of control frames, without losing any important information.
CMFBD: Identifying Control Frames

- Suppose $A_j \equiv A(y_j)$ are simultaneously diagonalizable (e.g. Fourier transforms for circulant matrices)

\[ A_j = F^* \Lambda_j F \]

- Consider noise free data, and the $j$-th frame:

\[ A_j x = b_j \quad \Rightarrow \quad \Lambda_j \hat{x} = \hat{b}_j \]

\[ \Rightarrow \quad \Lambda_j \text{diag}(\hat{x}) = \text{diag}(\hat{b}_j) \]

\[ \Rightarrow \quad \text{diag}(\hat{b}_j)^\dagger = \text{diag}(\hat{x})^\dagger \Lambda_j^\dagger \]

where $\hat{x} = Fx$ and $\hat{b}_j = Fb_j$ are unitary Fourier transforms.
CMFBD: Identifying Control Frames

- Assume there is a uniformly “best” conditioned matrix $A_k$. That is, there is a $\Lambda_k$ such that

$$[|\Lambda_k|]_{ii} \geq \tau \quad \text{if there exists } j \text{ with } [|\Lambda_j|]_{ii} \geq \tau$$

where $\tau > 0$ is a tolerance.

- In this case, where there is a single control frame, observe:

$$\text{diag}(\hat{b}_j) = \Lambda_j \text{diag}(\hat{x}) \quad \text{and} \quad \text{diag}(\hat{b}_k)^\dagger = \text{diag}(\hat{x})^\dagger \Lambda_k^\dagger$$

- This allows to compute spectral ratios

$$\underbrace{\text{diag}(\hat{b}_j) \text{ diag}(\hat{b}_k)^\dagger}_{\text{known}} = \underbrace{\Lambda_j \Lambda_k^\dagger}_{\text{unknown}}$$
WLOG, assume the control frame is $k = 1$, and observe:

$$\left\| \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_m \end{bmatrix} \hat{x} - \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{bmatrix} \right\|_2^2$$

$$= \left\| \begin{bmatrix} \Lambda_1 \Lambda_1^\dagger \\ \Lambda_2 \Lambda_1^\dagger \\ \vdots \\ \Lambda_m \Lambda_1^\dagger \end{bmatrix} \Lambda_1 \hat{x} - \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{bmatrix} \right\|_2^2$$
Compact Multi-Frame Blind Deconvolution

CMFBD: Exploiting Control Frames

- WLOG, assume the control frame is $k = 1$, and observe:

$$ \left\| \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_m \end{bmatrix} \hat{x} - \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{bmatrix} \right\|_2^2 $$

$$ = \left\| \begin{bmatrix} I \\ D_2 \\ \vdots \\ D_m \end{bmatrix} \Lambda_1 \hat{x} - \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{bmatrix} \right\|_2^2 $$

where $D_j = \text{diag}(\hat{b}_j) \text{diag}(\hat{b}_1)^\dagger$
CMFBD Observations

- The initial MFBD problem has unknowns:
  \[ A_1, A_2, \ldots, A_m, x \] or, equivalently \[ \Lambda_1, \Lambda_2, \ldots, \Lambda_m, \hat{x} \]

- After identifying a control frame, significantly fewer unknowns:
  \[ A_1, x \] or, equivalently \[ \Lambda_1, \hat{x} \]

- More control frames may be needed to capture all \[ |\Lambda_j|_{ii} \geq \tau \].

- For noisy data, algebra relating known and unknown information holds only approximately.

- Frame Selection: Based on heuristics
  - “Best” conditioned \( A_k \) \( \Leftrightarrow \) least blurred image
  - Many techniques can be used – we use a Fourier based power spectrum approach.
CMFBD Practical Details

- **Reduction of Single Frame Problem**: Use Givens rotations

\[
Q^* \begin{pmatrix} I & \text{D} & \cdots & \text{D}_m \end{pmatrix} \Lambda_1 \hat{x} - \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{pmatrix} = \begin{pmatrix} \text{D} \\ \text{0} \\ \vdots \\ \text{0} \end{pmatrix} \Lambda_1 \hat{x} - \begin{pmatrix} \text{d}_1 \\ \text{d}_2 \\ \vdots \\ \text{d}_m \end{pmatrix}
\]

Therefore, we need only consider

\[
\| \text{D} \Lambda_1 \hat{x} - \text{d}_1 \|_2^2 = \| \text{DFA}_1 \hat{x} - \text{d}_1 \|_2^2
\]

Thus, the MFBD problem

\[
\min_{y_j, \hat{x}} \sum_{j=1}^{m} \| A(y_j) \hat{x} - b_j \|_2^2
\]

reduces to the CSFBD problem

\[
\min_{y_1, \hat{x}} \| \text{WA}(y_1) \hat{x} - \text{d}_1 \|_2^2, \quad \text{W} = \text{DF}
\]
Numerical Illustration of Time Savings

![Graph showing timing in seconds for MFBD and CMFBD with respect to number of frames.](image)

- **Mean data frame**
- **MFBD**
- **CMFBD**

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The MFBD problem can be written as:

$$\min_{y_i, x} \sum_{i=1}^{m} \| b_i - A(y_i)x \|^2_2 + g(x)$$

where here we include an object regularization term, $g(x)$.

Remarks:

- Regularization $g(x)$ can be used to enforce nonnegativity, sparsity, etc.
- The unknown $x$ couples the objective terms $i = 1, \ldots, m$
- We can get a partial decoupling by reformulating as:

$$\min_{y_i, x_i} \sum_{i=1}^{m} \| b_i - A(y_i)x_i \|^2_2 + g(z) \quad \text{subject to } x_i = z, \ i = 1, \ldots, m$$
Global Variable Consensus

Using an augmented Lagrangian approach, and the Alternating Direction Method of Multipliers (ADMM)\(^1\), the optimization decouples:

$$\text{for } k = 1, 2, \ldots$$

$$\begin{bmatrix} y_i^{(k+1)} \ , \ x_i^{(k+1)} \end{bmatrix} = \underset{y_i, x_i}{\text{argmin}} \| b_i - A(y_i)x_i \|_2^2 + \frac{\beta}{2} \| x_i - z^{(k)} + u_i^{(k)} \|_2^2$$

$$\bar{x}^{(k+1)} = \frac{1}{m} \sum_{i=1}^{m} x_i^{(k+1)}$$

$$\bar{u}^{(k)} = \frac{1}{m} \sum_{i=1}^{m} u_i^{(k)}$$

$$z^{(k+1)} = \underset{z}{\text{argmin}} \left\{ g(z) + \frac{m\beta}{2} \| z - \bar{x}^{(k+1)} - \bar{u}^{(k)} \|_2^2 \right\}$$

$$u_i^{(k+1)} = u_i^{(k)} + x_i^{(k+1)} - z^{(k+1)}$$

end

Global Variable Consensus

Advantages:

- Decoupling allows for easy parallel processing of groups of frames

  subgroup 1 | subgroup 2 | subgroup 3 | subgroup 4

- Can either use standard MFBD on subgroups of frames, or
- Use CMFBD on subgroups of frames

- Regularization term is also decoupled, allowing users to plug in many options, and it simplifies the computation.

- Sliding window approach might be possible:
Global Variable Consensus

Global Variable Consensus: Numerical Illustration

50 total frames, split in three different ways
10 subgroups (5 frames each) 5 subgroups (10 frames each) 2 subgroups (25 frames each)

\[ J. \text{ D. Schmidt, Numerical Simulation of Optical Wave Propagation, SPIE Press Monograph Vol. PM199, 2010} \]
Higher Dimensional Image Reconstruction

Three dimensional reconstruction from two dimensional measurements:
Higher Dimensional Image Reconstruction

Some computational challenges:

- Requires processing many, many frames of data.
- Mathematical model is similar to MFBD, but
  - Number of unknowns for object significantly increases.
  - Additional unknowns associated with parameters defining object orientation.
- Some related work has been done for molecular structure determination, e.g. in Cryo-EM and x-ray crystallography

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Higher Dimensional Image Reconstruction

What further information can be used?

- Possibly assume blocks of data have constant orientation parameters.
  Idea like this was used in PET brain image reconstruction\textsuperscript{4}.

- Can use consensus ADMM type approach on blocks of data.
- Use other information (e.g., a frozen flow assumption), or technologies (e.g., laser guide stars).

Summary

- Big data, multi-frame image processing requires not only powerful computers, but also new approaches to process massive data sets.

  This is especially true for 3D/4D image reconstructions.

- Goal should be to extract as much information as possible from collected data, but to also do it quickly.

- Important to have synergistic collaborations with various expertise.