Deblurring Images
Matrices, Spectra and Filtering

Additional Material

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1. Some Useful Matrix Decompositions

This short summary of orthogonal matrices, eigenvalues and singular values is restricted to square real matrices $\mathbf{A}$ of dimension $N \times N$, as used in our book.

Orthogonal Matrices and Projections

A real, square matrix $\mathbf{U} \in \mathbb{R}^{N\times N}$ is orthogonal if its inverse equals its transpose, $\mathbf{U}^{-1} = \mathbf{U}^T$. Consequently we have the two relations

$$\mathbf{U}^T\mathbf{U} = \mathbf{I} \qquad \text{and} \qquad \mathbf{U}\,\mathbf{U}^T = \mathbf{I}.$$

The columns of $\mathbf{U}$ are orthonormal, i.e., they are orthogonal and the 2-norm of each column is one. To see this, let $\mathbf{u}_i$ denote the ith column of $\mathbf{U}$ so that $\mathbf{U} = [\,\mathbf{u}_1\,\mathbf{u}_2\,\ldots\,\mathbf{u}_N\,]$. Then the relation $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ implies that

$$\mathbf{u}_i^T\mathbf{u}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

An orthogonal matrix is perfectly well conditioned; its condition number (in any norm) is one. Moreover, any operation with its inverse merely involves a matrix product with its transpose.

An orthogonal transformation is accomplished by multiplication with an orthogonal matrix. Such a transformation leaves the 2-norm unchanged, because

$$\|\mathbf{U}\,\mathbf{x}\|_2 = \big((\mathbf{U}\,\mathbf{x})^T(\mathbf{U}\,\mathbf{x})\big)^{1/2} = (\mathbf{x}^T\mathbf{x})^{1/2} = \|\mathbf{x}\|_2.$$

An orthogonal transformation can be considered as a change of basis between the "canonical" basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ in $\mathbb{R}^N$ (where $\mathbf{e}_i$ is the ith column of the identity matrix) and the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ given by the orthonormal columns of $\mathbf{U}$. Specifically, for an arbitrary vector $\mathbf{x} \in \mathbb{R}^n$ we can find scalars $z_1, \ldots, z_n$ so that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \sum_{i=1}^{N} x_i\,\mathbf{e}_i = \sum_{i=1}^{N} z_i\,\mathbf{u}_i = \mathbf{U}\,\mathbf{z}$$

and it follows immediately that the coordinates z_i in the new basis are the elements of the vector

$$\mathbf{z} = \mathbf{U}^T\mathbf{x}.$$

Because they do not distort the size of vectors, orthogonal transformations are valuable tools in numerical computations.

For any k less than N the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ span a k-dimension subspace $\mathcal{S}_k \subset \mathbb{R}^N$. The orthogonal projection $\mathbf{x}_k \in \mathbb{R}^N$ of an arbitrary vector $\mathbf{x} \in \mathbb{R}^N$ onto this subspace is the unique vector in $\mathcal{S}_k$ which is closest to $\mathbf{x}$ in the 2-norm, and it is computed as

$$\mathbf{x}_k = \mathbf{U}_k\,\mathbf{U}_k^T\mathbf{x}, \qquad \text{with} \qquad \mathbf{U}_k = [\,\mathbf{u}_1\;\mathbf{u}_2\;\ldots\;\mathbf{u}_k\,]\,.$$

The matrix $\mathbf{U}_k\,\mathbf{U}_k^T$, which is $N\times N$ and has rank k, is called an orthogonal projector.

The Spectral Decomposition

A real, symmetric matrix $\mathbf{A} = \mathbf{A}^T$ always has an eigenvalue decomposition (or spectral decomposition) of the form

$$\mathbf{A} = \mathbf{U}\,\boldsymbol{\Lambda}\,\mathbf{U}^T,$$

where $\mathbf{U}$ is orthogonal, and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ is a diagonal matrix whose diagonal elements λ_i are the eigenvalues of $\mathbf{A}$. A real symmetric matrix always has real eigenvalues. The columns $\mathbf{u}_i$ of $\mathbf{U}$ are the eigenvectors of $\mathbf{A}$, and the eigenpairs $(\lambda_i, \mathbf{u}_i)$ satisfy

$$\mathbf{A}\,\mathbf{u}_i = \lambda_i\,\mathbf{u}_i, \qquad i = 1, \dots, N.$$

The matrix $\mathbf{A}$ represents a linear mapping from $\mathbb{R}^N$ onto itself, and the geometric interpretation of the eigenvalue decomposition is that $\mathbf{U}$ represents a new, orthonormal basis in which this mapping is the diagonal matrix $\boldsymbol{\Lambda}$. In particular, each basis vector $\mathbf{u}_i$ is mapped to a vector in the same direction, namely, the vector $\mathbf{A}\,\mathbf{u}_i = \lambda_i\,\mathbf{u}_i$.

A real square matrix is normal if it satisfies $\mathbf{A}\,\mathbf{A}^T = \mathbf{A}^T\,\mathbf{A}$. Important examples of normal matrices are symmetric, circulant and Hankel matrices. A normal matrix has a spectral decomposition of the form

$$\mathbf{A} = \widetilde{\mathbf{U}}\,\boldsymbol{\Lambda}\,\widetilde{\mathbf{U}}^*,$$

where the complex matrix $\widetilde{\mathbf{U}}$ is unitary, i.e.,

$$\widetilde{\mathbf{U}}^{-1} = \widetilde{\mathbf{U}}^* = \mathrm{conj}(\widetilde{\mathbf{U}})^T,$$

and $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ is a diagonal matrix containing the (possibly complex) eigenvalues of $\mathbf{A}$. (Note that orthogonal matrices are included in the set of unitary matrices.) If $\mathbf{A}$ is real and normal, then its eigenvalues are either real or appear in complex conjugate pairs. The columns $\tilde{\mathbf{u}}_i$ of $\widetilde{\mathbf{U}}$ are the eigenvectors of $\mathbf{A}$. We note that a unitary matrix $\widetilde{\mathbf{U}}$ has orthonormal columns: $\tilde{\mathbf{u}}_i^*\mathbf{u}_j = \mathrm{conj}(\mathbf{u}_i)^T\mathbf{u}_j = \delta_{ij}$. Also note that multiplication with $\widetilde{\mathbf{U}}$ leaves the 2-norm unchanged: $\|\widetilde{\mathbf{U}}\,\mathbf{x}\|_2 = \|\mathbf{x}\|_2$.

The Singular Value Decomposition (SVD)

A real matrix which is not normal CANNOT be diagonalized by an orthogonal or unitary matrix. It takes two orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ to diagonalize such a matrix, by means of the singular value decomposition,

$$\mathbf{A} = \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^T = \sum_{i=1}^{N} \mathbf{u}_i\,\sigma_i\,\mathbf{v}_i^T,$$

where $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_N)$ is a real diagonal matrix whose diagonal elements σ_i are the singular values of $\mathbf{A}$, while the singular vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are the columns of the orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$. The singular values are nonnegative and are typically written in nonincreasing order:

$$\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_N \ge 0.$$

We note that if $\mathbf{A}$ is normal, then its singular values are equal to the absolute values of its eigenvalues.

The geometric interpretation of the SVD is that it provides two sets of orthogonal basis vectors – the columns of $\mathbf{U}$ and $\mathbf{V}$ – such that the mapping represented by $\mathbf{A}$ becomes a diagonal matrix when expressed in these bases. Specifically, we have

$$\mathbf{A}\,\mathbf{v}_i = \sigma_i\,\mathbf{u}_i, \qquad i = 1,\ldots,N.$$

That is, σ_i is the "magnification" when mapping $\mathbf{v}_i$ onto $\mathbf{u}_i$. Any vector $\mathbf{x} \in \mathbb{R}^N$ can be written as $\mathbf{x} = \sum_{i=1}^{N} (\mathbf{v}_i^T\mathbf{x})\,\mathbf{v}_i$, and it follows that its image is given by

$$\mathbf{A}\,\mathbf{x} = \sum_{i=1}^{N} (\mathbf{v}_i^T\mathbf{x})\,\mathbf{A}\,\mathbf{v}_i = \sum_{i=1}^{N} \sigma_i\,(\mathbf{v}_i^T\mathbf{x})\,\mathbf{u}_i.$$

If $\mathbf{A}$ has an inverse, then the mapping of the inverse also defines a diagonal matrix:

$$\mathbf{A}^{-1}\,\mathbf{u}_i = \sigma_i^{-1}\,\mathbf{v}_i,$$

so that σ_i^{-1} is the "magnification" when mapping $\mathbf{u}_i$ back onto $\mathbf{v}_i$. Similarly, any vector $\mathbf{b} \in \mathbb{R}^N$ can be written as $\mathbf{x} = \sum_{i=1}^{N} (\mathbf{u}_i^T\mathbf{b})\,\mathbf{u}_i$, and it follows that the vector $\mathbf{A}^{-1}\,\mathbf{b}$ is given by

$$\mathbf{A}^{-1}\,\mathbf{b} = \sum_{i=1}^{N} (\mathbf{u}_i^T\mathbf{b})\,\mathbf{A}^{-1}\,\mathbf{u}_i = \sum_{i=1}^{N} \frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}\,\mathbf{v}_i.$$

Similar relations can easily be derived for the spectral decompositions.

Rank, Conditioning, and Truncated SVD

The rank of a matrix is equal to the number of nonzero singular values: $r = \mathrm{rank}(\mathbf{A})$ means that

$$\sigma_r > 0, \qquad \sigma_{r+1} = 0.$$

The matrix $\mathbf{A}$ has full rank (and, therefore, an inverse) only if all of its singular values are nonzero. If $\mathbf{A}$ is rank deficient then the system $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ may not be compatible; in other words, there may be no vector $\mathbf{x}$ that solves the problem. The columns of $\mathbf{U}_r = [\,\mathbf{u}_1\;\mathbf{u}_2\;\ldots\;\mathbf{u}_r\,]$ form an orthonormal basis for the range of $\mathbf{A}$, and the system $\mathbf{A}\,\mathbf{x} = \mathbf{b}_r$ with $\mathbf{b}_r = \mathbf{U}_r\,\mathbf{U}_r^T\mathbf{b}$ is the closest compatible system. This compatible system has infinitely many solutions, and the solution of minimum 2-norm is

$$\mathbf{x}_r = \sum_{i=1}^{r} \frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}\,\mathbf{v}_i.$$

Consider now a perturbed version $\mathbf{A}\,\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ of the original system $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, in which the perturbed right-hand side is given by $\tilde{\mathbf{b}} = \mathbf{b} + \mathbf{e}$. If $\mathbf{A}$ has full rank then the perturbed solution is given by $\tilde{\mathbf{x}} = \mathbf{A}^{-1}\tilde{\mathbf{b}} = \mathbf{x} + \mathbf{A}^{-1}\mathbf{e}$, and we need an upper bound for the relative perturbation $\|\mathbf{x} - \tilde{\mathbf{x}}\|_2/\|\mathbf{x}\|_2$. The worst-case situation arises when $\mathbf{b}$ is in the direction of the left singular vector $\mathbf{u}_1$ while the perturbation $\mathbf{e}$ is solely in the direction of $\mathbf{u}_N$, and it follows that the perturbation bound is given by

$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \leq \operatorname{cond}(\mathbf{A}) \frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2}, \qquad \text{where} \qquad \operatorname{cond}(\mathbf{A}) = \frac{\sigma_1}{\sigma_N}.$$

The quantity $\operatorname{cond}(\mathbf{A})$ is the condition number of $\mathbf{A}$. The larger the condition number, the more sensitive the system is to perturbations of the right-hand side.

The smallest singular value σ_N measures how "close" $\mathbf{A}$ is to a singular matrix (and $\sigma_N = 0$ when $\mathbf{A}$ is singular). A perturbation of $\mathbf{A}$ with a matrix $\mathbf{E}$, whose elements are of the order σ_N, can make $\mathbf{A}$ rank deficient. The existence of one or more small singular values (small compared to the largest singular value σ_1) therefore indicates that $\mathbf{A}$ is "almost" singular.

In this case, it is often recommended to replace the ill-conditioned matrix $\mathbf{A}$ with a nearby but exactly rank-deficient matrix $\mathbf{A}_k$ whose rank k cannot be reduced by small perturbations. The typical choice of $\mathbf{A}_k$ is the truncated SVD (TSVD) matrix

$$\mathbf{A}_k = \mathbf{U}_k\, \mathbf{U}_k^T\, \mathbf{A} = \sum_{i=1}^{k} \mathbf{u}_i\, \sigma_i\, \mathbf{v}_i^T.$$

The rank k of $\mathbf{A}_k$ is chosen such that σ_k – which measures how "close" $\mathbf{A}_k$ is to a singular matrix – is larger than the perturbations (the errors) in the original matrix $\mathbf{A}$. The minimum-norm solution to the corresponding compatible system $\mathbf{A}_k\, \mathbf{x} = \mathbf{b}_k$ with $\mathbf{b}_k = \mathbf{U}_k\, \mathbf{U}_k^T\, \mathbf{b}$ is called the TSVD solution, and it is given by

$$\mathbf{x}_k = \mathbf{U}_k\, \mathbf{U}_k^T\, \mathbf{x} = \sum_{i=1}^{k} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i}\, \mathbf{v}_i.$$

2. The DFT and Smoothing Norms

The following is a derivation of equation (7.9) in Section 7.3 for the efficient computation of the Tikhonov solution with a smoothing norm $\|\mathbf{D}\mathbf{x}\|_2$ that involves partial derivatives. We consider the case of periodic boundary conditions where the DFT matrix $\mathbf{F} = \mathbf{F}_{\mathrm{r}} \otimes \mathbf{F}_{\mathrm{c}}$ diagonalizes the matrix $\mathbf{A}$, i.e.,

$$\mathbf{A} = \mathbf{F}^* \boldsymbol{\Lambda}_{\mathbf{A}}\, \mathbf{F},$$

in which the diagonal matrix $\boldsymbol{\Lambda}_{\mathbf{A}}$ contains the eigenvalues λ_i of $\mathbf{A}$. These eigenvalues are computed as described in section 4.2.

For periodic boundary conditions, the one-dimensional DFT matrix $\mathbf{F}_{\mathrm{c}}$ diagonalizes the first and second derivative matrices $\mathbf{D}_{1,m}$ (7.7) and $\mathbf{D}_{2,m}$ (7.6), i.e.,

$$\mathbf{D}_{q,m} = \mathbf{F}_{\mathrm{c}}^* \,\boldsymbol{\Lambda}_{\mathbf{D}_{q,m}}\, \mathbf{F}_{\mathrm{c}}, \qquad q = 1, 2,$$

where the diagonal matrix $\boldsymbol{\Lambda}_{\mathbf{D}_{q,m}}$ contains the eigenvalues $\lambda_{q,k}$ of $\mathbf{D}_{q,m}$. These eigenvalues can be pre-computed by the following expressions

$$\lambda_{q,k} = \begin{cases} \exp(2k\pi\hat{\imath}/m) - 1, & q = 1 \\ 2\cos(2k\pi/m) - 2, & q = 2 \end{cases} \qquad \text{for} \qquad k = 1, \dots, m$$

in which $\hat{\imath} = \sqrt{-1}$ denotes the imaginary unit. It follows that several choices of the matrix $\mathbf{D}$ have simple expressions in terms of Kronecker products; for example

$$\begin{aligned} \mathbf{I}_n \otimes \mathbf{D}_{q,m} &= (\mathbf{F}_{\mathrm{r}}^* \,\mathbf{F}_{\mathrm{r}}) \otimes (\mathbf{F}_{\mathrm{c}}^* \,\boldsymbol{\Lambda}_{\mathbf{D}_{q,m}}\, \mathbf{F}_{\mathrm{c}}) \\ &= (\mathbf{F}_{\mathrm{r}}^* \otimes \mathbf{F}_{\mathrm{c}}^*)\, (\mathbf{I}_n \otimes \boldsymbol{\Lambda}_{\mathbf{D}_{q,m}})\, (\mathbf{F}_{\mathrm{r}} \otimes \mathbf{F}_{\mathrm{c}}) \\ &= \mathbf{F}^* (\mathbf{I}_n \otimes \boldsymbol{\Lambda}_{\mathbf{D}_{q,m}})\, \mathbf{F}. \end{aligned}$$

Similarly, we obtain

$$\mathbf{D}_{q,n} \otimes \mathbf{I}_m = \mathbf{F}^* (\boldsymbol{\Lambda}_{\mathbf{D}_{q,n}} \otimes \mathbf{I}_m)\, \mathbf{F}$$

$$\mathbf{I}_n \otimes \mathbf{D}_{2,m} + \mathbf{D}_{2,n} \otimes \mathbf{I}_m = \mathbf{F}^* (\mathbf{I}_n \otimes \boldsymbol{\Lambda}_{\mathbf{D}_{q,m}} + \boldsymbol{\Lambda}_{\mathbf{D}_{q,n}} \otimes \mathbf{I}_m) \mathbf{F}$$

$$\begin{bmatrix} \mathbf{I}_n \otimes \mathbf{D}_{q,m} \\ \mathbf{D}_{q,n} \otimes \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{F}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^* \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \otimes \boldsymbol{\Lambda}_{\mathbf{D}_{q,m}} \\ \boldsymbol{\Lambda}_{\mathbf{D}_{q,n}} \otimes \mathbf{I}_m \end{bmatrix} \mathbf{F}.$$

We can use the above relations to derive a simple expression for the Tikhonov solution to (7.3). We need the following result:

$$\begin{aligned} \mathbf{A}^T &= \mathbf{A}^* \\ &= \mathbf{F}^* \mathrm{conj}(\boldsymbol{\Lambda}_{\mathbf{A}})\, \mathbf{F} \\ &= \mathbf{F}^* \mathrm{conj}(\boldsymbol{\Lambda}_{\mathbf{A}})\, \boldsymbol{\Lambda}_{\mathbf{A}}\, \boldsymbol{\Lambda}_{\mathbf{A}}^{-1} \mathbf{F} \\ &= \mathbf{F}^* |\boldsymbol{\Lambda}_{\mathbf{A}}|^2 \boldsymbol{\Lambda}_{\mathbf{A}}^{-1} \mathbf{F} \end{aligned}$$

where $|\boldsymbol{\Lambda}_{\mathbf{A}}|^2$ denotes a diagonal matrix whose elements are $|\lambda_i|^2$. It follows immediately that

$$\mathbf{A}^T \mathbf{A} = \mathbf{F}^* \mathrm{conj}(\boldsymbol{\Lambda}_{\mathbf{A}})\, \boldsymbol{\Lambda}_{\mathbf{A}}\, \mathbf{F} = \mathbf{F}^* |\boldsymbol{\Lambda}_{\mathbf{A}}|^2 \mathbf{F}.$$

A similar result holds for the matrix $\mathbf{D}$, depending on its form. For example, if $\mathbf{D} = \mathbf{I}_n \otimes \mathbf{D}_{q,m} = \mathbf{F}^*(\mathbf{I}_n \otimes \mathbf{\Lambda}_{\mathbf{D}_{q,m}})\,\mathbf{F}$ then

$$\mathbf{D}^T = \mathbf{D}^* = \mathbf{F}^* \left(\mathbf{I}_n \otimes \mathrm{conj}(\mathbf{\Lambda}_{\mathbf{D}_{q,m}})\right)\, \mathbf{F}$$

and hence

$$\begin{aligned}\mathbf{D}^T\mathbf{D} &= \mathbf{F}^* \left(\mathbf{I}_n \otimes \mathrm{conj}(\mathbf{\Lambda}_{\mathbf{D}_{q,m}})\right) \left(\mathbf{I}_n \otimes \mathbf{\Lambda}_{\mathbf{D}_{q,m}}\right) \mathbf{F} \\ &= \mathbf{F}^* \left(\mathbf{I}_n \otimes \mathrm{conj}(\mathbf{\Lambda}_{\mathbf{D}_{q,m}})\mathbf{\Lambda}_{\mathbf{D}_{q,m}}\right) \mathbf{F} \\ &= \mathbf{F}^* \left(\mathbf{I}_n \otimes |\mathbf{\Lambda}_{\mathbf{D}_{q,m}}|^2\right) \mathbf{F}.\end{aligned}$$

Putting the above relations together, we arrive at the following expression for the Tikhonov solution

$$\begin{aligned}\mathbf{x}_{\alpha,\mathbf{D}} &= (\mathbf{A}^T\mathbf{A} + \alpha^2\,\mathbf{D}^T\mathbf{D})^{-1}\mathbf{A}^T\mathbf{b} \\ &= \mathbf{F}^*\Big(|\mathbf{\Lambda}_\mathbf{A}|^2 \left(|\mathbf{\Lambda}_\mathbf{A}|^2 + \alpha^2\,(\mathbf{I}_n \otimes |\mathbf{\Lambda}_{\mathbf{D}_{q,m}}|^2)\right)^{-1}\Big)\, \mathbf{\Lambda}_\mathbf{A}^{-1}\,\mathbf{F}\,\mathbf{b}.\end{aligned}$$

There are similar expressions for the other choices of the matrix $\mathbf{D}$. If $\mathbf{D} = \mathbf{D}_{q,n} \otimes \mathbf{I}_m = \mathbf{F}^*(\mathbf{\Lambda}_{\mathbf{D}_{q,n}} \otimes \mathbf{I}_m)\,\mathbf{F}$ then it follows immediately that

$$\mathbf{D}^T\mathbf{D} = \mathbf{F}^* \left(|\mathbf{\Lambda}_{\mathbf{D}_{q,n}}|^2 \otimes \mathbf{I}_m\right) \mathbf{F},$$

and if we use a sum of squared norm, represented by

$$\mathbf{D} = \begin{bmatrix} \mathbf{I}_n \otimes \mathbf{D}_{q,m} \\ \mathbf{D}_{q,n} \otimes \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{F}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^* \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \otimes \mathbf{\Lambda}_{\mathbf{D}_{q,m}} \\ \mathbf{\Lambda}_{\mathbf{D}_{q,n}} \otimes \mathbf{I}_m \end{bmatrix} \mathbf{F},$$

then we obtain

$$\mathbf{D}^T\mathbf{D} = \mathbf{F}^* \left(\mathbf{I}_n \otimes |\mathbf{\Lambda}_{\mathbf{D}_{q,m}}|^2 + |\mathbf{\Lambda}_{\mathbf{D}_{q,n}}|^2 \otimes \mathbf{I}_m\right) \mathbf{F}.$$

Finally if $\mathbf{D} = \mathbf{I}_n \otimes \mathbf{D}_{2,m} + \mathbf{D}_{2,n} \otimes \mathbf{I}_m = \mathbf{F}^*(\mathbf{I}_n \otimes \mathbf{\Lambda}_{\mathbf{D}_{2,m}} + \mathbf{\Lambda}_{\mathbf{D}_{2,n}} \otimes \mathbf{I}_m)\,\mathbf{F}$ (approximating the Laplacian) then we obtain

$$\mathbf{D}^T\mathbf{D} = \mathbf{F}^* \Big(\mathbf{I}_n \otimes \mathbf{\Lambda}^2_{\mathbf{D}_{2,m}} + \mathbf{\Lambda}^2_{\mathbf{D}_{2,n}} \otimes \mathbf{I}_m + 2\,\mathbf{\Lambda}_{\mathbf{D}_{2,n}} \otimes \mathbf{\Lambda}_{\mathbf{D}_{2,m}}\Big)\, \mathbf{F}.$$

The absolute value is not necessary here because the eigenvalues are real.

We can summarize these results in the expression (7.9) for the Tikhonov solution

$$\mathbf{x}_{\alpha,\mathbf{D}} = \mathbf{F}^*\Big(|\mathbf{\Lambda}_\mathbf{A}|^2 \left(|\mathbf{\Lambda}_\mathbf{A}|^2 + \alpha^2\,\mathbf{\Delta}\right)^{-1}\Big)\, \mathbf{\Lambda}_\mathbf{A}^{-1}\,\mathbf{F}\,\mathbf{b},$$

where the diagonal matrix $\mathbf{\Delta}$ takes one of the four forms shown in the last column of Table 7.2. Note that the filtering matrix $|\mathbf{\Lambda}_\mathbf{A}|^2\,(|\mathbf{\Lambda}_\mathbf{A}|^2 + \alpha^2\,\mathbf{\Delta})^{-1}$ is diagonal.