Regularization and Perturbation Errors

The purpose of this challenge is to illustrate the behavior of the errors in the filtered solution – more precisely, the regularization errors and the perturbation errors as discussed in §6.3. We shall also relate this analysis to the choice of the regularization parameter based on the "Picard plot," the L-curve and the GCV-function. We focus on the TSVD method, and it is suggested to use the same small test problem as in the "SVD Analysis and TSVD Solutions" challenge, to keep the computing time reasonable.

1. Create the test problem $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ and a noise vector $\mathbf{e}$.

2. Consider the total error $\mathbf{x}_{\text{exact}} - \mathbf{x}_k$, the regularization error $(\mathbf{I}_N - \mathbf{V}\,\mathbf{\Phi}\,\mathbf{V}^T)\,\mathbf{x}_{\text{exact}}$ and the perturbation error $\mathbf{V}\,\mathbf{\Phi}\,\mathbf{\Sigma}^{-1}\,\mathbf{U}^T\mathbf{e}$ for $k = 1, 2, \ldots$. Derive simple expressions for the 2-norm of these errors, and plot the norms versus k. You should observe that the minimum of the total error occurs near the point where the regularization and perturbation errors take the same size. What is the optimal truncation parameter k_{opt}?

3. Now form the Picard plot, and locate the transition where the coefficients $|\mathbf{u}_i^T\mathbf{b}|$ start to level off. Is the corresponding index close to k_{opt}?

4. Form the L-curve and find a truncation parameter at its corner (it is OK to "eyeball" the corner). Again, is this index close to k_{opt}?

5. Finally, compute and plot the GCV function as a function of the truncation parameter k; note that equation (5.13) holds for all spectral filtering methods. Then find the value of k that minimizes the GCV function, and compare with k_{opt}.

6. Try to make a conclusion about the different approaches to choosing the truncation parameter.