# What You Can See in Limited Data Tomography

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- *Motion-Compensated CT:* CT when the body moves during the scan.

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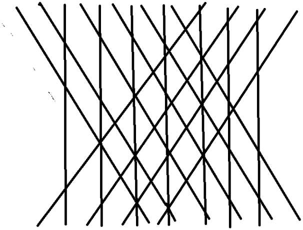
*Limited angle data over "somewhat" vertical lines.*



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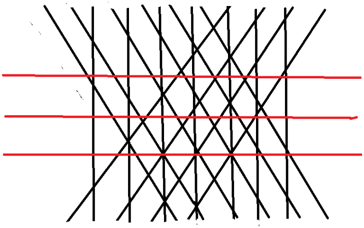

∼Horizontal lines are missing.

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2. Understand, geometrically, how this depends on the data.

# The Model of X-ray CT

$f$ a function in the plane representing the density of an object
$L$ a line in the plane over which the photons travel.
The X-ray (Radon) Line Transform:

$$\text{Tomographic Data} \sim Rf(L) = \int_{x \in L} f(x)\, ds$$

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*The goal: Recover a picture of the body (values of $f(x)$), from X-ray CT data over a finite number of lines.*

With *complete data* (lines throughout the object in fairly evenly spaced directions), good reconstruction methods exist (e.g., Filtered Backprojection [Natterer, Natterer-Wübbling]).

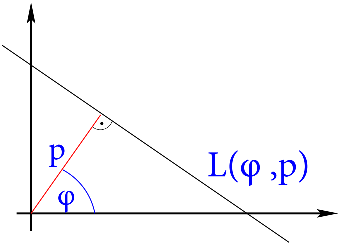
## GE scanner

## GE Reconstruction

# Parallel Beam Geometry ($\sim$fan beam but simpler):

**The angle:** $\varphi \in [0, 2\pi]$, $\quad \theta(\varphi) = (\cos(\varphi), \sin(\varphi))$

**The line over which X-rays travel:** $L(\varphi, p)$ is the line perpendicular to $\varphi$ and $p$ units from the origin (in opposite direction if $p < 0$)



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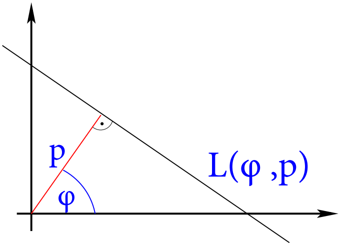
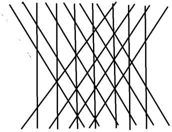
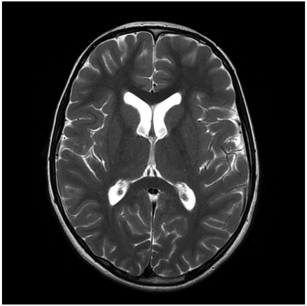


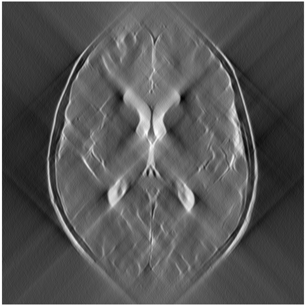

**The object:** $f$ is the density function of an object in the plane.

**Tomographic data:** $Rf(\varphi, p) = \int_{x \in L(\varphi,p)} f(x)ds$ is given when X-rays travel along the line $L(\varphi, p)$.

# Limited Angle Tomography, $\varphi^{-1}[-\pi/4, \pi/4]$







Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

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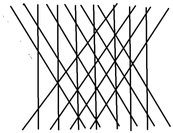

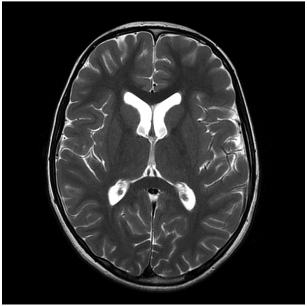

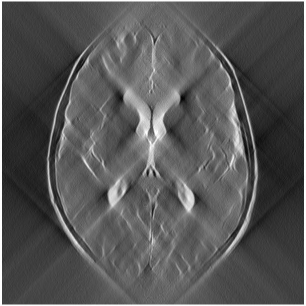

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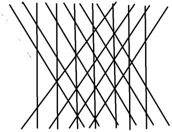

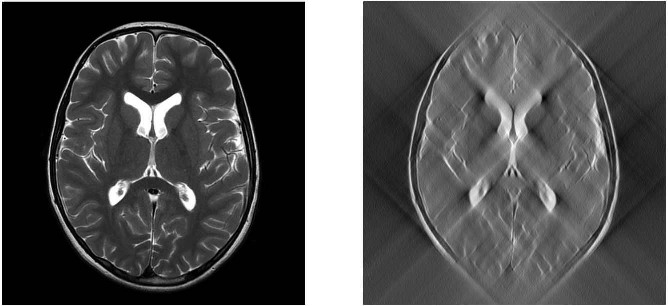

Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

- *Which features of the object are visible in the reconstruction? Which are invisible?*
- *Are there added artifacts?*

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# Big Question

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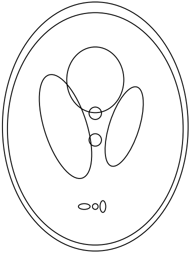
   - *Practically:* Density jumps, boundaries between regions, discontinuities of $f$.
   - *Mathematically:* Where the function is not $C^\infty$ smooth.

The function



Its singularities (sing. supp.)

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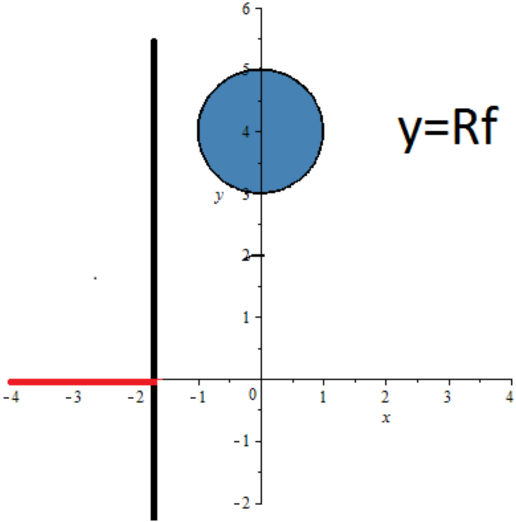

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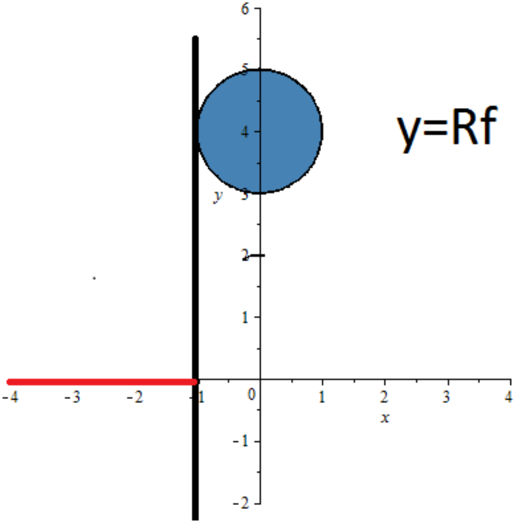

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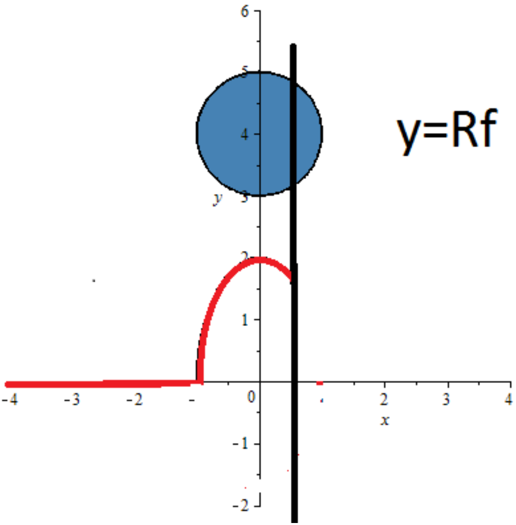

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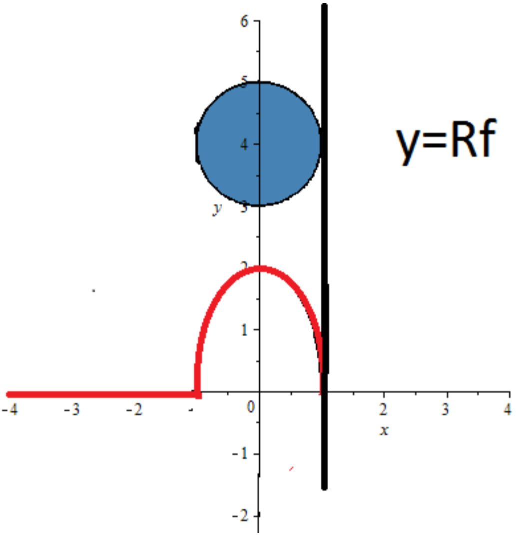

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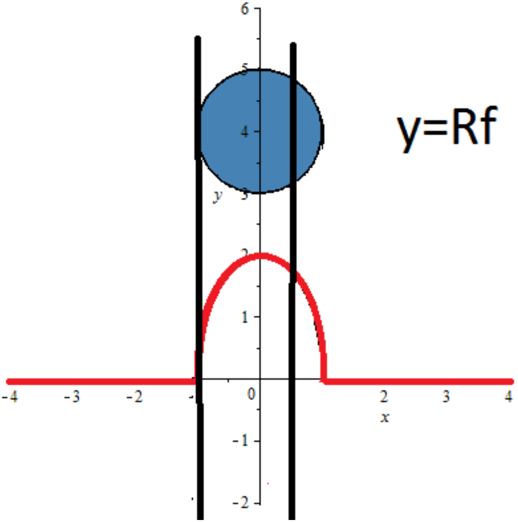

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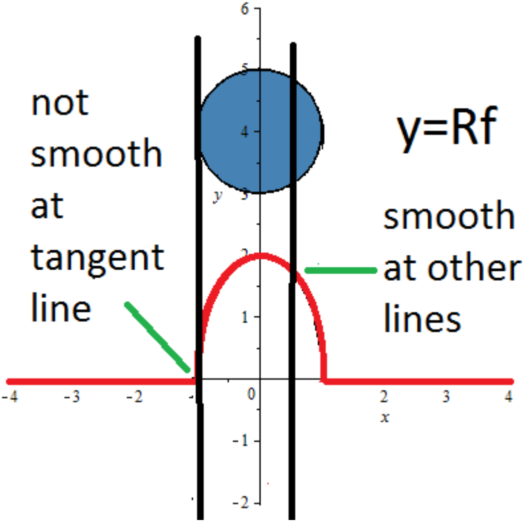

# Characterization of Visible Singularities

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**In terms of wavefront sets:** If some wavefront direction of $f$ is *perpendicular* to $L(\varphi_0, p_0)$, then $Rf$ has WF above $(\varphi_0, p_0)$.

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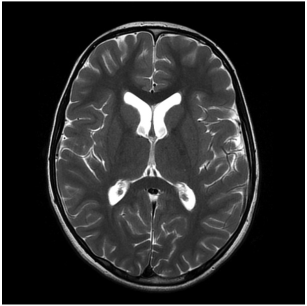
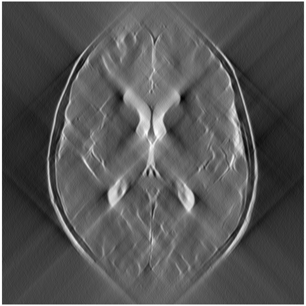
   *and harder to reconstruct stably from that limited data (it could be "blurred out")*.

***Moral for limited data CT:*** *If the line $L_0$ is in a limited data set, then singularities of $f$ tangent to $L_0$ should be "easy" to reconstruct from that data.*
*Singularities of $f$ not tangent to any line in the data set will be harder to reconstruct (less stable).*

# Limited Angle Reconstruction Revisited

Reconstruction for lines with $\varphi \in [-\pi/4, \pi/4]$ [Frikel, Q 2013]. $\longrightarrow$

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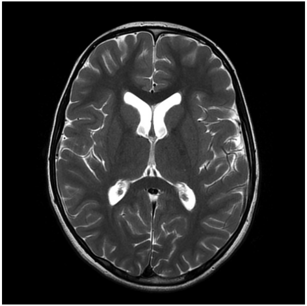

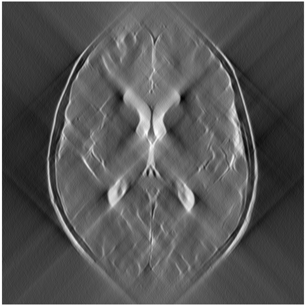

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- Singularities not tangent to lines in the data set–the "$\sim$horizontal" boundaries–are blurred.

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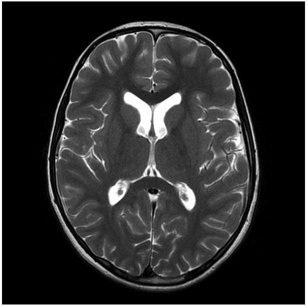

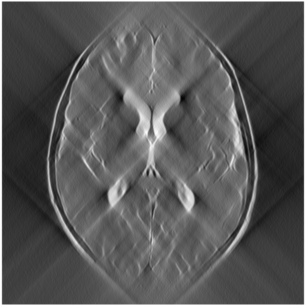

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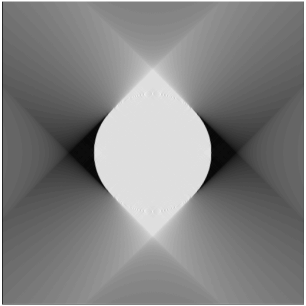
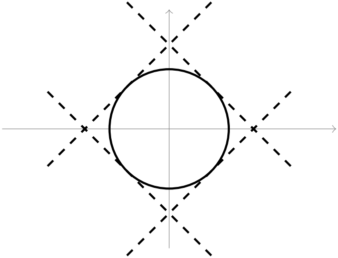
- Singularities not tangent to lines in the data set–the "$\sim$horizontal" boundaries–are blurred.
- But, what about the streaks.....?

# The Added Artifacts for data with $\phi \in [-\pi/4, \pi/4]$





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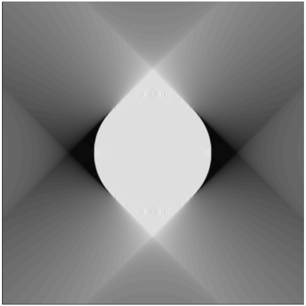

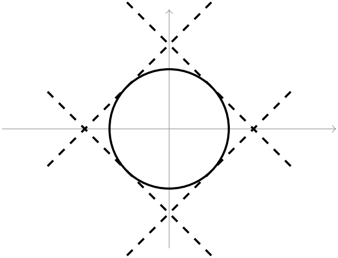

Note how the singularities of $f$ tangent to lines at the ends of the angular range, $L(\pm\pi/4, p)$, generate added artifacts all along the lines tangent to them.

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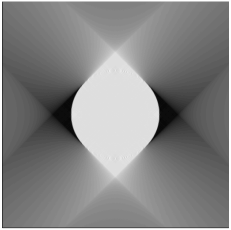 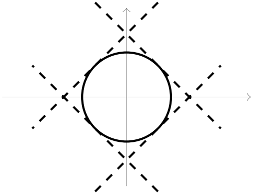

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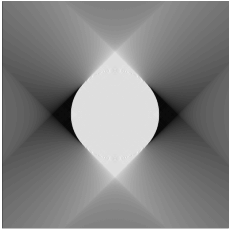
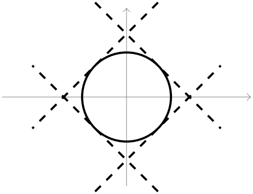




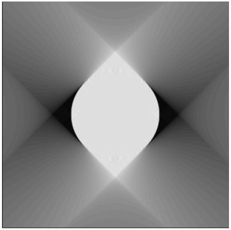
**Theorem ([Frikel Q 2013])**

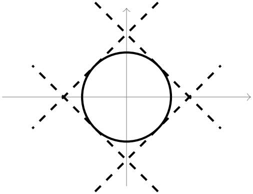
*For limited angle tomography, added artifacts will occur on lines at the end of the angular range*

*–from X-rays at the start and end of the scan–*

*that are tangent to some singularity in the object.*

# Added Artifacts





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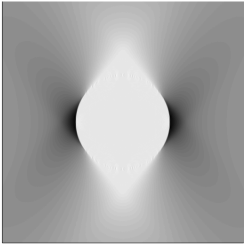
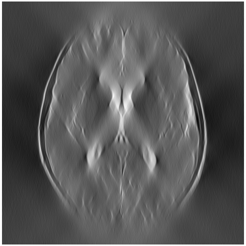
# Our (simple practical) Artifact Reduction Procedure

Assume the limited angle data are given for $\varphi \in [a, b]$.

$$\text{modified data} = [\kappa(\varphi) Rf]\,(\varphi, p)$$

where $\kappa$ is a smooth cutoff function equal to zero off of $[a, b]$ and equal to one on most of $[a, b]$.



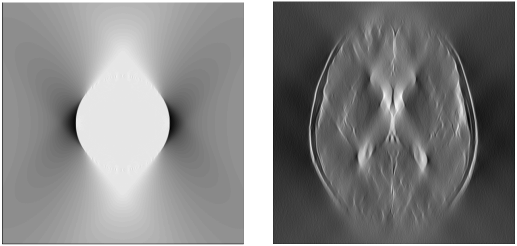


# Our (simple practical) Artifact Reduction Procedure

Assume the limited angle data are given for $\varphi \in [a, b]$.

$$\text{modified data} = [\kappa(\varphi) Rf](\varphi, p)$$

where $\kappa$ is a smooth cutoff function equal to zero off of $[a, b]$ and equal to one on most of $[a, b]$.



then there will be no added streak artifacts and most visible singularities will be recovered [Frikel Q 2013,2015].

# Hybrid imaging: Thermoacoustic Tomography (TAT)

- Pulsed electromagnetic (EM) radiation (radio waves (PAT: laser light)) is beamed into a part of the body (e.g., breast).
- The body heats up and generates sound pressure waves that are measured by acoustic transducers.

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- Sometimes TAT/PAT transducers are collimated to a plane and they move along the unit circle [Razansky 2009, Elbau 2012].

- These transducers measure the sound pressure over time and, by solving the wave equation (with constant sound speed), this can be reduced to the integrals over circles of the initial value of the acoustic pressure, $f$.

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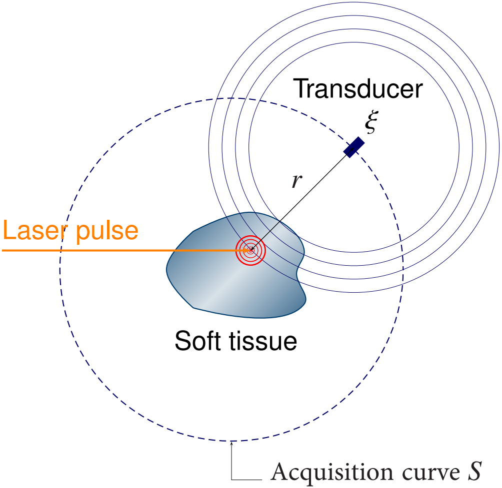
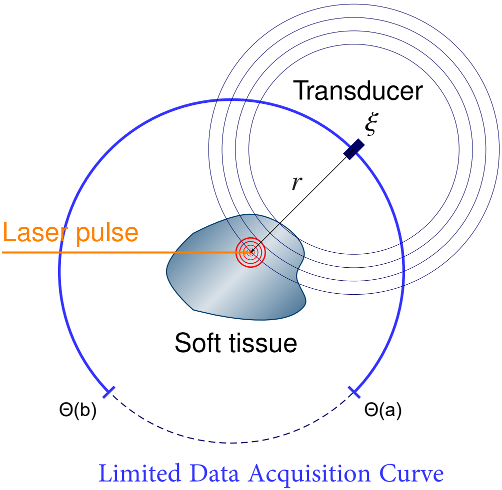
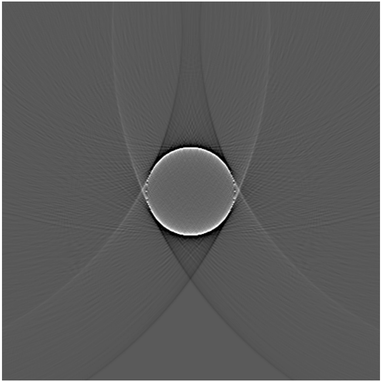

Transducer
$\xi$
$r$
Laser pulse
Soft tissue
Acquisition curve $S$

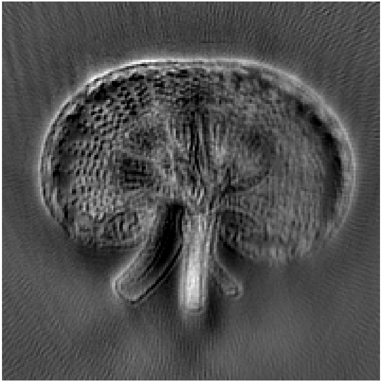



*Limited Data PAT:* When transducers cannot scan all around the object (e.g., because of specimen holder), so data are given only for centers $\theta(\varphi)$ for $a \leqslant \varphi \leqslant b$.

# Limited data PAT reconstructions



Simulated data, $\varphi \in [25^\circ, 155^\circ]$



Real data, $\varphi \in [-45^\circ, 225^\circ]$

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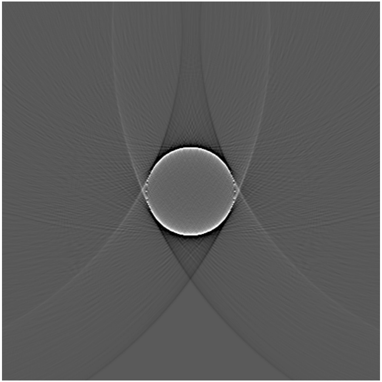

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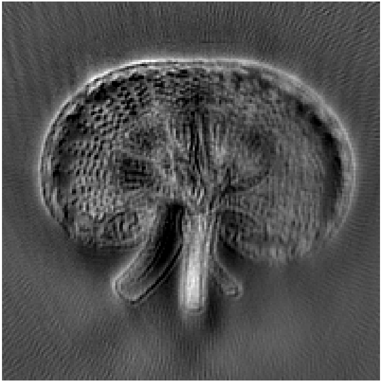

Real data, $\varphi \in [-45^\circ, 225^\circ]$

- Why is the circle not completely imaged?

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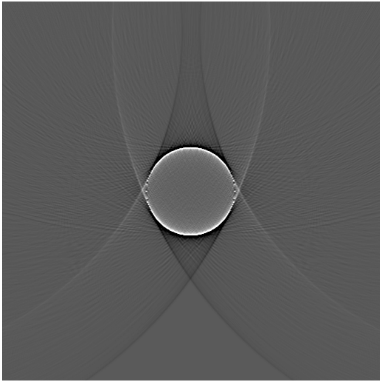
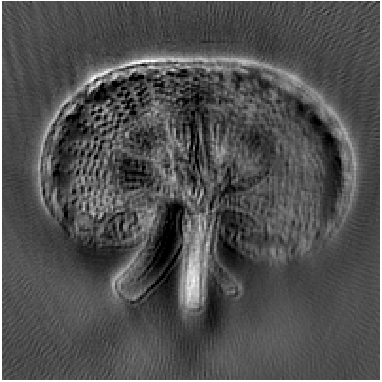




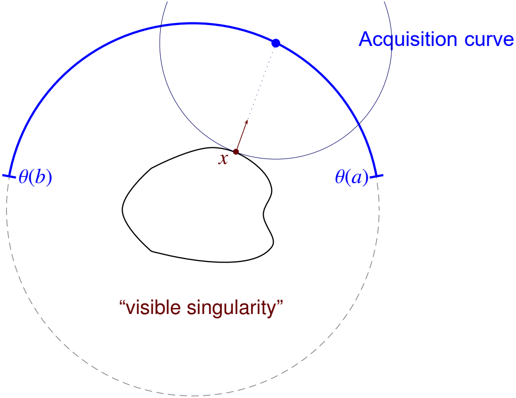
Simulated data, $\varphi \in [25^\circ, 155^\circ]$ Real data, $\varphi \in [-45^\circ, 225^\circ]$

- Why is the circle not completely imaged?
- Why are there streak artifacts in both reconstructions?

PAT data are ∼averages over circles.

**Theorem (Frikel Q 2015)**

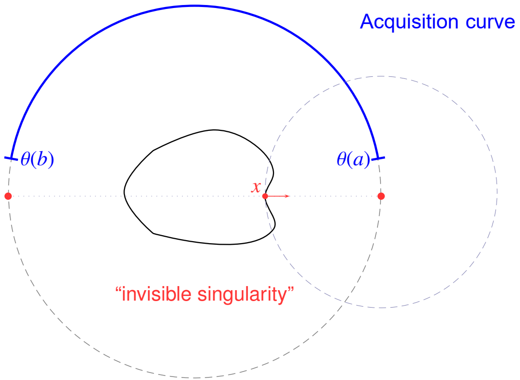
***Visible singularities*** *of f are tangent to circles in the data set*



PAT data are ∼averages over circles.

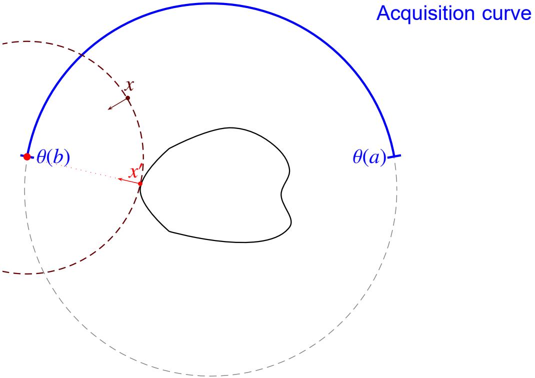
**Theorem (Frikel Q 2015)**

***Visible singularities*** *of f are tangent to circles in the data set*
***Invisible singularities*** *are tangent to* ***NO*** *circle in the data set.*



## Theorem ([Frikel Q 2015])

***Added artifacts*** *occur when a circle at the ends of the data set (center $\theta(a)$ or $\theta(b)$) is tangent to a singularity of $f$.*



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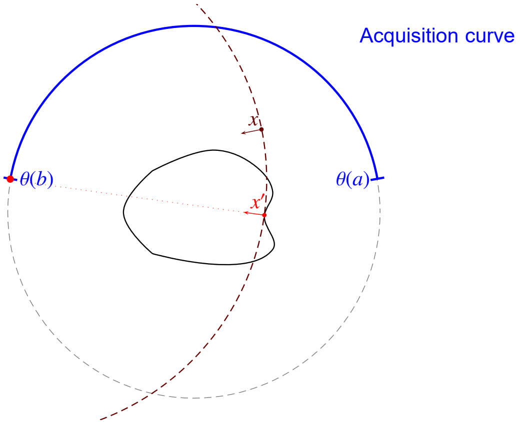
*The singularity spreads along the entire circle!*



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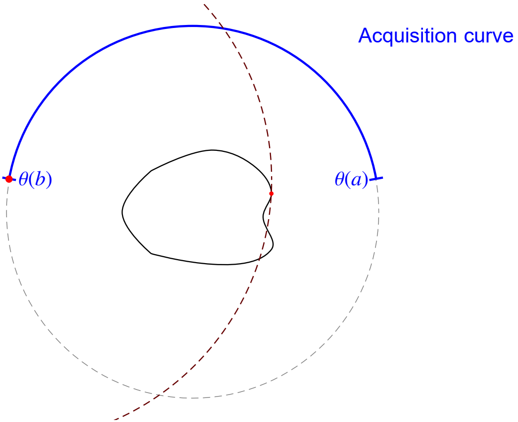

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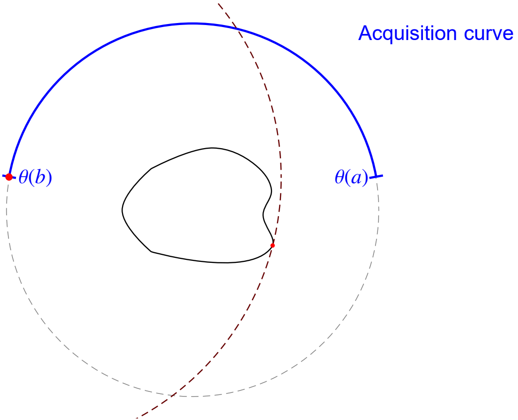
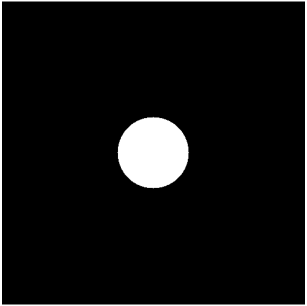
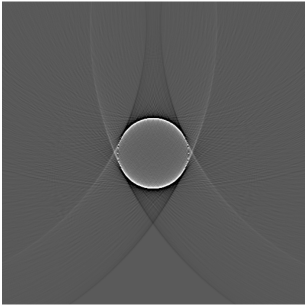


# Limited view reconstructions revisited



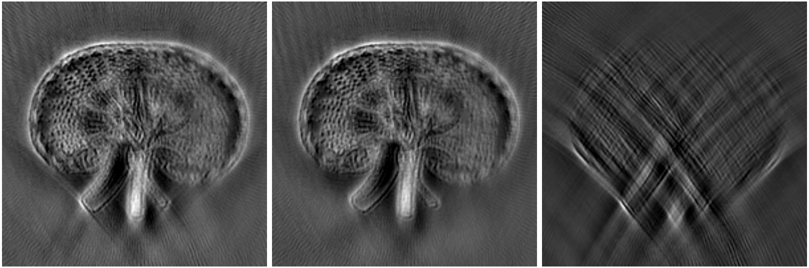
(g) $f$



(h) $\Lambda g = \mathcal{M}^* \left( -\frac{\mathrm{d}^2}{\mathrm{d}r^2} g \right)$

Lambda reconstruction for **range of view** $[25^\circ, 155^\circ]$. Note the added artifacts are along circles centered at $\theta(25^\circ)$ and $\theta(155^\circ)$.

# Real data reconstructions, $\varphi$ between $-45^\circ$ to $225^\circ$



No artifact reduction    With artifact reduction    Difference Image

*Paper phantom with ink as acoustic absorber*[1] *.*

[1] Data by courtesy of Prof. Daniel Razansky (Institute of Biological and Medical Imaging, Helmholtz Zentrum München).

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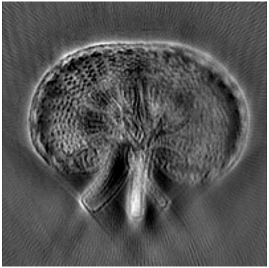

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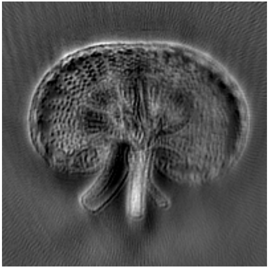

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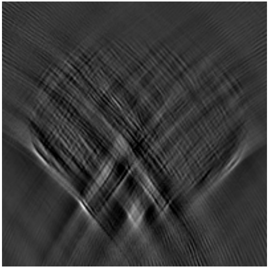

Difference Image

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*The added artifacts are exactly as predicted–they occur on circles at the ends of the data set*

# Summary

**The Paradigm:** $f$ is the function to be reconstructed.
If the tomography problem is modeled by a transform that averages over curves (e.g., X-ray CT, TAT/PAT, Motion compensated CT), then:

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Our reconstructions are of filtered backprojection type. ***Other reconstruction methods might reconstruct the invisible singularities better, but invisible singularities will always be difficult to reconstruct (highly ill-posed).***

# The Proof

**[Q 1993, Frikel Q 2013, Frikel Q 2015, Hahn Q 2016] use the following keys.**

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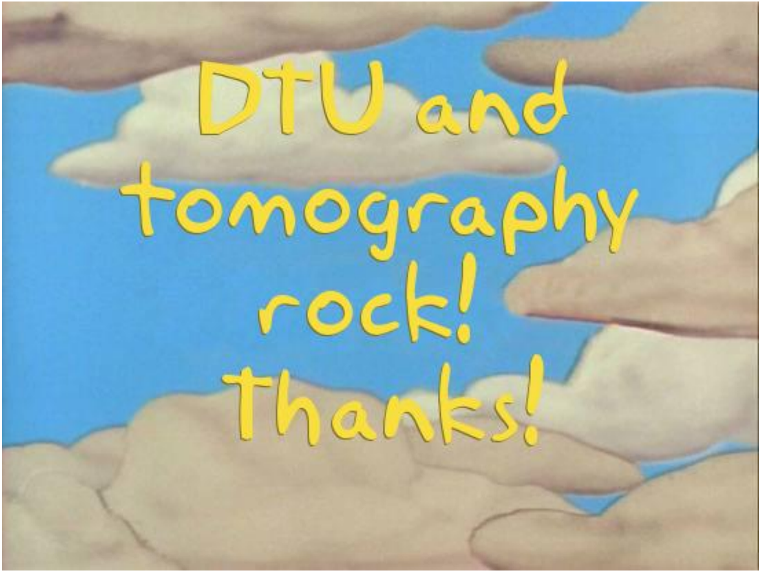
*Final word: Invisible singularities and added artifacts are intrinsic to limited data tomography and they can be understood using the geometry of the data set.*

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# The reconstruction operator for Limited angle CT

$$B_\Phi f = R^* \left( \sqrt{-d^2/dp^2} \chi_{[a,b]} Rf \right)$$

where $R^*$ is the X-ray backprojection operator.

- In [FrQu2013], we prove that $B_\Phi$ is a singular pseudodifferential operator, and we use a theorem of Hörmander to characterize the added artifacts. ▸ ΨDOs.

$$V_\Phi = \left\{ s\theta(\varphi) \middle| s \neq 0,\ \varphi \in (-\Phi, \Phi) \right\}$$

$$B_\Phi f(x) = \frac{1}{2\pi} \int_{\xi \in V_\Phi} e^{ix\cdot\xi}\ 1\ \mathcal{F} f(\xi) d\xi$$

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# Observations about the artifact reduction procedure

If $\kappa$ is the smooth function supported in $(-\Phi, \Phi)$ and equal to one on $(-\Phi + \varepsilon, \Phi - \varepsilon)$, then we prove that

$$B_{\Phi,\kappa} f = \frac{1}{2\pi} \int_{\xi \in V_\Phi} e^{ix\cdot\xi} \, \kappa\left(\frac{\xi}{\|\xi\|}\right) \mathcal{F}f(\xi) d\xi$$

- Note that the symbol of $B_\Phi$ as a pseudodifferential operator is $p(x, \xi) = \kappa\left(\frac{\xi}{\|\xi\|}\right)$, which is elliptic, at least on $\mathcal{V}_{(-\Phi+\varepsilon, \Phi-\varepsilon)}$.

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- Furthermore, since the symbol is smooth, $B_{\Phi,\kappa}$ is a standard pseudodifferential operator and does not adds singularities.

# Fourier Integral Operators

$Z$ and $X$ are open subsets of $\mathbb{R}^n$:

$$F(f)(z) = \int_{x \in X, \omega \in \mathbb{R}^n} e^{i\phi(z,x,\omega)} p(z,x,\omega) f(x) dx\, d\omega$$

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Canonical Relation:

$\mathcal{C} = \{(z, \partial_z \phi(z, x, \omega); x, -\partial_x \phi(z, x, \omega)) | \partial_\omega \phi(z, x, \omega) = 0\}$

$$\begin{array}{ccc} & \mathcal{C} & \\ \Pi_L \swarrow & & \searrow \Pi_R \\ Z \times (\mathbb{R}^n \backslash \{\mathbf{0}\}) & & X \times (\mathbb{R}^n \backslash \{\mathbf{0}\}) \end{array}$$

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WF relation: $\mathrm{WF}(F(f)) \subset \Pi_L\left(\Pi_R^{-1}(\mathrm{WF}(f))\right)$.

What it means: FIO change singularities in specific ways determined by the geometry of $\mathcal{C}$. Back

# Pseudodifferential operators

$$P(f)(z) = \int e^{i(z-x)\cdot\omega} p(z,x,\omega) f(x)\, dx\, d\omega$$

Phase Function: $\phi(z,x,\omega) = (z-x)\cdot\omega$ is linear in $\omega$, smooth.

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What it means: ΨDO do not move wavefront set.

Back

# Limited Angle Operators

**Limited Angular Range:** $\Phi \in (0, \pi/2)$

$\varphi \in [-\Phi, \Phi]: \quad \theta(\varphi) = (\cos(\varphi), \sin(\varphi))$

**Lines:** $L(\varphi, p)$ perpendicular to $\varphi$ and $p$ units from the origin, $\varphi \in [-\Phi, \Phi]$, $p \in \mathbb{R}$.

# Limited Angle Operators

**Limited Angular Range:** $\Phi \in (0, \pi/2)$
$\varphi \in [-\Phi, \Phi]: \quad \theta(\varphi) = (\cos(\varphi), \sin(\varphi))$

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**Limited Angle Filtered Back Projection Operator:**

$$f(x) = \frac{1}{4\pi} R^* \left(\Lambda_p R f\right)(x) \quad B_\Phi f(x) := \frac{1}{4\pi} R^* \left(\Lambda_p 1_{[-\Phi,\Phi]} R f\right)(x)$$

Where $1_{[-\Phi,\Phi]}(\phi)$ is 1 on the interval $[-\Phi, \Phi]$ and 0 elsewhere. It sets data outside the known region to zero.

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# Microlocal Analysis of $B_\Phi$

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## Theorem (Frikel, Q 2013)

*Let $f \in \mathcal{E}'(\mathbb{R}^2)$. Then*

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