A Frame Theoretic View on Inverse Problems

Jakob Lemvig — Technical University of Denmark (DTU)

joint work with Jürgen Frikel (OTH Regesburg)

DTU Compute
Department of Applied Mathematics and Computer Science

Outline

- Frame Theory
  - Frames in abstract Hilbert spaces
  - Structured frames for $L^2(\mathbb{R}^d)$
- Inverse Problems in Abstract Settings: Dual frame based regularizations
- Shearlet Frames in $L^2(\mathbb{R}^2)$
- Inversion of the Radon Transform by Shearlet Frames

Convenient Expansions of Functions

- Expansions of signals f of finite energy, i.e., $f \in L^2(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|^2 \, dx < \infty\}$

$$f(x) = \sum_{j=1}^{\infty} c_k \varphi_k(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x) + \dots$$

  in terms of convenient building blocks $\varphi_k \in L^2(\mathbb{R}^d)$.

- Shearlet analysis: An alternative to Fourier analysis, where the building blocks $\{\varphi_k\}$ are dilations (scales), shears and translations of a single function $\psi \in L^2(\mathbb{R}^2)$:

$$\{\varphi_k\} \sim \left\{2^{3j/4} \psi(2^j x_1 + k 2^{j/2} x_2 - m_1, 2^{j/2} x_2 - m_2)\right\}_{j \in \mathbb{Z}, k \in \mathbb{Z}, m \in \mathbb{Z}^2}$$

- Frames: A generalization of orthonormal bases (ONB) with more flexibility and freedom.

Beyond Orthonormal Bases: Frames

It is not always possible/desirable to require ONB.

- Problems:
  - Non-existence of Gabor ONB with good time-frequency localization
  - Non-existence of ONB sensitive to curvilinear singularities
  - Non-resilience to erasures/noise of expansions in an ONB

Beyond Orthonormal Bases: Frames

It is not always possible/desirable to require ONB.

- Problems:
  - Non-existence of Gabor ONB with good time-frequency localization
  - Non-existence of ONB sensitive to curvilinear singularities
  - Non-resilience to erasures/noise of expansions in an ONB
- Solution: Frames – a standard tools in applied mathematics and engineering.

Beyond Orthonormal Bases: Frames

It is not always possible/desirable to require ONB.

- Problems:
  - Non-existence of Gabor ONB with good time-frequency localization
  - Non-existence of ONB sensitive to curvilinear singularities
  - Non-resilience to erasures/noise of expansions in an ONB
- Solution: Frames – a standard tools in applied mathematics and engineering.
- Key Property of Frames:

Redundancy!

What is a Frame?

Definition

A sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is a frame for a separable Hilbert space X if

$$\exists A, B > 0 : \quad A\,\|f\|^2 \leqslant \sum_{k=1}^{\infty} |\langle f, \varphi_k\rangle|^2 \leqslant B\,\|f\|^2 \qquad \text{for all } f \in X.$$

If the upper bound holds, then $\{\varphi_k\}$ is said to be a Bessel sequence.

Definition

Two Bessel sequences $\{\varphi_k\}$ and $\{\psi_k\}$ are said to be dual frames if

$$f = \sum_{k=1}^{\infty} \langle f, \varphi_k\rangle \psi_k \quad \text{for all } f \in X.$$

Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \to \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k\rangle\}_{k\in\mathbb{N}}$$

and the Synthesis operator:

$$D_\Phi : \ell^2(\mathbb{N}) \to X, \quad D_\Phi\{c_k\}_{k\in\mathbb{N}} = \sum_{k=1}^{\infty} c_k\varphi_k$$

Operators associated with Frames/Bessel sequences

DTU

For $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \to \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k\rangle\}_{k\in\mathbb{N}}$$

and the Synthesis operator:

$$D_\Phi : \ell^2(\mathbb{N}) \to X, \quad D_\Phi\{c_k\}_{k\in\mathbb{N}} = \sum_{k=1}^{\infty} c_k\varphi_k$$

- $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$ is Bessel $\Leftrightarrow$ C_Φ is a bounded operator. Here: $(C_\Phi)^* = D_\Phi$.

Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \to \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k \rangle\}_{k\in\mathbb{N}}$$

and the Synthesis operator:

$$D_\Phi : \ell^2(\mathbb{N}) \to X, \quad D_\Phi \{c_k\}_{k\in\mathbb{N}} = \sum_{k=1}^{\infty} c_k \varphi_k$$

- $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$ is Bessel $\Leftrightarrow C_\Phi$ is a bounded operator. Here: $(C_\Phi)^* = D_\Phi$.
- The frame inequalities mean $A \|f\|^2 \leqslant \|C_\Phi f\|^2 \leqslant B \|f\|^2$ for all $f \in X$.

Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \to \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k\rangle\}_{k\in\mathbb{N}}$$

and the Synthesis operator:

$$D_\Phi : \ell^2(\mathbb{N}) \to X, \quad D_\Phi\{c_k\}_{k\in\mathbb{N}} = \sum_{k=1}^{\infty} c_k\varphi_k$$

- $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$ is Bessel $\Leftrightarrow$ C_Φ is a bounded operator. Here: $(C_\Phi)^* = D_\Phi$.
- The frame inequalities mean $A\|f\|^2 \leqslant \|C_\Phi f\|^2 \leqslant B\|f\|^2$ for all $f \in X$.
- Φ and Ψ are dual frames $\Leftrightarrow$ $D_\Psi C_\Phi = I_X$, i.e., $f = \sum_{k=1}^{\infty}\langle f, \varphi_k\rangle\psi_k \quad \forall f \in X$

Operators associated with Frames/Bessel sequences

For $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$, define the Analysis operator:

$$C_\Phi : X \to \ell^2(\mathbb{N}), \quad C_\Phi f = \{\langle f, \varphi_k\rangle\}_{k\in\mathbb{N}}$$

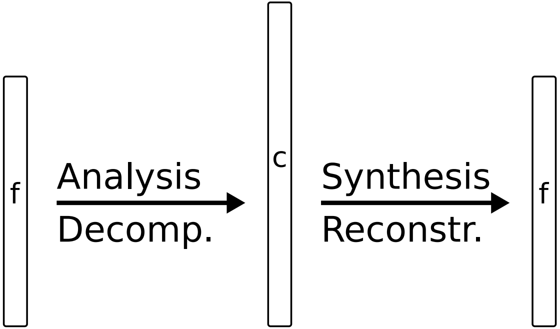
and the Synthesis operator:

$$D_\Phi : \ell^2(\mathbb{N}) \to X, \quad D_\Phi\{c_k\}_{k\in\mathbb{N}} = \sum_{k=1}^{\infty} c_k\varphi_k$$

- $\Phi = \{\varphi_k\}_{k\in\mathbb{N}}$ is Bessel $\Leftrightarrow$ C_Φ is a bounded operator. Here: $(C_\Phi)^* = D_\Phi$.
- The frame inequalities mean $A\|f\|^2 \leqslant \|C_\Phi f\|^2 \leqslant B\|f\|^2$ for all $f \in X$.
- Φ and Ψ are dual frames $\Leftrightarrow$ $D_\Psi C_\Phi = I_X$, i.e., $f = \sum_{k=1}^{\infty}\langle f, \varphi_k\rangle\psi_k \quad \forall f \in X$
- A frame has at least one dual frame: the canonical dual $\{(D_\Phi C_\Phi)^{-1}\varphi_k\}_{k\in\mathbb{N}}$.

The Picture of Frame Expansions

The Picture of Frame Expansions:



Signal f **Coefficients** $c_k = \langle f, \varphi_k \rangle$ **Signal** $f = \sum_k c_k \psi_k$

- Want: Analysis & synthesis to be linear and continuous operations, often assuming some structure on φ_k and/or ψ_k

Structered Frames in $L^2(\mathbb{R}^d)$

Generalized Shift-invariant (GSI) systems are of the form: $\{T_{C_p k} g_p\}_{k \in \mathbb{Z}^d, p \in P}$, where $g_p \in L^2(\mathbb{R}^d), C_p \in \mathrm{GL}(d, \mathbb{R}), P$ a countable index set.

Structured Frames in $L^2(\mathbb{R}^d)$

Generalized Shift-invariant (GSI) systems are of the form: $\{T_{C_p k} g_p\}_{k \in \mathbb{Z}^d, p \in P}$, where $g_p \in L^2(\mathbb{R}^d), C_p \in \mathrm{GL}(d, \mathbb{R}), P$ a countable index set.

Necessary condition for Frame (covering of frequency domain)

If the GSI system is a frame with bounds A and B, then

$$A \leqslant \sum_{p \in P} \frac{1}{|\det C_p|} |\hat{g}_p(\gamma)|^2 \leqslant B \quad a.e.\ \gamma \in \mathbb{R}^d$$

Here we ignore a technical Local Integrability Condition. Result for $d = 1$ due to [Christensen, Hasannasab, L.] and for general $d \in \mathbb{N}$ by [Führ, Jakobsen, L.]

Structered Frames in $L^2(\mathbb{R}^d)$

Generalized Shift-invariant (GSI) systems are of the form: $\{T_{C_p k} g_p\}_{k\in\mathbb{Z}^d, p\in P}$, where $g_p \in L^2(\mathbb{R}^d), C_p \in \mathrm{GL}(d,\mathbb{R}), P$ a countable index set.

Necessary condition for Frame (covering of frequency domain)

If the GSI system is a frame with bounds A and B, then

$$A \leqslant \sum_{p\in P} \frac{1}{|\det C_p|} |\hat{g}_p(\gamma)|^2 \leqslant B \quad a.e.\ \gamma \in \mathbb{R}^d$$

Sufficient condition for Bessel (not too much overlap)

If

$$B := \operatorname*{ess\,sup}_{\gamma\in\mathbb{R}^d} \sum_{p\in P} \sum_{\alpha\in\mathbb{Z}^d} \frac{1}{|\det C_p|} \left|\hat{g}_p(\gamma)\hat{g}_p(\gamma + C_p^{\#}\alpha)\right| < \infty$$

then the GSI system is Bessel with bounds B. Here $C_p^{\#} := (C_p^T)^{-1}$

An Inverse Problem

Setup: Let X, Y be separable (inf. dim.) Hilbert spaces. Let

$$K : D(K) \to Y, X = \overline{D(K)}, Y = \overline{R(K)}$$

be an injective, closed operator (typically, it is compact).

Problem

Given $g \in Y$ and $\varepsilon > 0$ s.t. $\|Kf - g\|_Y < \varepsilon$, recover f.

An Inverse Problem

Setup: Let X, Y be separable (inf. dim.) Hilbert spaces. Let

$$K : D(K) \to Y, X = \overline{D(K)}, \ Y = \overline{R(K)}$$

be an injective, closed operator (typically, it is compact).

Problem

Given $g \in Y$ and $\varepsilon > 0$ s.t. $\|Kf - g\|_Y < \varepsilon$, recover f.

- It holds $(K^{-1})^* = (K^*)^{-1}$; for short, we write this operator as $K^{-*} : X \to Y$.
- $D(K^{-*}) = X \Leftrightarrow K^{-1}$ bounded. If K is compact, K^{-1} is unbounded.

Inversion Formula based on Dual Frames

- Take dual frames $\{\varphi_k\}$ and $\{\psi_k\}$ for X s.t. $\varphi_k \in D(K^{-*})$:

$$f = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle_X \psi_k \qquad \text{for all } f \in X.$$

- Note that

$$\langle f, \varphi_k \rangle_X = \langle f, K^* K^{-*} \varphi_k \rangle_X = \langle Kf, K^{-*} \varphi_k \rangle_Y$$

- This gives the inversion formula:

$$f = \sum_{k=1}^{\infty} \langle Kf, K^{-*} \varphi_k \rangle_Y \psi_k \qquad \text{for all } f \in X.$$

Regularization Strategy based on Dual Frames I

- Set $w_k = \kappa_k K^{-*}\varphi_k$. Pick weights $\kappa_k > 0$ s.t. $W = \{w_k\}_{k\in\mathbb{N}}$ is a Bessel sequence in X.

This is indeed always possible:

Lemma

Let $\{\theta_k\}$ be a sequence of positive numbers such that $\sum_{k\in\mathbb{N}} \theta_k < \infty$. Take $\kappa_k = \sqrt{\theta_k} / \|K^{-*}\varphi_k\|_Y$. Then $W = \{w_k\}_{k\in\mathbb{N}}$ is Bessel with bound $B = \sum_{k\in\mathbb{N}} \theta_k$.

DTU

Regularization Strategy based on Dual Frames I

- Set $w_k = \kappa_k K^{-*} \varphi_k$. Pick weights $\kappa_k > 0$ s.t. $W = \{w_k\}_{k \in \mathbb{N}}$ is a Bessel sequence in X.

This is indeed always possible:

Lemma

Let $\{\theta_k\}$ be a sequence of positive numbers such that $\sum_{k \in \mathbb{N}} \theta_k < \infty$. Take $\kappa_k = \sqrt{\theta_k} / \left\| K^{-*} \varphi_k \right\|_Y$. Then $W = \{w_k\}_{k \in \mathbb{N}}$ is Bessel with bound $B = \sum_{k \in \mathbb{N}} \theta_k$.

- The inversion formula is now:

$$f = \sum_{k=1}^{\infty} \frac{1}{\kappa_k} \langle Kf, w_k \rangle_Y \psi_k \qquad \text{for all } f \in D(K).$$

- Or in terms of analysis, synthesis and multiplication operators:

$$f = D_\Psi M_{1/\kappa} C_W K f$$

where $M_{1/\kappa}$ is a (often unbounded) multiplication operator on $\ell^2(\mathbb{N})$ defined by $M_{1/\kappa} \{c_k\}_{k \in \mathbb{N}} = \{c_k / \kappa_k\}_{k \in \mathbb{N}}$.

Regularization Strategy based on Dual Frames II

- We define the recovery operator $R = D_\Psi M_{1/\kappa} C_W$:

$$R : D(R) \to X,\ Rg = \sum_{k\in\mathbb{N}} \frac{1}{\kappa_k} \langle g, w_k \rangle_Y \psi_k \tag{1}$$

  where the domain is determined by a Picard condition:

$$D(R) = \left\{ g \in Y : \sum_{k\in\mathbb{N}} \frac{\left|\langle g, w_k\rangle_Y\right|^2}{\kappa_k^2} < \infty \right\}.$$

- The recovery strategy from $\|Kf - g\| < \varepsilon$ is:

$$D_\Psi M_{1/\kappa} S C_W g$$

  where $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is a threshold procedure.

Shearlet Systems in $L^2(\mathbb{R}^2)$

- Anisotropic scaling A:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2^{1/2} \end{pmatrix},$$

- Shearing S_k (direction parameter $\leftrightarrow$ rotations):

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

- The shearlet system generated by $\psi \in L^2(\mathbb{R}^2)$ is

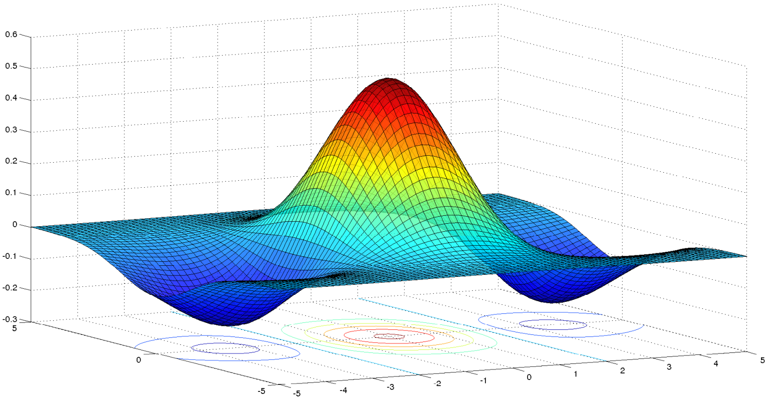
$$\left\{ \psi_{j,k,m} = D_{S_k A^j} T_m \psi = 2^{3j/4} \psi(S_k A^j \cdot - m) : j \in \mathbb{Z}, k \in \mathbb{Z}, m \in \mathbb{Z}^2 \right\}$$

- Frequency localization of ψ:

$$\left| \hat{\psi}(\gamma_1, \gamma_2) \right| \leqslant C \min(1, |2\gamma_1|^{\alpha}) \min(1, |\gamma_1|^{-\delta}) \min(1, |\gamma_2|^{-\delta})$$

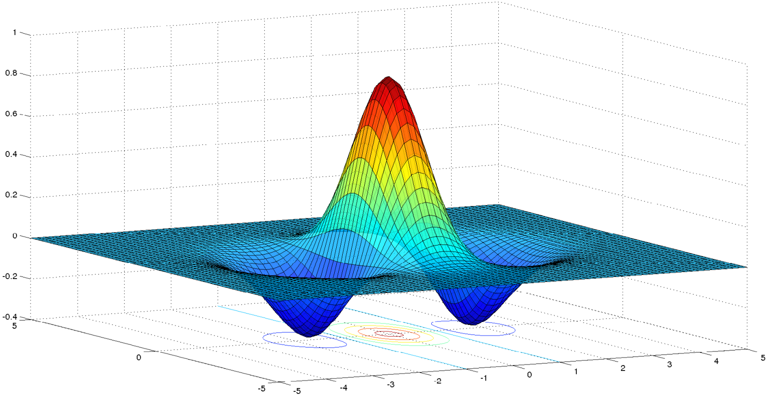
for some $C > 0$, $\alpha > \delta > 3$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



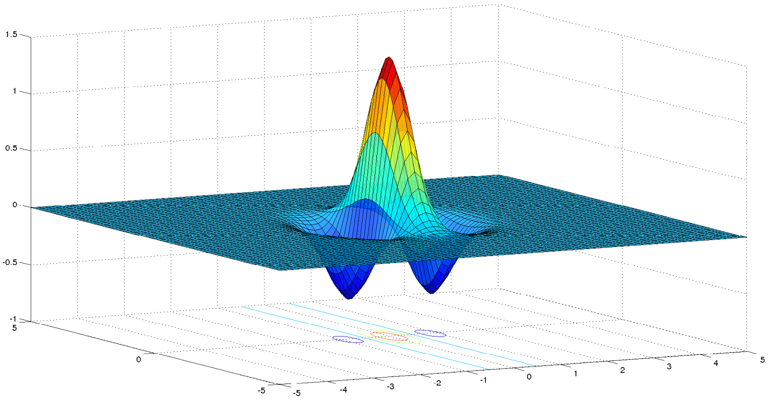
$\psi_{j,k,m}$ for $j = 0, k = 0, m = (0, 0)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



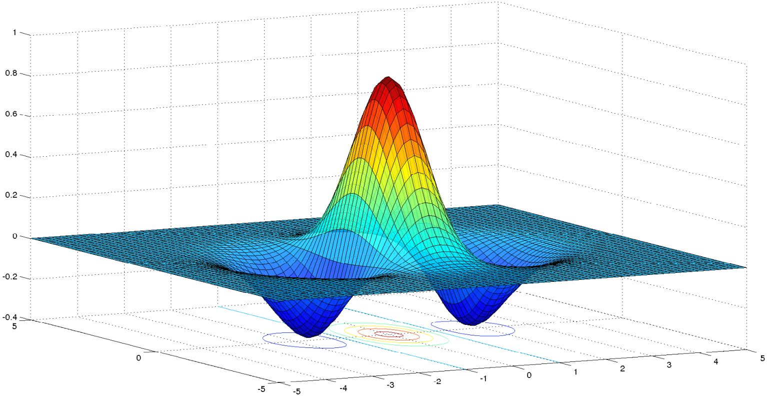
$\psi_{j,k,m}$ for $j = 1, k = 0, m = (0, 0)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



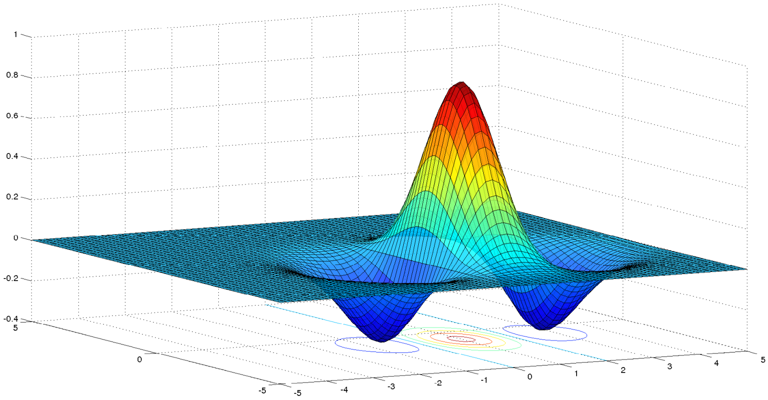
$\psi_{j,k,m}$ for $j = 2, k = 0, m = (0,0)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



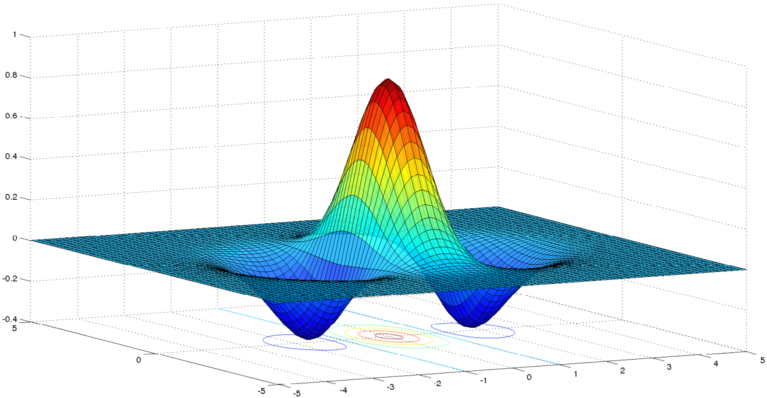
$\psi_{j,k,m}$ for $j = 1, k = 0, m = (0,0)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



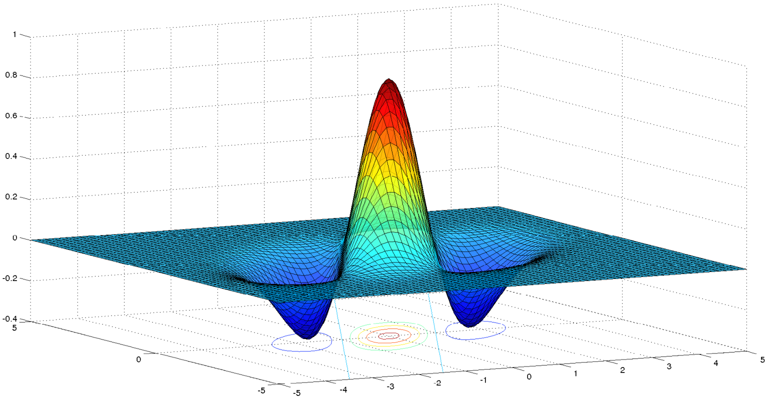
$\psi_{j,k,m}$ for $j = 1, k = 0, m = (1, -1)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



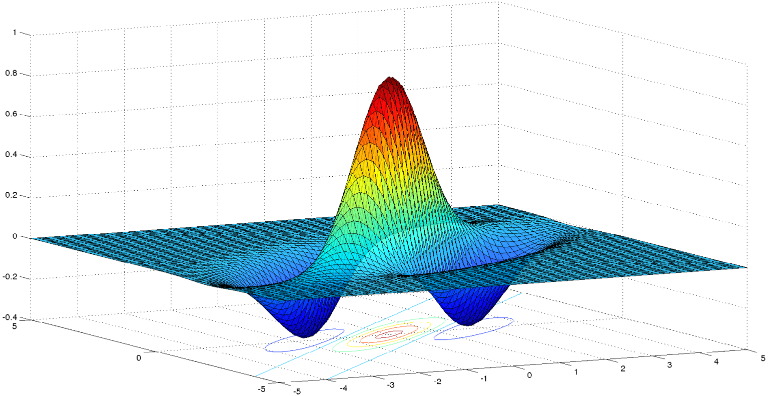
$\psi_{j,k,m}$ for $j = {\color{red}1}, k = 0, m = (0,0)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



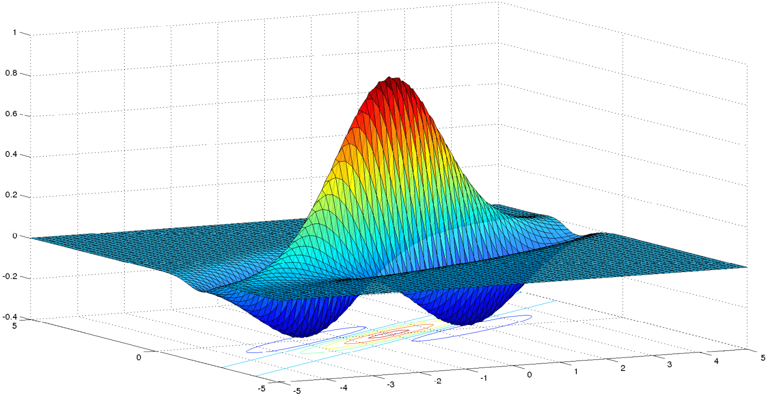
$\psi_{j,k,m}$ for $j = 1, k = -1, m = (0, 0)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



$\psi_{j,k,m}$ for $j = 1, k = -2, m = (0,0)$

Action of Anisotropic Scaling, Shearing, and Translation in 2D



$\psi_{j,k,m}$ for $j = 1, k = -3, m = (0,0)$

Cone-adapted Shearlet systems

- Problem: "Length" of $\operatorname{supp} \hat{\psi}_{j,k,m}$ goes to ∞ as $|k| \to \infty$.

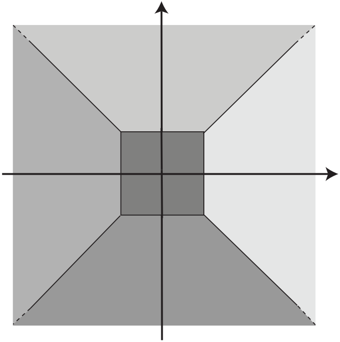
Cone-adapted Shearlet systems

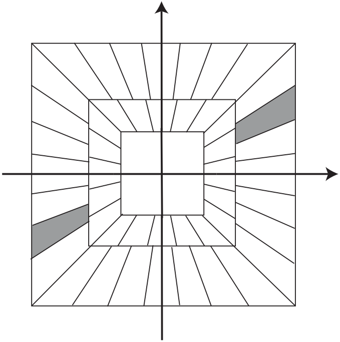
- Problem: "Length" of $\operatorname{supp} \hat{\psi}_{j,k,m}$ goes to ∞ as $|k| \to \infty$.
- Solution: Cone-adapted shearlet system

Definition

For $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$, the cone-adapted shearlet system $\mathrm{SH}(\phi, \psi, \tilde{\psi})$ is the union:

$$\{T_k \phi\}_{k \in \mathbb{Z}^2} \cup \{D_{S_k A^j} T_m \psi\}_{j \geqslant 0, |k| \leqslant \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2} \cup \{D_{\tilde{S}_k \tilde{A}^j} T_m \tilde{\psi}\}_{j \geqslant 0, |k| \leqslant \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2}$$





Shearlet systems as GSI systems

By the commutator relations

$$D_{A^j S_k} T_m = T_{S_{-k} A^{-j} m} D_{A^j S_k},$$

it follows that

$$\psi_{j,k,m} = D_{S_k A^j} T_m \psi = T_{S_{-k} A^{-j} m} D_{A^j S_k} \psi$$

Hence, the shearlet system on horizontal is a GSI system $\{T_{C_p k} g_p\}_{k \in \mathbb{Z}^d, p \in P}$ with

$$C_p = C_{(j,k)} = S_{-k} A^{-j} \quad \text{and} \quad g_p = g_{(j,k)} = D_{A^j S_k} \psi$$

where

$$P = \bigcup_{j \in \mathbb{N}_0} \{j\} \times \left[-\left\lceil 2^{j/2} \right\rceil, \left\lceil 2^{j/2} \right\rceil \right].$$

Similar for the vertical cones and the central box.

Necessary and Sufficient Conditions

Necessary condition for Frame (covering of frequency domain)

If the shearlet system $\mathrm{SH}(\phi, \psi, \tilde{\psi})$ is a frame with bounds A and B, then

$$A \leqslant \left|\hat{\phi}(\gamma)\right|^2 + \sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} \left|\hat{\psi}(S_{-k}^T A^j \gamma)\right|^2 + \sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} \left|\hat{\tilde{\psi}}(\tilde{S}_{-k}^T \tilde{A}^j \gamma)\right|^2 \leqslant B$$

for a.e. $\gamma \in \mathbb{R}^2$.

Sufficient condition for Bessel (not too much overlap)

If

$$B := \operatorname*{ess\,sup}_{\gamma \in \mathbb{R}^2} \Bigg(\sum_{\alpha \in \mathbb{Z}^2} \left|\hat{\phi}(\gamma)\hat{\phi}(\gamma + \alpha)\right| + \sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} \left|\hat{\psi}(S_{-k}^T A^j \gamma)\hat{\psi}(S_{-k}^T A^j \gamma + \alpha)\right|$$

$$+ \sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} \left|\hat{\tilde{\psi}}(\tilde{S}_{-k}^T \tilde{A}^j \gamma)\hat{\tilde{\psi}}(\tilde{S}_{-k}^T \tilde{A}^j \gamma + \alpha)\right| \Bigg) < \infty,$$

then $\mathrm{SH}(\phi, \psi, \tilde{\psi})$ is a Bessel system with bounds B. Here $C_p^{\#} := (C_p^T)^{-1}$

What is the weighted system W?

- Setup: $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^{\perp})dt$$

What is the weighted system W?

- Setup: $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^{\perp})dt$$

- It can be shown that $R^{-*} = \Lambda_s R$, where $\Lambda f = \mathcal{F}^{-1}(|\gamma| \, \hat{f}(\gamma))$ is a Riesz potential (here on the s-variable)

What is the weighted system W?

- Setup: $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^{\perp})dt$$

- It can be shown that $R^{-*} = \Lambda_s R$, where $\Lambda f = \mathcal{F}^{-1}(|\gamma|\,\hat{f}(\gamma))$ is a Riesz potential (here on the s-variable)
- A intertwining relation gives $R^{-*} = R\Lambda$

What is the weighted system W?

- Setup: $X = L^2(\mathbb{R}^2)$, $Y = L^2(S^1 \times \mathbb{R})$, and

$$Rf(\theta, s) = \int_{-\infty}^{\infty} f(s\theta + t\theta^{\perp})dt$$

- It can be shown that $R^{-*} = \Lambda_s R$, where $\Lambda f = \mathcal{F}^{-1}(|\gamma|\, \hat{f}(\gamma))$ is a Riesz potential (here on the s-variable)
- A intertwining relation gives $R^{-*} = R\Lambda$
- Since R is bounded, we only have to make

$$\{\kappa_{j,k,m}\Lambda\psi_{j,k,m}\}_{j\geqslant 0, |k|\leqslant \lceil 2^{j/2} \rceil, m\in\mathbb{Z}^2}$$

  a Bessel system for some choice of $\kappa_{j,k,m} > 0$. And similar for the vertical cones and the central box.

Riesz Potential of a GSI system

- Since Λ is a Fourier multiplier, it commutes with translation:

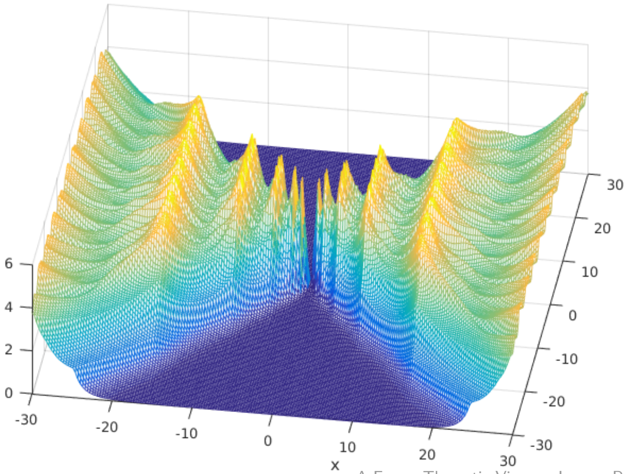
$$\Lambda T_{C_p k} g_p = T_{C_p k} \Lambda g_p$$

- Hence, Λ maps the shearlet system to a GSI system (that is not a shearlet system!) with generators $g_p = \Lambda D_{A^j S_k} \psi$.

Covering of the Horizontal Cone

Plot of

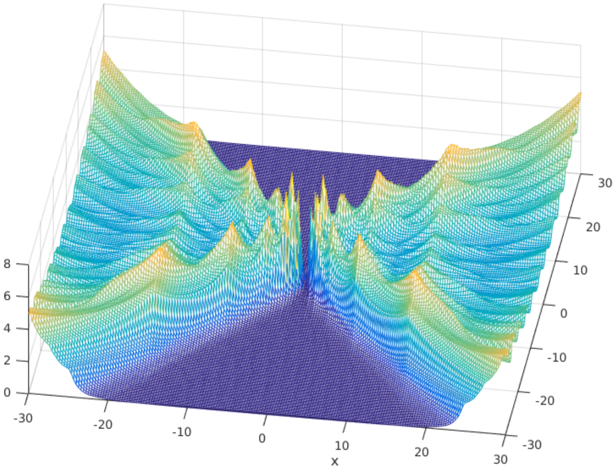
$$\sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} |\hat{\psi}(S_{-k}^T A^j \gamma)|^2$$



Covering of the Horizontal Cone

Plot of

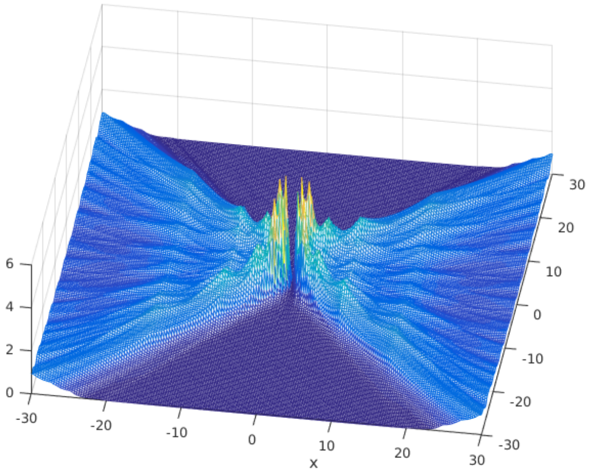
$$\sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} \kappa_j^2 \, |\gamma|^2 \, \left|\hat{\psi}(S_{-k}^T A^j \gamma)\right|^2, \quad \kappa_j = 2^{-j},$$



Covering of the Horizontal Cone

Plot of

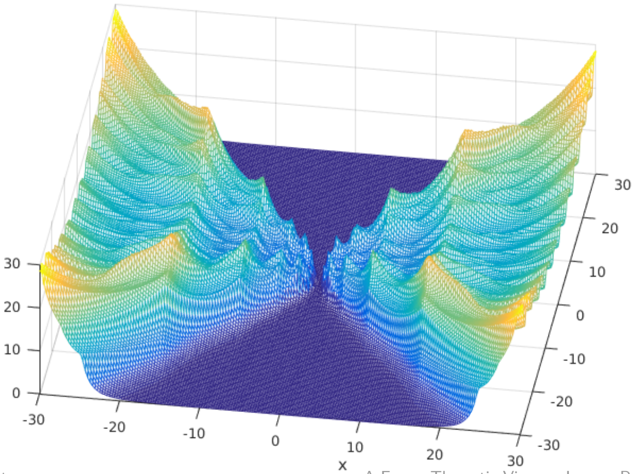
$$\sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} \kappa_j^2 \, |\gamma|^2 \, |\hat{\psi}(S_{-k}^T A^j \gamma)|^2, \quad \kappa_j = 2^{-5j/4},$$



Covering of the Horizontal Cone

Plot of

$$\sum_{j=0}^{\infty} \sum_{|k| \leqslant \lceil 2^{j/2} \rceil} \kappa_j^2 \, |\gamma|^2 \, \left| \hat{\psi}(S_{-k}^T A^j \gamma) \right|^2, \quad \kappa_j = 2^{-3j/4},$$



Future tasks

- We really want the solve the inverse problem under the prior information that $f \in C \subset X$ for some image class C, e.g., cartoon-like images.
- If $\{\langle f, \phi_k \rangle\}_{k \in \mathbb{N}}$ belongs (after reordering descending in absolute value) to a weak ℓ_p space for some small $p > 0$, then so does $\{\langle Kf, K^{-*}\phi_k \rangle\}_{k \in \mathbb{N}}$.
- Understand the role of the weights κ_k

References

E. J. Candès and D. L. Donoho.
Recovering edges in ill-posed inverse problems: optimality of curvelet frames.
Ann. Statist., 30(3):784–842, 2002.

Thank You!

Jakob Lemvig – Technical University of Denmark (DTU)
Group: Harmonic Analysis – Theory and Applications
Email: jakle@dtu.dk.
Preprints: www.compute.dtu.dk/people/J.Lemvig