Proving Paraconsistent, Many-Valued and Modal Logics by Handling Polynomials: Some Perspectives on Polynomizing Logics

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Outline

1. Polynomial expansions as logic tools
   - Boole's views on Algebra & Logic
   - Polynomizing=Algebra+Calculus+Logic
   - Deductions as solving equations
2. Several logics in polynomial format
   - PC, FOL, Belnap-Dunn's logic, *mbC*, C_1 in polynomial form
   - Boole's analysis of syllogistic in polynomial format
   - Modal Logic in polynomial form
3. Polynomials as heuristic machines
   - Half-logics and quarter-logics
   - The "translation paradox"
   - Polynomizing: perspectives and problems

Boole's dream of algebrizing logic

- *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities* (1854): ordinary algebra + Aristotelian Logic
- Boole was more interested in the *algebra of logic* than in the logic of algebra
- In in this sense, he was concerned in solving equations, while Aristotle was concerned with predication

Why was Boole mixing everything?

- However, his first publication on mathematics was a paper on the Theory of Analytical Transformations (*Cambridge Math. J.* in 1840;
- And also Boole was much involved with his "Differential Equations" of 1859 and his "Finite Differences" of 1860.
- How did Boole ***unify*** all this?

Polynomizing=Algebra+Calculus+Logic

- Develops some ideas on recovering Logic + Algebra in a *wide* sense
- Reasons with polynomials as a guiding model
- But departs from Boolean rings and their generalization, instead of Boolean algebras
- Gives new proof theory (or semantics) to classical and to seveal non-classical logics, and lead to the clarification of some ideas of Boole.

Polynomial representations: the "complex" made simple (but infinite)

- Functions $f(x)$ rewritten as infinite polynomials (close to a base point x_0):

$$f(x) = \alpha_0(x_0) + \alpha_1(x_0) \cdot (x - x_0) + \ldots \alpha_n(x_0) \cdot (x - x_0)^n + \ldots$$

- Coefficients $\alpha_k(x_0)$ coincide with the derivatives of $f(x)$ in x_0

Polynomial expansions can be enlightening: Euler

- Leonhard Euler (1707-1783), in comparing infinite sums and products: $\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \ldots}{1 \cdot 2 \cdot 4 \cdot 10 \ldots} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots$
- In contemporary notation:

$$\prod_p \frac{p}{p-1} = \sum_n \frac{1}{n} \quad \text{for } p \text{ primes, } n \geq 1$$

- This gives another proof of the infinity of primes: the right-hand harmonic series is divergent.
- Euler's proof talks about distribution, not counting.

Are deductions and solving equations incompatible?

Against Boole:

- Some authors see solving equations as opposed to performing deductions
- e.g. Corcoran, p. 281:

  ".... There is no such thing as indirect equation-solving, of course.´´

- **Not so sure!** What about conditional equation solving (solving under constraints)?

And a shortcoming?

Not sure!

> *"According to our ideas there was one serious shortcoming in Boole's calculus, considered as a system of logic; it contained no quantifiers, and therefore could not deal with some of the most interesting questions..."*

W. Kneale. Boole and the Revival of Logic. *op. cit.*

Boole's unifying approach

- Boole is reasoning at the same time with algebra and with classes, anticipating the results by M. Stone...
- ..or, if you prefer, the work by Stone justifies his intuitions
- But more: Boole mixed ideas of Differential Calculus, Logic, Algebra and Probability

Boole's idea on the 'index law'

- *The Laws of Thought*: great importance to the "index law" $x^2 = x$ "...a fundamental law of Metaphysics is but the consequence of a law of thought."

  $x(x-1) = 0$: Law of Non-Contradiction.
- Boole thought of generalizing the 'index law" to $x^n = x$, but rejected it as meaningless
- However, this is **totally meaningful** using polynomials over finite fields (Carnielli, 2001)

PC in polynomial form

Definition

The translation $* : \mathbf{PC} \mapsto \mathbf{Z_2}[\mathbf{X}]$ of **PC** into the Boolean ring $\mathbf{Z_2}[\mathbf{X}]$ produces the following interpretation for Classical Logic:

- $x^2 \rightsquigarrow x$
- $x + x \rightsquigarrow 0$
- $p_i \rightsquigarrow x_i$ for each atomic variable p_i
- $\neg\alpha \rightsquigarrow 1 + x$
- $\alpha \wedge \beta \rightsquigarrow x \cdot y$
- $\alpha \vee \beta \rightsquigarrow x \cdot y + x + y$
- $\alpha \rightarrow \beta \rightsquigarrow x \cdot y + x + 1$

Proving *reductio ad absurdum*

Example

$\alpha \to \beta, \alpha \to \neg\beta \vdash_{PC} \neg\alpha$

Proof.

In polynomial form, we have to check that:

$$(x \cdot y + x + 1) \cdot (x \cdot (y + 1) + x + 1) \cdot x \vdash_{\approx} 0$$

But easily:

$(xy + x + 1)(x(y + 1) + x + 1)x \approx (xy + x + 1)(xy + 1)x \approx$
$(x^2y^2 + xy + x^2y + x + xy + 1)x \approx$
$(\overbrace{xy} + \overbrace{xy} + \overbrace{xy} + x + \overbrace{xy} + 1)x \approx (x + 1)x \approx x^2 + x \approx 0$ □

Completeness for **PC**

Theorem (Weak Completeness for *PC*)

$$\vdash_{PC} \alpha \textit{ iff } (\alpha)^* \vdash_{\approx} 1$$

Theorem (Strong Completeness for *PC*)

$$\Gamma \vdash_{PC} \alpha \textit{ iff } \prod_{i=1,n} (\gamma_i)^* \cdot ((\alpha) + 1)^* \vdash_{\approx} 0$$

for $\Gamma_0 = \{\gamma_1, \ldots, \gamma_n\}$ *where* $\Gamma_0 \subseteq \Gamma$

That is,

$$\gamma_1 \wedge \cdots \wedge \gamma_n \wedge (\neg\alpha) = 0$$

Previous intuitions on 'polynomizing'

- E. Schröder used $\sum$ and $\prod$ to represent quantifiers.
- Some methods for Boolean reasoning were developed by the Russian logician Platon Poretsky (known in digital circuitry) in the 19th century.
- Also, Gégalkine, 1927,*Mat. Sbornik* (in Russian) shows a translation of sentences of *Principia Mathematica* into polynomials.

Gröbner bases and complexity

- Clegg, Edmonds, and Impagliazzo: Gröbner bases algorithm to find proofs of unsatisfiability, 1996.
- Wu, Tan and Li: polynomials over Q to represent truth-tables and decide many-valued logics, 1998.
- However, nobody used polynomial ring properties, nor extended the method to all finite-valued logics, to non-finite valued logics or to FOL...
- ... or to modal logics!

Polynomials instead of formulas

- Given a propositional logic **L**, a *polynomial interpretation* for **L** is a translation $* : \mathbf{L} \mapsto \mathbf{F}[X]$ of wffs into the ring $\mathbf{F}[X]$
- $\alpha \in \mathbf{L}$ is *satisfiable* if its traduct $\alpha^* \in \mathbf{F}[X]$ gets values in a certain $D \subseteq \mathbf{F}$ when evaluated in the field **F**
- $D \subseteq \mathbf{F}$ are the *distinguished truth-values*

PRC Rules for many-valued logics

For general (many-valued) logics formulas are interpreted within the polynomial rings over Galois fields $GF(p^n)[X]$:

Index rules:

1. $x + x + \ldots x \vdash_{\approx} 0$ (summing p times)
2. $x^{p^n} \vdash_{\approx} x$

Ring rules:

1. $f + (g + h) \vdash_{\approx} (f + g) + h$
2. $(f + g) \vdash_{\approx} (g + f)$
3. $f + 0 \vdash_{\approx} f$
4. $f + (-f) \vdash_{\approx} 0$
5. $f \cdot (g \cdot h) \vdash_{\approx} (f \cdot g) \cdot h$
6. $f \cdot (g + h) \vdash_{\approx} (f \cdot g) + (f \cdot h)$

PRC Rules: Metarules

Substituting "inside" and "outside" For $f, g, h \in \mathbf{F}[X]$:

1. Uniform Substitution:

$$\frac{f \vdash_{\approx} g}{f[x:h] \vdash_{\approx} g[x:h]}$$

2. Leibniz Rule:

$$\frac{f \vdash_{\approx} g}{h[x:f] \vdash_{\approx} h[x:g]}$$

Proofs and deductions in PRC

Definition (Weak Completeness for L)

$\vdash_L \alpha$ iff $\alpha^* \vdash_\approx d$, where $d \in D$ (i.e., d ranges over distinguished truth-values).

- That is: the polynomial rules *prove* that the polynomial α^* never outputs values outside the set D of distinguished truth values

Definition (Strong Completeness for L)

$\Gamma \vdash_L \alpha$ iff $\alpha^* \vdash_\approx d \in D$, under the constraints $\Gamma^* \approx d \in D$

Why do we need Galois fields $GF(p^n)$?

Theorem (Representing finite functions)

Any k-ary finite functions can be represented as polynomials over $GF(p^n)[x_1, \cdots, x_k]$.

- As Z_m is not a field if m is not a prime number, $Z_m[X]$ does not suffice
- For example: $Z_4[x, y]$ cannot represent $f(x, y) = max\{x, y\}$, but $GF(2^2)[x, y]$ can

But also

Theorem (Representing non-deterministic finite functions)

Any k-ary bounded non-deterministic finite functions can be represented as polynomials over $GF(p^n)[x_1, \cdots, x_k]$ with extra (hidden) variables.

4-valued logics and the Galois field $GF(2^2)$

4-valued logics are well represented in polynomials over $GF(2^2)$:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\odot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

The (paraconsistent) 4-valued logic of Belnap and Dunn

Tables in $GF(2^2)$:

$$B_4 = \langle \{0, 1, 2, 3\}, \{\neg, \wedge, \vee\}, \{2, 3\} \rangle$$

	$\neg$
0	0
1	2
2	1
3	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

Belnap-Dunn's logic B_4 in polynomial form

B_4 is translated into $GF(2^2)[X]$ as:

- $\neg$: $p_\neg(x)$ becomes $2x^2$
- $\wedge$: $p_\wedge(x, y)$ becomes $x^2y^2 + 3x^2y + xy^2$
- $\vee$: $p_\vee(x, y)$ becomes $x^2y^2 + 3x^2y + xy^2 + x + y$
- Rules: $x + x \approx 0$ and $x^4 \approx x$

Using the Galois field $GF(2^2)$ in 4-valued logics

Example (Deciding in Belnap-Dunn's logic)

$\alpha \vee \neg\alpha$ translates (using the $GF(2^2)[X]$ arithmetic to):

$$x + x^2 + 3x^3$$

It can be easily seen that:

- $x + x^2 + 3x^3 \in \{2, 3\}$ for $x \neq 0$,

but

- $0 + 0^2 + 3 \times 0^3 \approx 0$,
- hence $\alpha \vee \neg\alpha$ is not a B_4 tautology

Monadic **FOL** in polynomial format

Definition

Monadic **FOL** is represented in $\mathbf{Z_2}[\mathbf{X}]$ by adding clauses:

1. $(A(c_i))^* = x_i^A$, for each constant c_i (in a denumerable universe), where x_i^A is a variable in $\mathbf{Z_2}[\mathbf{X}]$
2. $(\forall z A(z))^* = \prod_{i=1}^{\infty} x_i^A$

As a consequence:

Definition

$(\exists z A(z))^* = (\neg\forall z \neg A(z))^* = 1 + \prod_{1=1}^{\infty}(1 + x_i^A)$

Note that now polynomials are infinite (i.e, formal series in $\mathbf{Z_2}[\mathbf{X}]$) Simplified notation:

- $(\forall z A(z))^* = \prod x_i$
- $(\exists z A(z))^* = 1 + \prod(1 + x_i)$

Examples of proofs in **FOL**

Example

- $\forall z A(z) \rightarrow \exists z A(z)$:
  $(\prod x_i) \cdot (1 + \prod(1 + x_i)) + \prod x_i + 1 \approx$

  $(\prod x_i) \cdot (\prod(1 + x_i)) + \prod x_i + \prod x_i + 1 \approx$

  $(\prod x_i \cdot (1 + x_i)) + \prod x_i + \prod x_i + 1 \approx 1$

  since $\prod x_i + \prod x_i \approx 0$ and $x_i \cdot (1 + x_i) \approx 0$ for each x_i

- We can also easily find counter-models in **FOL**, by using this method.

Boole's analysis of Syllogism in polynomial format

The four categorical forms:

A	All *A* is *B*	$\forall z(A(z) \rightarrow B(z))$
I	Some *A* is *B*	$\exists z(A(z) \wedge B(z))$
E	No *A* is *B*	$\forall z(A(z) \rightarrow \neg B(z))$
O	Some *A* is not *B*	$\exists z(A(z) \wedge \neg B(z))$

- **A** and **I** are affirmative (resp., universal and existential)
- **E** and **O** are negative (resp., universal and existential)
- **O** $=$ ¬**A** and **E** $=$ ¬**I**

Recovering Boole's interpretation

- **A** holds iff

$$\prod(ab + a + 1) = 1 \text{ iff } ab + a + 1 = 1$$

for every a, b iff $ab + a = 0$ for every a, b iff $ab = a$ for every a, b, which coincides with Boole's formalization of **A** as "$AB = A$"
in *The Mathematical Analysis of Logic* of 1847.

- **I** holds iff

$$1 + \prod(1 + ab) = 1 \text{ iff } \prod(1 + ab) = 0 \text{ iff } 1 + a_0 b_0 = 0$$

for some a_0, b_0 iff $a_0 b_0 = 1$ for some a_0, b_0 which coincides with Boole's formalization of **I** as "$AB = V$"
in *The Calculus of Logic* of 1848.

A Polynomial Ring Calculus for *S5*

Definition (*PRC* for *S5*)

- Translation function ($*$: $ForS5 \to \mathbb{Z}_2[X \cup X']$, where $X = \{x_1, \ldots\}$ and $X' = \{x_{\Box\alpha_1}, \ldots\ x_{\neg\Box\alpha_1}, \ldots\}$):

Reduction rules and translations for connectives are the same for *CPL*, plus:

$(\Box\alpha)^* = x_{\Box\alpha}$, where $x_{\Box\alpha}$ is a hidden variable, plus constraints:

(cK)	$x_{\Box(\alpha\to\beta)}(x_{\Box\alpha}(x_{\Box\beta}+1)) \approx 0$	$\Box(\alpha \to \beta) \to \Box\alpha \to \Box\beta$
(cT)	$x_{\Box\alpha}(\alpha^* + 1) \approx 0$	$\Box\alpha \to \alpha$
(cB)	$\alpha^*(x_{\Box\Diamond\alpha} + 1) \approx 0$	$\alpha \to \Box\Diamond\alpha$
(c4)	$x_{\Box\alpha}(x_{\Box\Box\alpha} + 1) \approx 0$	$\Box\alpha \to \Box\Box\alpha$
(cNec)	$\alpha^* \approx 1$ implies $x_{\Box\alpha} \approx 1$	$\vdash \alpha$ implies $\vdash \Box\alpha$

A Polynomial Ring Calculus for *S*5

Lemma

$$x_{\Box\bot} \approx 0, \tag{a}$$

$$x_{\Box\alpha}x_{\Box\neg\alpha} \approx 0, \tag{b}$$

$$x_{\Box\neg\neg\alpha} \approx x_{\Box\alpha}, \tag{c}$$

$$x_{\Box\alpha} \approx 1 \textit{ or } x_{\Box\beta} \approx 1 \textit{ implies } x_{\Box(\alpha\vee\beta)} \approx 1, \tag{d}$$

$$x_{\Box(\alpha\wedge\beta)} \approx x_{\Box\alpha}x_{\Box\beta}, \tag{e}$$

$$x_{\Box\alpha} \approx x_{\Box\Box\alpha} \approx x_{\Diamond\Box\alpha}, \tag{f}$$

$$x_{\Diamond\alpha} \approx x_{\Diamond\Diamond\alpha} \approx x_{\Box\Diamond\alpha}. \tag{g}$$

A Polynomial Ring Calculus for *S5*

Theorem (Soundness)

If $\Gamma \vdash_{S5} \alpha$ *then* $\Gamma \approx_{S5} \alpha$.

Proof.

Deduction theorem plus the following fact: constraints (cK)-(c4) establish validity of axioms **K**, **T**, **B** and **4**. Constraint (cNec) establishes validity preservation under necessitation rule. □

Theorem (Strong completeness)

$\Gamma \approx_{S5} \alpha$ *then* $\Gamma \vdash_{S5} \alpha$

Proof.

Adapting the familiar Lindenbaum-Asser argument for *CPL*. □

A Polynomial Ring Calculus for *S5*

Example

$\approx_{S5} (\Diamond p \to p) \vee (\Diamond p \to \Box\Diamond p)$:

$$
\begin{aligned}
&((\Diamond p \top) \vee (\Diamond p \to \Box\Diamond p))^* \\
&= (\Diamond p \to p)^*(\Diamond p \to \Box\Diamond p)^* + (\Diamond p \to p)^* + (\Diamond p \to \Box\Diamond p)^* \\
&\approx (\Diamond p \to \Box\Diamond p)^*((\Diamond p \to p)^* + 1) + (\Diamond p \to p)^* \\
&\approx ((\Diamond p)^*((\Box\Diamond p)^* + 1) + 1)((\Diamond p)^*(p^* + 1)) + (\Diamond p)^*(p^* + 1) + 1 \\
&\approx ((x_{\Box\neg p} + 1)(x_{\Box\Diamond p} + 1) + 1)((x_{\Box\neg p} + 1)(x_p + 1)) + (x_{\Box\neg p} + 1)(x_p + 1) + 1 \\
&\approx ((x_{\Box\neg p} + 1)(x_{\Diamond p} + 1) + 1)((x_{\Box\neg p} + 1)(x_p + 1)) + (x_{\Box\neg p} + 1)(x_p + 1) + 1 \\
&\approx ((x_{\Box\neg p} + 1)(x_{\Box\neg p}) + 1)((x_{\Box\neg p} + 1)(x_p + 1)) + (x_{\Box\neg p} + 1)(x_p + 1) + 1 \\
&\approx (x_{\Box\neg p} + 1)(x_p + 1) + (x_{\Box\neg p} + 1)(x_p + 1) + 1 \\
&\approx 1.
\end{aligned}
$$

A Polynomial Ring Calculus for *S5*

Example

$\approx_{S5} \Box(\Box(p \to \Box p) \to p) \to \Box(\Diamond\Box p \to p)$:

$$\begin{aligned}
&(\Box(\Box(p \to \Box p) \to p) \to \Box(\Diamond\Box p \to p))^* \\
&= (\Box(\Box(p \to \Box p) \to p))^*((\Box(\Diamond\Box p \to p))^* + 1) + 1 \\
&= x_{\Box(\Box(p\to\Box p)\to p)}(x_{\Box(\Diamond\Box p\to p)} + 1) + 1.
\end{aligned}$$

But we also have that:

$$\begin{aligned}
(\Diamond\Box p \to p)^* &= (\Diamond\Box p)^*(p^* + 1) + 1 \\
&= (x_{\Box\neg\Box p} + 1)(p^* + 1) + 1 \\
&\approx (x_{\Box\Diamond\neg p} + 1)(\neg p)^* + 1 \\
&\approx 1 \text{ (by polynomial constraint (cB)).}
\end{aligned}$$

Then, by polynomial constraint (cNec) we obtain $x_{\Box(\Diamond\Box p\to p)} \approx 1$. Consequently, $(\Box(\Box(p \to \Box p) \to p) \to \Box(\Diamond\Box p \to p))^* \approx 1$.

The relationship with modal algebras

Theorem

The structure $\mathcal{Z} = \langle \mathbb{Z}_2[X \cup X']/\cong, \sqcup', \sqcap', -', \mathbf{n}' \rangle$, with the order $\lesssim'$, is a normal-epistemic-symmetric-transitive modal algebra.

Proof.

The definitions below define a modal algebra.

- $[P] \sqcup' [Q] = [PQ + P + Q]$,
- $[P] \sqcap' [Q] = [PQ]$,
- $-' [P] = [P + 1]$,
- $\mathbf{n}'([P]) = [x_{\Box f(P)}]$.

The order relation $\lesssim'$ is defined by $[P] \lesssim' [Q]$ if $P \lesssim Q$. □

Non-truth functionality in polynomial form

Definition (Semi-negations)

- $\neg_1(p) = \begin{cases} 1 & \text{if } p = 0 \\ \text{undetermined in } \{0, 1\} & \text{if } p = 1 \end{cases}$
- $\neg_2(p) = \begin{cases} 0 & \text{if } p = 1 \\ \text{undetermined in } \{0, 1\} & \text{if } p = 0 \end{cases}$

Lemma (Semi-negations in polynomial form)

1. $\neg_1 p = xp + 1$
2. $\neg_2 p = x(p + 1)$

Proof.

1. $\neg_1(0) = 1$, while $\neg_1(1) = x + 1$
2. $\neg_2(1) = 0$, while $\neg_2(0) = x$

Logics of Formal Inconsistency (*LFIs*)

Definition

LFIs are paraconsistent logics that define connectives of consistency $\circ$ (and also inconsistency $\bullet$) at the object language.

- Most of the *LFIs* cannot be characterized by finite matrices.
- Some *LFIs* can be characterized by non-truth-functional 2-valued valuation semantic.

Example (Valuations for *mbC*, a simple *LFI*)

(1) $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$;
(2) $v(\varphi \vee \psi) = 1$ iff $v(\varphi) = 1$ or $v(\psi) = 1$;
(3) $v(\varphi \to \psi) = 1$ iff $v(\varphi) = 0$ or $v(\psi) = 1$;
(4) $v(\neg\varphi) = 0$ implies $v(\varphi) = 1$;
(5) $v(\circ\varphi) = 1$ implies $v(\varphi) = 0$ or $v(\neg\varphi) = 0$.

Polynomial ring calculus with hidden variables

Example (Application *mbC*)

- Translation function $*$: $For \to \mathbb{Z}_2[X]$.

$p_i^* = x_i$ (if p_i is a variable);
$(\varphi \wedge \psi)^* = \varphi^*\psi^*$;
$(\varphi \vee \psi)^* = \varphi^*\psi^* + \varphi^* + \psi^*$;
$(\varphi \to \psi)^* = \varphi^*\psi^* + \varphi^* + 1$;
$(\neg\varphi)^* = \varphi^* x_\varphi + 1$ (x_φ is a **hidden variables**);
$(\circ\varphi)^* = (\varphi^*(x_\varphi + 1) + 1)x_{\varphi'}$ (x_φ, $x_{\varphi'}$ are **hidden variables**);

- Reduction rules: $2x = 0$ and $x^2 = x$.
- $\vdash_{mbC} \varphi$ iff φ^* reduces by *PRC* rules to the constant polynomial 1.

An example in *mbC*

Example

The PRC shows easily that

1. $\alpha \wedge \neg\alpha$ is not a bottom particle in *mbC*,
2. $\alpha \wedge \neg\alpha \wedge \circ\alpha$ is a bottom particle in *mbC*

Proof.

Indeed, translating the wffs we have:

1. $\alpha^*(\alpha^*(x_{\alpha^*} + 1)) \approx \alpha^* x_{\alpha^*} \approx \alpha^*(x_{\alpha^*} + 1) \not\approx 0$
2. $\alpha^*(x_{\alpha^*} + 1)(\alpha^*(x_{\alpha^*} + 1) + 1)(x'_{\alpha^*}) \approx 0(x'_{\alpha^*}) \approx 0$

Notice that x_{α^*} and x'_{α^*} are independent **hidden variables**; □

The case of da Costa's C_1: a particular *LFI*

Example (Bivaluations for C_1)

(1) $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$;
(2) $v(\varphi \vee \psi) = 1$ iff $v(\varphi) = 1$ or $v(\psi) = 1$;
(3) $v(\varphi \rightarrow \psi) = 1$ iff $v(\varphi) = 0$ or $v(\psi) = 1$;
(4) $v(\neg\varphi) = 0$ implies $v(\varphi) = 1$;
(5) $v(\neg\neg\varphi) = 1$ implies $v(\varphi) = 1$;
(6) $v(\circ\varphi) = v(\psi \rightarrow \varphi) = v(\psi \rightarrow \neg\varphi) = 1$ implies $v(\psi) = 0$;
(7) $v(\circ(\varphi \# \psi)) = 0$ implies $v(\circ\varphi) = 0$ or $v(\circ\psi) = 0$.

Polynomial ring calculus for C_1

Example (Translation function $*$: $For \to \mathbb{Z}_2[X]$)

(1) $p_i^* = x_i$ if p_i is a propositional variable;
(2) $(\varphi \wedge \psi)^* = \varphi^*\psi^*$;
(3) $(\varphi \vee \psi)^* = \varphi^*\psi^* + \varphi^* + \psi^*$;
(4) $(\varphi \to \psi)^* = \varphi^*\psi^* + \varphi^* + 1$;
(5) $(\neg\varphi)^* = \varphi^* x_\varphi + 1$;
(6) $(\circ\varphi)^* = (\varphi^* x_\varphi x'_\varphi + \varphi^* x'_\varphi + 1) + 1)x'_\varphi$;
(7) $\circ(\varphi\#\psi)$ is a bit too complicated....

(5) $x_\varphi = 0$ implies $x_{\neg\varphi} = 1$;
(6) $x_\varphi = 0$ implies $x'_\varphi = 1$;
$x_{\circ\varphi} = 1$ and $x_{\circ\psi} = 1$ imply $x_{\circ(\varphi\#\psi)} = 1$

- Reduction rules: $2x = 0$ and $x^2 = x$.
- $\vdash_{C1} \varphi$ iff φ^* reduces to 1.

Half-logics

Lemma (Béziau)

$\neg_2$ recovers classical negation through $\sim P = P \to \neg_2 P$.

Proof.

In polynomial format: $P \to \neg_2 P$ is computed as $p(x(p+1)) + (p+1) = p+1$, but $p+1$ represents $\sim$. □

- So we recover classical logic, in the language of implication $\to$ and negation $\sim$, characterized by two-valued valuations v s.t.: (1) $v(P \to Q) = 1$ iff $v(P) = 0$ or $v(Q) = 1$; (2) $v(\sim P) = 0$ iff $v(P) = 1$

The "translation paradox"

A phenomenon?

- A subclassical logic as $K/2$ (in $\{\rightarrow, \neg_1\}$) turns out to be superclassical in $\{\rightarrow, \sim, \neg_1\}$
- Moreover, *PC* can be strongly translated within $K/2$:

Definition

1. $(P)^* = P$, for P atomic;
2. $(A \rightarrow B)^* = (A)^* \rightarrow (B)^*$;
3. $(\sim A)^* = A \rightarrow \neg_1 A$

More half-logics!

Example ($\neg_1$ is the negation of da Costa's C_1)

1. $v(\neg_1 p) = \begin{cases} 1 & \text{if } p = 0 \\ \textit{undetermined} & \text{if } p = 1 \end{cases}$
2. $v(p \stackrel{*}{\leftarrow} q) = 1$ iff $v(p) = 1$ and $v(q) = 0$;

The connectives $\neg_1$ and $\stackrel{*}{\leftarrow}$ in polynomial terms:

1. $\neg_1 P = px + 1$
2. $P \stackrel{*}{\leftarrow} Q = p(q + 1)$

$\neg_1(P) \stackrel{*}{\leftarrow} P$ defines classical negation $\sim$. Indeed,
$(px + 1)(p + 1) = \underbrace{p^2x}_{px} + px + p + \underbrace{1^2}_{1} = p + 1$.

And a "three-quarter" logic

Definition (A logic $K3/4$ in the signature $\{\rightarrow, \rightharpoonup\}$)

Consider a binary connective in p and q: $x(p+1)q$, corresponding to a non-truth-functional connective $\rightharpoonup$ whose valuation is:

$$v(P \rightharpoonup Q) = \begin{cases} 0 & \text{if } v(P) = 1 \text{ or } v(Q) = 0 \\ \mathit{undetermined} & \text{otherwise} \end{cases}$$

$\rightharpoonup$	0	1
0	0	x
1	0	0

A "three-quarter" logic, continued

Lemma

Classical negation $\sim p$ can be defined by $p \rightarrow (p \rightharpoonup q)$

Proof.

In fact, this formula in polynomial format turns out to be:
$p(x(p+1)q) + p + 1 = p + 1$, □

Hence full *PC* is recovered in the signature $\{\rightarrow, \rightharpoonup, \sim\}$.

More "three-quarter" logics

Definition (A logic $K3/4$ in the signature $\{\rightarrow, \rightharpoondown\}$)

Consider a binary connective in p and q: $xp(q+1)$, corresponding to a non-truth-functional connective $\rightharpoondown$ whose valuation is:

$$v(P \rightharpoondown Q) = \begin{cases} 0 & \text{if } v(P) = 0 \text{ or } v(Q) = 1 \\ \mathit{undetermined} & \text{otherwise} \end{cases}$$

$\rightharpoondown$	0	1
0	0	0
1	x	0

More “three-quarter” logics, continued

Lemma

Classical negation $\sim q$ *can be defined by* $q \rightarrow (p \rightharpoondown q)$

Proof.

In fact, this formula in polynomial format turns out to be:
$qxp(q+1)+(q+1)=q+1$, □

Hence, again, full *PC* is recovered in the signature $\{\rightarrow, \rightharpoondown, \sim\}$.

Polynomials as a "heuristic machine"

There are more "paradoxical" connectives...

- ...than we ever expected:
- At least 32 binary connectives which may define such "quarter" logics
- And many more in other arities!

Polynomizing: perspectives

- Recover the tradition from Leibniz, Boole, Schröder, etc, incorporating Taylor and features of 17th century thinking and certain ancient (Indian and Chinese) tradition.

- Most fundamental notions of contemporary classical *propositional* logic go back to the Stoics, not to Aristotle

  > *"Boole rehabilitated Stoic logic, rather than Stoicism supported Boole"*

  Cf. B. Mates, *Stoic Logic* of 1953

Polynomizing: problems

Which algebra fits logic?

- Can we obtain a new algebraic approach to logic, for multiple-valued and non-finite valued logics?
- Could Differential Calculus and Finite Differences be used to treat full **FOL** and **HOL** in polynomial form?

Papers available:

- Polynomizing: Logic Inference in Polynomial Format and the Legacy of Boole
  In Model-Based Reasoning in Science, Technology, and Medicine. Studies in Comp. Intell. v.64 (Eds. L. Magnani, Lorenzo; P. Li) Springer, 2007
  Pre-print at *CLE e-Prints* vol. 6(3), 2006. `http://www.cle.unicamp.br/e-prints/vol_6,n_3,2006.html`
- Polynomial ring calculus for many-valued logics. Proc. of the 35th Intl. Symp. on Mult.-Valued Logic. IEEE Comp. Soc. Calgary, Canadá, pp. 20-25, 2005.
  Pre-print at *CLE e-Prints* vol. 5(3), 2005 as “Polynomial Ring Calculus for Logical Inference” `http://www.cle.unicamp.br/e-prints/vol_5,n_3,2005.html`