Light propagation in and outside a sphere illuminated by plane waves of light.

by

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As long as we consider light to be rays that mutually interfere, refract, and reflect in the surfaces of bodies according to certain laws, our understanding of light propagation is still only elementary and piecewise since we decompose the general fundamental law of the entire light propagation into individual laws and separate phenomena although they essentially belong together. This elementary approach has and will always have its own great importance, but as long as we were unable to go beyond it many problems in optics would be left unsolved and indissoluble.

The general fundamental law of light propagation is like the laws for transmission of electricity and elastic forces of simple form, since it is expressible by three concurrent partial linear differential equations of the second order in which the three oscillatory components are the dependent while the coordinates of space and time are the independent variables. All the problems in formal optics must be subject to integration of these equations.

In a treatise "Ueber die Reflexion an einer Kugelfläche", A. Clebsch[1] tried to determine the reflection of light from *perfectly* reflective spherical surfaces by taking the differential equations of the theory of elasticity as a starting point, but this skilled mathematician did not succeed in surmounting the actual main difficulty. The author expresses this in the introduction using the words: "Die Resultate der ganzen Untersuchung sind sehr verwickelt, und namentlich für den in der Optik wichtigen Fall einer sehr kleinen Wellenlänge scheint es sehr schwer dieselben einfach in passender Form darzustellen".[2] Whereas the following is added: "Der entgegengesetzte Fall eines gegen die Wellenlänge sehr kleinen Radius der reflectirenden Kugel ist dagegen für eine Annäherung sehr geeignet".[3]

The differential equations from which the present investigation takes its starting point have been presented and substantiated in several of my previous works. They differ from the theory of elasticity by the fact that they rule out the possibility of longitudinal oscillations, and, since they are valid

[1] Crelles Journal, Vol. 61, p. 195. 1863. ["On the reflection from a spherical surface", translated title.]

[2] "The result of the entire investigation is very complicated, and especially the case most important for optics, where a very short wavelength is radiated, is very difficult to put into a suitable form" (translator's note).

[3] "The opposite case, where, as opposed to the wavelength, the reflecting sphere has a very small radius, is on the other hand very suitable for approximation" (translator's note).

for every point in any transparent heterogeneous medium, the boundary conditions at the transition from one body to another can be derived from the differential equations themselves.

In a previous work “Farvespredningens Theori”,[1] I have from the same differential equations derived formulae that serve to compute the light propagation in a medium consisting of concentric spherical layers. The computation was here applied to a system of *small* spheres mutually separated by *large* distances of “empty” space, with the goal in mind to determine the dependency of light refraction on the density of the system. Later I have employed the same series expansions to solve the problem I here have in mind, namely the computation of light propagation which appears when a *homogeneous, transparent, and isotropic sphere is illuminated by plane, parallel waves of light*, and I have in this way also succeeded in arriving at the same results which should be reported here. But, in the following, I have preferred a different and simpler way of presentation where I, also to ease the reading, shall avoid assuming knowledge of my previous work.

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1 Boundary Conditions

Let ξ, η, ζ denote the components of the light oscillations, corresponding to the time and space coordinates t, x, y, z. Moreover, introducing the notation[2]

$$\Delta_2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\,, \quad \theta = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\,,$$

the laws of light propagation in any transparent medium can be expressed by the three differential equations

$$\Delta_2\xi - \frac{d\theta}{dx} = \frac{1}{\omega^2}\frac{d^2\xi}{dt^2}\,, \quad \Delta_2\eta - \frac{d\theta}{dy} = \frac{1}{\omega^2}\frac{d^2\eta}{dt^2}\,, \quad \Delta_2\zeta - \frac{d\theta}{dz} = \frac{1}{\omega^2}\frac{d^2\zeta}{dt^2}\,, \tag{1}$$

since ω is in general a variable dependent of x, y, z which corresponds to the velocity of light in the point x, y, z, in so far as you can consider this constant within a very small volume.

The present task is to integrate these equations under the assumption that ω has a constant value inside the surface of a given sphere and a different constant value outside said surface with a discontinuous transition in the spherical surface itself. This discontinuous transition is considered to be produced by a surface layer of finite

[1] Vidensk. Selsk. Skr. 6. Række, p. 167. 1883. [“Theory of colour dispersion”, translated title.]

[2] In modern terminology, Δ_2 is written ∇^2 and is referred to as the Laplacian. (Translator’s note.)

thickness and continuous change of ω, considered to be a function of the distance r from the centre of the sphere, that goes to a layer of thickness zero. At this transition, the oscillatory components must here as everywhere stay finite, whereas the differential coefficients with respect to r might become infinite. The components and their differential coefficients therefore, in general in the boundary surface when the thickness of the boundary layer is reduced to 0, go discontinuously from one value to another, while some combinations of these might really keep their value unchanged.

Since I shall seek these out, I prefer instead of the components with respect to a fixed axial system to employ the projection of the oscillatory deflection on the radius, the projection orthogonal to this and positioned in the plane through radius and the x-axis, and the projection orthogonal to the two preceding and thus orthogonal to the x-axis.

Defining in polar[1] coordinates

$$x = r\,\cos\varphi\,,\quad y = r\,\sin\varphi\,\cos\phi\,,\quad z = r\,\sin\varphi\,\sin\phi\,,$$

and letting $\overline{\xi}$, $\overline{\eta}$, $\overline{\zeta}$ denote the new components, these are given by

$$\left.\begin{array}{rcr} \overline{\xi} &=& \cos\varphi\,\xi + \sin\varphi\,\cos\phi\,\eta + \sin\varphi\,\sin\phi\,\zeta\,, \\ \overline{\eta} &=& -\sin\varphi\,\xi + \cos\varphi\,\cos\phi\,\eta + \cos\varphi\,\sin\phi\,\zeta\,, \\ \overline{\zeta} &=& -\sin\phi\,\eta + \cos\phi\,\zeta\,. \end{array}\right\} \tag{2}$$

When the equations (1) are multiplied by x, y, and z, respectively, and added together, one obtains

$$\Delta_2 r\overline{\xi} - \frac{dr^2\theta}{rdr} = \frac{1}{\omega^2}\frac{d^2r\overline{\xi}}{dt^2}\,.$$

From this we seen, when Δ_2 is expressed in polar coordinates, that

$$\frac{d^2r^2\,\overline{\xi}}{dr^2} - \frac{dr^2\theta}{dr}$$

is expressible by quantities that remain finite, even when the thickness of the boundary layer is reduced to zero.

Hence, it follows that

$$\frac{dr^2\,\overline{\xi}}{dr} - r^2\theta$$

is a continuous function which thus also remains finite in the boundary layer as it is finite on both sides outside this layer. Consequently,

$$\frac{d\,\overline{\xi}}{dr} - \theta$$

is also everywhere a finite quantity.

Furthermore, multiplying the equations (1) by $-\sin\varphi$, $\cos\varphi\cos\phi$, $\cos\varphi\sin\phi$, respectively, and adding them together, one obtains

[1]In modern terminology, one would write *spherical polar* or simply *spherical* coordinates as the coordinate system is three-dimensional. (Translator's note.)

$$\frac{d^2 r\,\overline{\eta}}{dr^2} - \frac{d\theta}{d\varphi}$$

expressed by quantities that remain finite everywhere. Similarly, we find by multiplication of the equations (1) by 0, $-\sin\phi$, $\cos\phi$, and addition,

$$\frac{d^2 r\,\overline{\zeta}}{dr^2} - \frac{d\theta}{\sin\varphi\, d\phi}$$

expressed by finite quantities everywhere.

In this way, we have found three combinations that are finite everywhere. Eliminating θ from these, it is seen that the quantities

$$\frac{d^2 r\,\overline{\eta}}{dr^2} - \frac{d^2\,\overline{\xi}}{d\varphi dr} \quad \text{and} \quad \frac{d^2 r\,\overline{\zeta}}{dr^2} - \frac{d^2\,\overline{\xi}}{\sin\varphi\, d\phi dr}$$

are finite everywhere, from which it follows that

$$\frac{dr\,\overline{\eta}}{dr^2} - \frac{d\,\overline{\xi}}{d\varphi} \quad \text{and} \quad \frac{dr\,\overline{\zeta}}{dr} - \frac{d\,\overline{\xi}}{\sin\varphi\, d\phi}$$

are continuous functions that thus remain unchanged at the transition from one side of the boundary surface of the sphere to the other. I will express this by

$$\left[\frac{dr\,\overline{\eta}}{dr^2} - \frac{d\,\overline{\xi}}{d\varphi}\right] = 0\,, \quad \left[\frac{dr\,\overline{\zeta}}{dr} - \frac{d\,\overline{\xi}}{\sin\varphi\, d\phi}\right] = 0\,. \tag{3}$$

In addition, note that the same quantities, as continuous functions and finite everywhere outside the boundary surface, must also be finite in the boundary surface. Hence, it follows that $r\,\overline{\eta}$ and $r\,\overline{\zeta}$ must be continuous such that one, using the same notation as above, would have

$$\left[\,\overline{\eta}\,\right] = 0\,, \quad \left[\,\overline{\zeta}\,\right] = 0\,. \tag{4}$$

The boundary conditions corresponding to $r = 0$ and $r = \infty$ are given by the fact that light propagation is finite everywhere, thus also for $r = 0$, and that at an infinite distance from the sphere we only find, besides the given incident light, light that is propagated from the sphere, but none that propagates *toward* it.

2 Expansion in Terms of Spherical Functions

The light incident on the sphere is taken to consist of plane, parallel waves of light. In general, these waves could contain a collection of oscillations, different with respect to amplitude, direction inside the wave plane, period of oscillation, and phase. But this general case is easily derived from the simple one in which the oscillatory components that we shall denote by ξ_0, η_0, ζ_0, are given outside the sphere by

$$\xi_0 = 0\,, \quad \eta_0 = e^{(kt-lx)\,i}\,, \quad \zeta_0 = 0\,. \tag{5}$$

Here the exponential form is chosen as the simplest, the oscillations with amplitude 1 occur along the y-axis and propagate along the x-axis with the constant velocity $\frac{k}{l} = \Omega$, with wavelength $\frac{2\pi}{l} = \lambda$, and period of oscillation $\frac{2\pi}{k} = T$.

Since we, *outside* the sphere, in this way separate the incident light from the other light, generated by the change of velocity in the spherical surface, we here set

$$\xi = \xi_0 + \xi_e\,, \quad \eta = \eta_0 + \eta_e\,, \quad \zeta = \zeta_0 + \zeta_e\,, \tag{6}$$

while we, *inside* the spherical surface, set

$$\xi = \xi'\,, \quad \eta = \eta'\,, \quad \zeta = \zeta'\,, \tag{7}$$

where also l', Ω', λ' replace the corresponding unmarked quantities outside the sphere. Furthermore, letting N (the refractive index of the sphere) denote the ratio between the two velocities Ω and Ω', one has

$$\Omega = N\Omega'\,, \quad l' = Nl\,, \quad \lambda = N\lambda'\,. \tag{8}$$

The components ξ, η, ζ are mutually connected outside as well as inside the spherical surface by the equation $\theta = 0$, which for constant ω appears from the equations (1), and thus they could be represented as depending on two quantities alone: Q and S outside the sphere, or Q' and S' inside the sphere. That is, one would be able to set

$$\left.\begin{aligned} \xi_e &= \frac{dC}{dy} - \frac{dB}{dz}\,, \quad \eta_e = \frac{dA}{dz} - \frac{dC}{dx}\,, \quad \zeta_e = \frac{dB}{dx} - \frac{dA}{dy}\,, \\ A &= z\frac{dQ}{dy} - y\frac{dQ}{dz} + xS\,, \quad B = x\frac{dQ}{dz} - z\frac{dQ}{dx} + yS\,, \quad C = y\frac{dQ}{dx} - x\frac{dQ}{dy} + zS\,, \end{aligned}\right\} \tag{9}$$

just as also ξ', η', ζ' could be expressed in the same way. The equations (1) would then be satisfied under the assumption that one has

$$\Delta_2 Q + l^2 Q = 0\,, \qquad \Delta_2 S + l^2 S = 0\,, \tag{10}$$

$$\Delta_2 Q' + l'^2 Q' = 0\,, \qquad \Delta_2 S' + l'^2 S' = 0\,. \tag{11}$$

It can be noticed here that the two radial projections

$$x\xi_e + y\eta_e + z\zeta_e$$

and
$$x\left(\frac{d\zeta_e}{dy} - \frac{d\eta_e}{dz}\right) + y\left(\frac{d\xi_e}{dz} - \frac{d\zeta_e}{dx}\right) + z\left(\frac{d\eta_e}{dx} - \frac{d\xi_e}{dy}\right)$$

by means of equations (9) could be transformed into

$$-r^2\Delta_2 Q + r\frac{d^2 rQ}{dr^2} = -\frac{1}{\sin\varphi}\frac{d}{d\varphi}\sin\varphi\frac{dQ}{d\varphi} - \frac{d^2 Q}{\sin^2\varphi\, d\phi^2}$$

and
$$-r^2\Delta_2 S + r\frac{d^2 rS}{dr^2} = -\frac{1}{\sin\varphi}\frac{1}{d\varphi}\sin\varphi\frac{dS}{d\varphi} - \frac{d^2 S}{\sin^2\varphi\, d\phi^2}\,.$$

Hence, it is seen that when Q and S are expanded in series in terms of spherical functions Q_n and S_n, namely

$$Q = \sum Q_n\,, \quad S = \sum S_n\,,$$

then the above-mentioned radial projections would be given, respectively, by

$$\sum n(n+1)Q_n \quad \text{and} \quad \sum n(n+1)S_n\,.$$

The same holds true in the space inside the sphere.

In analogy with (6) for points outside the sphere, we express the components $\overline{\xi}$, $\overline{\eta}$, $\overline{\zeta}$ introduced in the preceding section by

$$\overline{\xi} = \overline{\xi}_0 + \overline{\xi}_e\,, \quad \overline{\eta} = \overline{\eta}_0 + \overline{\eta}_e\,, \quad \overline{\zeta} = \overline{\zeta}_0 + \overline{\zeta}_e\,, \tag{12}$$

as these new components are given by

$$\overline{\xi}_0 = \sin\varphi\cos\phi\, e^{(kt-lx)\,i}\,, \quad \overline{\eta}_0 = \cos\varphi\cos\phi\, e^{(kt-lx)\,i}\,, \quad \overline{\zeta}_0 = -\sin\phi\, e^{(kt-lx)\,i}\,, \tag{13}$$

$$\left.\begin{array}{rcl} \overline{\xi}_e &=& \cos\varphi\,\xi_e + \sin\varphi\cos\phi\,\eta_e + \sin\varphi\sin\phi\,\zeta_e\,, \\ \overline{\eta}_e &=& -\sin\varphi\,\xi_e + \cos\varphi\cos\phi\,\eta_e + \cos\varphi\sin\phi\zeta_e\,, \\ \overline{\zeta}_e &=& -\sin\phi\,\eta_e + \cos\phi\,\zeta_e\,. \end{array}\right\} \tag{14}$$

Now, introducing for the sake of brevity the following notation

$$lr = a\,, \quad l'r = a'\,, \quad lQ = K\,, \quad l'Q' = K' \tag{15}$$

and, as R is the radius of the given sphere,

$$lR = \alpha\,, \quad l'R = \alpha'\,. \tag{16}$$

Then, by the equations (9) and by using the equations (10), one obtains

$$\left.\begin{array}{rcl} \overline{\xi}_e &=& \dfrac{d^2aK}{da^2} + aK\,, \\ \overline{\eta}_e &=& \dfrac{d^2aK}{a\,d\varphi\,da} + \dfrac{dS}{\sin\varphi\,d\phi}\,, \\ \overline{\zeta}_e &=& \dfrac{d^2aK}{a\sin\varphi\,d\phi\,da} - \dfrac{dS}{d\varphi}\,, \end{array}\right\} \tag{17}$$

just as, for an interior point, one correspondingly has

$$\left.\begin{array}{rcl} \overline{\xi'} &=& \dfrac{d^2a'K'}{da'^2} + a'K'\,, \\ \overline{\eta'} &=& \dfrac{d^2a'K'}{a'\,d\varphi\,da'} + \dfrac{dS'}{\sin\varphi\,d\phi}\,, \\ \overline{\zeta'} &=& \dfrac{d^2a'K'}{a'\sin\varphi\,d\phi\,da'} - \dfrac{dS'}{d\varphi}\,. \end{array}\right\} \tag{18}$$

The task is now to develop these components in series in terms of spherical functions. If it is at all possible to expand a function $f(x)$ in terms of spherical functions, the expansion is, as is well-known, the following:

$$f(x) = \sum_0^\infty \frac{2n+1}{2} P_n(x) \int_{-1}^1 f(u)P_n(u)\,du\,,$$

as the sum is taken for all values of n from $n=0$ to $n=\infty$, and

$$P_n(x) = \frac{1\,.\,3\ldots 2n-1}{1\,.\,2\ldots n}\left(x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4(2n-1)(2n-3)}x^{n-4} - \ldots\right).$$

If we now first seek to expand the expressions for $\overline{\xi}_0$, $\overline{\eta}_0$, $\overline{\zeta}_0$ given in the equations (13), wherein we set $lx = a\cos\varphi$, then according to the above-mentioned we have

$$e^{-a\cos\varphi\, i} = \sum_0^\infty \frac{2n+1}{2} P_n(\cos\varphi) \int_{-1}^1 e^{-aui} P_n(u)\, du \ .$$

The definite integral involved herein is expressible by the Bessel function $J_{n+\frac{1}{2}}(a)$, or, as I shall here prefer, by another function denoted $v_n(a)$ which differs from the Bessel function by only a factor since we set

$$v_n(a) = \sqrt{\frac{\pi a}{2}} J_{n+\frac{1}{2}}(a)\,.$$

One will then, as is well-known from the theory of Bessel functions, be able to define $v_n(a)$ by

$$v_n(a) = \frac{a^{n+1}}{2^{n+1}[n]} \int_{-1}^1 e^{-aui}(1-u^2)^n\, du \ .$$

This integral transforms by n-fold integration by parts to

$$v_n(a) = \frac{a}{2^{n+1}[n] i^n} \int_{-1}^1 e^{-aui} \frac{d^n(1-u^2)^n}{du^n}\, du\,,$$

which by using another familiar expression for P_n, namely

$$P_n(u) = \frac{(-1)^n}{2^n[n]} \frac{d^n(1-u^2)^n}{du^n}\,,$$

can also be given the form

$$v_n(a) = \frac{a}{2} i^n \int_{-1}^1 e^{-aui} P_n(u)\, du \ . \tag{19}$$

In this way, we obtain

$$e^{-a\cos\varphi\, i} = \frac{1}{a} \sum_0^\infty (2n+1) P_n(\cos\varphi) e^{-\frac{n\pi}{2} i}\, v_n(a)\,. \tag{20}$$

It is noted that the function $v_n(a)$ satisfies the differential equation

$$\frac{d^2 v_n(a)}{da^2} = \left(\frac{n(n+1)}{a^2} - 1\right) v_n(a)\,, \tag{21}$$

and that it, expanded in powers of a, gives the series

$$v_n(a) = \frac{a^{n+1}}{1\,.\,3\ldots 2n+1}\left(1 - \frac{a^2}{2(2n+3)} + \frac{a^4}{2\,.\,4(2n+3)(2n+5)} - \ldots\right)\,. \tag{22}$$

Another series expansion well-known from the theory of Bessel functions, where the number of terms is finite, is

$$\left.\begin{aligned} v_n(a) &= g_n(a)\sin\left(a - \frac{n\pi}{2}\right) + h_n(a)\cos\left(a - \frac{n\pi}{2}\right)\,, \\ g_n(a) &= 1 - \frac{(n-1)n(n+1)(n+2)}{2\,.\,4\,a^2} + \frac{(n-3)(n-2)\ldots(n+4)}{2\,.\,4\,.\,6\,.\,8\,a^4} - \ldots\,, \\ h_n(a) &= \frac{n(n+1)}{2\,a} - \frac{(n-2)(n-1)\ldots(n+3)}{2\,.\,4\,.\,6\,a^3} + \ldots\ . \end{aligned}\right\} \tag{23}$$

Furthermore, denoting by $w_n(a)$ another particular integral of Equation (21), and defining this integral more specifically by the series expansion

$$w_n(a) = \frac{1\,.\,3\ldots 2n-1}{a^n}\left(1+\frac{a^2}{2(2n-1)}+\frac{a^4}{2\,.\,4(2n-1)(2n-3)}+\ldots\right), \tag{24}$$

this function will likewise differ only by a factor from a Bessel function, namely $J_{-n-\frac{1}{2}}(a)$, and with the series for g_n and h_n given above it will also be expressible by

$$w_n(a) = g_n(a)\cos\left(a-\frac{n\pi}{2}\right) - h_n(a)\sin\left(a-\frac{n\pi}{2}\right). \tag{25}$$

From the expansion (20), the expressions given in the equations (13) can now be determined in the following way. One extracts from the series (20) the term corresponding to $n=0$ and sets

$$P_n(\cos\varphi) = -\frac{1}{n(n+1)}\cdot\frac{1}{\sin\varphi}\frac{d}{d\varphi}\sin\varphi\frac{dP_n(\cos\varphi)}{d\varphi},$$

whereby one obtains

$$e^{-a\cos\varphi\, i} = \frac{\sin a}{a} - \frac{1}{a}\sum_1^\infty \frac{2n+1}{n(n+1)}\frac{1}{\sin\varphi}\frac{d}{d\varphi}\sin\varphi\frac{P_n(\cos\varphi)}{d\varphi}e^{-\frac{n\pi}{2}i}\,v_n(a).$$

Introducing herein for brevity the notation

$$\left.\begin{aligned} K_0 &= -i\frac{\cos\phi}{a}\frac{d}{d\varphi}\sum_1^\infty\frac{2n+1}{n(n+1)}P_n(\cos\varphi)e^{\left(kt-\frac{n\pi}{2}\right)i}\,v_n(a),\\ S_0 &= -\frac{\sin\phi}{a}\frac{d}{d\varphi}\sum_1^\infty\frac{2n+1}{n(n+1)}P_n(\cos\varphi)e^{\left(kt-\frac{n\pi}{2}\right)i}\,v_n(a),\end{aligned}\right\} \tag{26}$$

one obtains by multiplication of the equation by $\cos\phi\, e^{kti}\sin\varphi\, d\varphi$ or by $-\sin\phi\, e^{kti}\sin\varphi\, d\varphi$, and by integration from $\varphi=0$ to $\varphi=\varphi$ of the two equations obtained in this way,

$$\left.\begin{aligned} K_0 &= \frac{\cos\phi}{a\sin\varphi}(i\sin a\cos\varphi - \cos a + e^{-a\cos\varphi\, i})\,e^{kti},\\ S_0 &= -\frac{\sin\phi}{a\sin\varphi}(-\sin a\cos\varphi - i\cos a + i\,e^{-a\cos\varphi\, i})\,e^{kti}.\end{aligned}\right\} \tag{27}$$

From this we find in conclusion

$$\left.\begin{aligned} \frac{d^2aK_0}{da^2}+aK_0 = \sin\varphi\cos\phi\, e^{(kt-a\cos\varphi)\,i} &= \overline{\xi}_0,\\ \frac{d^2aK_0}{a\,d\varphi\,da}+\frac{dS_0}{\sin\varphi\, d\phi} = \cos\varphi\cos\phi\, e^{(kt-a\cos\varphi)\,i} &= \overline{\eta}_0,\\ \frac{d^2aK_0}{a\sin\varphi\, d\phi\, da}-\frac{dS_0}{d\varphi} = -\sin\phi\, e^{(kt-a\cos\varphi)\,i} &= \overline{\zeta}_0.\end{aligned}\right\} \tag{28}$$

These expressions for the components $\overline{\xi}_0$, $\overline{\eta}_0$, $\overline{\zeta}_0$ correspond to the expressions presented in (17) for the components $\overline{\xi}_e$, $\overline{\eta}_e$, $\overline{\zeta}_e$, as K_0 and S_0 take the place of K and S in the equations (17). For K_0 and S_0, we have in (26) the expansions in terms of spherical functions, and these must, as one can also easily convince oneself, satisfy the same differential equations as K and S, namely according to (10) $\Delta_2 K_0 + l^2 K_0 = 0\,,\ \Delta_2 S_0 + l^2 S_0 = 0$. The expansions of K and S in terms of spherical functions must consequently be analogous with the expansions (26), as one would here instead of the particular integral $v_n(a)$ of the equation (21) insert the ordinary integral expressed linearly by $v_n(a)$ and $w_n(a)$. Thus, one obtains with the as yet undetermined constants k_n, x_n, s_n, σ_n,

$$\left.\begin{aligned} K &= -i\,\frac{\cos\phi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(k_n v_n(a) + x_n w_n(a)\right)\,, \\ S &= -\,\frac{\sin\phi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(s_n v_n(a) + \sigma_n w_n(a)\right)\,, \end{aligned}\right\} \tag{29}$$

and for a corresponding interior point

$$\left.\begin{aligned} K' &= -i\,\frac{\cos\phi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(k'_n v_n(a') + x'_n w_n(a')\right)\,, \\ S' &= -\,\frac{\sin\phi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(s'_n v_n(a') + \sigma'_n w_n(a')\right)\,. \end{aligned}\right\} \tag{30}$$

Herein $P_n(\cos\varphi)$ is shortened to P_n.

If we now first use the boundary condition corresponding to $a' = 0$, it is seen from (24) that $w_n(a')$ becomes ∞ for $a' = 0$ and $n > 0$, and that the finiteness condition therefore requires

$$x'_n = 0\,, \quad \sigma'_n = 0\,.$$

According to (23) and (25), $a = \infty$ corresponds to $v_n(a) = \sin\left(a - \frac{n\pi}{2}\right)\,,\quad w_n(a) = \cos\left(a - \frac{n\pi}{2}\right)$. At an infinite distance from the sphere one therefore has

$$2(k_n v_n(a) + x_n w_n(a))e^{\left(kt-\frac{n\pi}{2}\right)i} = (-k_n\,i + x_n)e^{(kt+a-n\pi)\,i} + (k_n\,i + x_n)e^{(kt-a)\,i}\,.$$

From this it is seen that the light propagation in general at this distance appears to be a periodic function of $kt + a$ and $kt - a$, corresponding to two opposite wave propagations, one propagating toward the centre of the sphere, the other in the direction away from the centre. Since now only the latter, according to the assumed conditions, is really permitted, one must have

$$-k_n\,i + x_n = 0\,, \quad \text{as also correspondingly} \quad -\,s_n\,i + \sigma_n = 0\,.$$

In this way, the series (29) and (30) are reduced to

$$\left.\begin{aligned} K &= -i\,\frac{\cos\phi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} k_n(v_n(a) + w_n(a)\,i)\,, \\ S &= -\,\frac{\sin\phi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} s_n(v_n(a) + w_n(a)\,i)\,, \\ K' &= -i\,\frac{\cos\phi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} k'_n v_n(a')\,, \\ S' &= -\,\frac{\sin\phi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} s'_n v_n(a')\,. \end{aligned}\right\}^{1} \tag{31}$$

[1]The original reads s_n instead of s'_n in the equation for S'. This must be a typo. (Translator's note.)

Finally, we also have the boundary conditions presented in (3) and (4) which can be expressed by

$$\left.\begin{aligned}\overline{\eta} = \overline{\eta'}\,, \quad \overline{\zeta} = \overline{\zeta'}&\\ \frac{da\overline{\eta}}{da} - \frac{d\overline{\xi}}{d\varphi} &= \frac{da'\overline{\eta'}}{da'} - \frac{d\overline{\xi'}}{d\varphi}\\ \frac{da\overline{\zeta}}{da} - \frac{d\overline{\xi}}{\sin\varphi\, d\phi} &= \frac{da'\overline{\zeta'}}{da'} - \frac{d\overline{\xi'}}{\sin\varphi\, d\phi}\end{aligned}\right\}\begin{aligned} a &= \alpha\\ a' &= \alpha'\,.\end{aligned}$$

Inserting herein the expressions for $\overline{\xi}$, $\overline{\eta}$, $\overline{\zeta}$ given by the equations (12), (17), and (28), and for $\overline{\xi'}$, $\overline{\eta'}$, $\overline{\zeta'}$ the expressions (18), these conditions can be transformed into

$$\left.\begin{array}{c} a(K_0+K) = a'K'\,, \quad S_0+S = S'\\ \dfrac{da(K_0+K)}{a\,da} = \dfrac{da'K'}{a'\,da'}\,, \quad \dfrac{da(S_0+S)}{da} = \dfrac{da'S}{da'}\end{array}\right\}\begin{aligned} a &= \alpha\\ a' &= \alpha'\,.\end{aligned} \tag{32}$$

Herein K_0, S_0, K, S, K', S' are expanded by the series given in (26) and (31), whereby 4 equations between the coefficients are obtained. Denoting for the sake of brevity the derived functions $\frac{dv_n(\alpha)}{da}$, $\frac{dw_n(\alpha)}{da}$, $\frac{dv_n(\alpha')}{da'}$ by $v_n'(\alpha)$, $w_n'(\alpha)$, $v_n'(\alpha')$, these equations become

$$\begin{aligned} N\,(v_n'(\alpha) + k_n(v_n'(\alpha) + w_n'(\alpha)\,i)) &= k_n' v_n'(\alpha')\\ N\,(v_n(\alpha) + s_n(v_n(\alpha) + w_n(\alpha)\,i)) &= s_n' v_n(\alpha')\\ v_n(\alpha) + k_n(v_n(\alpha) + w_n(\alpha)\,i) &= k_n' v_n(\alpha')\\ v_n'(\alpha) + s_n(v_n'(\alpha) + w_n'(\alpha)\,i) &= s_n' v_n'(\alpha')\,.\end{aligned}$$

From this we can determine the four coefficients. By introducing a small reduction by means of the equation

$$w_n(\alpha)v_n'(\alpha) - w_n'(\alpha)v_n(\alpha) = 1\,,$$

one will thus obtain

$$\left.\begin{aligned} 2k_n &= -1 - \frac{(v_n(\alpha) - w_n(\alpha)\,i)\,v_n'(\alpha') - N\,(v_n'(\alpha) - w_n'(\alpha)\,i)\,v_n(\alpha')}{(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - N\,(v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,,\\ 2s_n &= -1 - \frac{N\,(v_n(\alpha) - w_n(\alpha)\,i)\,v_n'(\alpha') - (v_n'(\alpha) - w_n'(\alpha)\,i)\,v_n(\alpha')}{N\,(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - (v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,,\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} k_n' &= \frac{Ni}{(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - N\,(v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,,\\ s_n' &= \frac{Ni}{N\,(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - (v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,.\end{aligned}\right\} \tag{34}$$

The posed task is hereby solved in so far as the oscillatory components have been determined everywhere in space by infinite series with known coefficients. It shall be seen that the series in the given form are well-suited for the calculation when either α, which corresponds to the circumference of the sphere measured in units of the wavelength λ, is a small number or when the considered point is close to the centre, whereas when α is a very large number, which we can almost say is the case for all spheres that are visible to the naked eye, it will in general be necessary to transform the series such that the summations can be done with sufficient approximation. I shall now first produce the summation formulae that would here be brought into play.

3 Summation Formulae

In the following section, sums will be produced that can be referred to the form

$$\sum_{n_1}^{n_2} A_n e^{F_n\, i}\,, \tag{35}$$

where n runs through the sequence from $n = n_1$ to $n = n_2$.

The two functions A_n and F_n are constituted in such a way that when inserting $n = \nu + z$, where both the new variables are also considered to be integers, one will obtain the following series, convergent within the given bounds,

$$A_n = A + B\frac{z}{\alpha} + C\frac{z^2}{\alpha^2} + \ldots\,, \quad F_n = F\alpha + Gz + H\frac{z^2}{\alpha} + I\frac{z^3}{\alpha^2} + \ldots\,. \tag{36}$$

The terms have here been ordered according to increasing powers of z and decreasing powers of the quantity α. The latter is considered to be a *very large*, however, not infinitely large, number, and all quantities will in the following be ordered according to powers of α such that the quantity which contains the larger power of α is considered to be a quantity of higher order. Here the coefficients $A, B, \ldots, F, G, \ldots$ *in the highest* are quantities of the same order as the unit (α^0). The calculation shall now aim at production of the results with such an accuracy that only quantities which are of lower order than the unit are considered small enough to be discarded.

The number of terms in the series (35) is itself a very large number, of the same order as α. The bounds n_1 and n_2 are *indeterminate* and *to a certain degree arbitrary*, that is, they are only constrained on the one hand by the convergence conditions for the series (36), on the other hand by the demand that $n_2 - n_1$ must be a very large number. This kind of indeterminate, arbitrary quantities which is here introduced, and for which I in the following shall use the common mark ω, are defined by the concept that a function of this quantity denotes the bound whereto the average of the same function of a particular quantity x converges when one lets x run through the gradually larger and larger sequence of values within the bounds delimited by ω.

Thus, taking our starting point in the two well-known integrals

$$\int_0^\infty e^{-x} x^{\mu-1}\, dx = \Gamma(\mu)\,, \ldots \quad (37) \qquad\qquad \int_0^\infty e^{xi} x^{\mu-1}\, dx = \Gamma(\mu) e^{\frac{\mu\pi}{2} i}\,, \ldots \quad (38)$$

the first one valid for all positive values of μ, the second one only for the positive values which are smaller than 1, then it is seen that one, also for $\mu < 1$, must have

$$\int_0^\omega e^{xi} x^{\mu-1}\, dx = \Gamma(\mu) e^{\frac{\mu\pi}{2} i}\,, \tag{39}$$

since

$$\int_0^\omega e^{xi} x^{\mu-1}\, dx = \int_0^\infty e^{xi} x^{\mu-1}\, dx - \int_\omega^\infty e^{xi} x^{\mu-1}\, dx\,,$$

where the latter integral can be expanded by integration by parts into a semi-convergent series whose average value, as corresponding to different values of ω, converges to 0 when the average value is taken between wider and wider bounds in the way stated above. Furthermore, if $\mu > 1$ in the integral (39), the exponent can be reduced by integration by parts to become smaller than 1, and the average value of the periodical terms appearing outside the integral will likewise converge to 0. Consequently, the equation (39) with the agreed meaning of the upper bound ω is valid for *all positive* values of μ.

As another example that will of use in the following, we take the sum (35) reduced to the simplest form

$$\sum_{n_1}^{n^2} e^{ani} = \frac{e^{an_1 i} - e^{a(n_2+1)i}}{1 - e^{ai}}$$

Here, the right-hand side must also disappear, assuming that a is not 0 or a multiple of 2π since in this case the sum becomes $n_2 - n_1 + 1$ which presumably is indeterminate, but in any case cannot be equal to zero. Furthermore, if a is very small or very close to a multiple of 2π, one dare not consider the sum to be zero since the number of terms is assumed very large, but not infinitely large.

If the sum is zero, it will continue to be so when differentiated an arbitrary number of times with respect to a. Hence, more generally, one has

$$\sum_{n_1}^{n_2} n^m e^{ani} = 0\,, \tag{40}$$

when m is an integer or 0, and when a is not equal to or lying very close to 0 or a multiple of 2π.

Now, considering the sum given by the expansions (35) and (36), it is seen that it can be changed into a convergent series with terms that, when omitting constant factors, have the form

$$\sum_{n_1-\nu}^{n_2-\nu} z^m e^{Gzi}$$

That is, if one does *not* have

$$G = 2p\pi\,, \tag{41}$$

for $p = 0$ or an integer, and neither $G - 2p\pi$ very close to 0, then the entire sum (35) will disappear.

Conversely, if one is able to find a value of ν that enables satisfaction of the above-mentioned condition (41), then Gz can be omitted from the exponent, and the summation can now be changed to integration without appreciable error. Hence, the sum (35) can be given the form

$$\int_{-(\nu-n_1)}^{n_2-\nu} dz\left(A+B\frac{z}{\alpha}+\ldots\right)e^{\left(F\alpha+H\frac{z^2}{\alpha}+I\frac{z^3}{\alpha^2}+\ldots\right)i}\,, \tag{42}$$

where we restrict ourselves to the assumption that ν is situated between n_1 and n_2 and, thus, that both $\nu-n_1$ and $n_2-\nu$ will have to belong to the kind of indeterminate quantities defined above. Changing in this integral the sign of z for $z<0$, and afterwards setting $Hz^2=\alpha x$, the bounds of x, assuming that H is not 0 or very small, belongs to the kind of quantities denoted above by the common mark ω, and the integral will by series expansion become

$$\int_0^\omega \frac{dx}{2}\left(A\sqrt{\frac{\alpha}{Hx}}+\frac{B}{H}\ldots+\frac{AIxi}{H^2}+\ldots\right)e^{(F\alpha+x)i}+\int_0^\omega \frac{dx}{2}\left(A\sqrt{\frac{\alpha}{Hx}}-\frac{B}{H}\ldots-\frac{AIxi}{H^2}+\ldots\right)e^{(F\alpha+x)i}\,.$$

These integrals will according to (39), as $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, together become

$$A\sqrt{\frac{\alpha\pi}{H}}e^{\left(F\alpha+\frac{\pi}{4}\right)i}\,, \tag{43}$$

since the terms which are of the order $\alpha^{-\frac{1}{2}}$ and of lower order are discarded. This result is also valid for negative values of H when it is taken care of that one in this case must set

$$\frac{1}{\sqrt{-1}}=-i=e^{-\frac{\pi}{2}i}\,.$$

The result becomes invalid for

$$H=0\,. \tag{44}$$

To further generalise it, we can in this case assume that $G-2p\pi$ is a very small quantity. Also in this case, the summation will change to integration and instead of (42) one will obtain the integral

$$\int_{-(\nu-n_1)}^{n_2-\nu} dz\left(A+B\frac{z}{\alpha}+C\frac{z^2}{\alpha^2}+\ldots\right)e^{\left(F\alpha+(G-2p\pi)z+I\frac{z^3}{\alpha^2}+K\frac{z^4}{\alpha^3}+L\frac{z^5}{\alpha^4}\ldots\right)i}\,. \tag{45}$$

Changing herein too the sign of z for $z<0$, and afterwards setting $\pm Iz^3=\alpha^2 x$, where the double sign is determined such that $\pm I$ becomes positive. Introducing for the sake of brevity the notation

$$G-2p\pi=-\varepsilon\sqrt[3]{\frac{I}{\alpha^2}}\,, \tag{46}$$

$$\int_0^\omega x^{-\frac{2}{3}}\cos(-\varepsilon\, x^{\frac{1}{3}}+x)\,dx=Q\,, \tag{47}$$

as well as

$$A = A_1 I\,, \quad B = B_1 I\,, \quad C = C_1 I\,, \quad K = K_1 I\,, \quad L = L_1 I\,, \tag{48}$$

one will without difficulty be able to give the integral (45) the form

$$\begin{aligned}\pm\tfrac{2}{3}e^{F\alpha i}\Bigg[(\alpha I)^{\frac{2}{3}}A_1 Q + (\alpha I)^{\frac{1}{3}}i\left(B_1\frac{dQ}{d\varepsilon} + A_1 K_1\frac{d^4Q}{d\varepsilon^4}\right) - C_1\frac{d^2Q}{d\varepsilon^2}\\ -(A_1L_1 + B_1K_1)\frac{d^5Q}{d\varepsilon^5} - \tfrac{1}{2}A_1K_1^2\frac{d^8Q}{d\varepsilon^8}\Bigg]\,,\end{aligned} \tag{49}$$

as the terms of the order $\alpha^{-\frac{1}{2}}$ or less are discarded.

In case of one having $\varepsilon = 0$, one obtains by means of (39)

$$\Gamma\left(\frac{1}{3}\right)\cos\frac{\pi}{6} = Q = -3\frac{d^3Q}{d\varepsilon^3}\,, \quad \Gamma\left(\frac{2}{3}\right)\cos\frac{\pi}{6} = \frac{dQ}{d\varepsilon} = -\frac{2}{3}\frac{d^4Q}{d\varepsilon^4}\,,$$
$$0 = \frac{d^2Q}{d\varepsilon^2} = \frac{d^5Q}{d\varepsilon^5} = \frac{d^8Q}{d\varepsilon^8}\,,$$

where

$$\Gamma\left(\frac{1}{3}\right) = 2.67894\ldots\,, \quad \Gamma\left(\frac{2}{3}\right) = 1.35412\ldots\,,$$

or by the common logarithms

$$\text{Log}\;\Gamma\left(\frac{1}{3}\right) = 0.4279627\ldots\,, \quad \text{Log}\;\Gamma\left(\frac{2}{3}\right) = 0.1316565\ldots\,.$$

Hereby (49) passes to

$$\pm\frac{1}{\sqrt{3}}e^{F\alpha i}\left[(\alpha I)^{\frac{2}{3}}A_1\,\Gamma\left(\frac{1}{3}\right) + (\alpha I)^{\frac{1}{3}}i\left(B_1 - \frac{2}{3}A_1K_1\right)\,\Gamma\left(\frac{2}{3}\right)\right]\,. \tag{50}$$

The integral Q (47) has under a somewhat different form been calculated numerically by *Airy*[1], who for the integral

$$\int_0^\infty d\omega\,\cos\frac{\pi}{2}(\omega^3 - m\omega) = W$$

has provided the following table

m	W	m	W
-5	0.00041	0	0.66527
-4	0.00298	1	1.00041
-3	0.01730	2	0.56490
-2	0.07908	3	-0.56322
-1	0.27283	4	-0.47446
		5	0.68182.

[1]On the intensity of Light in the neighbourhood of a Caustic. Trans. of the Cambr. Soc. t. VI, p. 379, t. VIII, p. 595.

By means hereof one can also calculate Q, as one has

$$\varepsilon = \left(\frac{\pi}{2}\right)^{\frac{2}{3}} m\,, \quad Q = 3\left(\frac{\pi}{2}\right)^{\frac{1}{3}} W\,.^{1}$$

Going from $m = 0$ to the negative side, W keeps decreasing until 0. Going to the positive side, W is first increasing, reaching a maximum at $m = 1.08$, and hereafter also approaches 0 through a periodic motion around the zero point. The first and largest maximum of W is 1.504 times larger than the value of W for $m = 0$.

Stokes[2] has extended the calculation of Airy to the first 50 roots of the equation $W = 0$ and the first 10 roots of $\frac{dW}{dm} = 0$. Thus, the sequence corresponding to $W = 0$ is

$$m = 2.4955; 4.3631; 5.8922; 7.2436; 8.4788; \ldots$$

in which the qth root for growing q converges to $3\left(q - \frac{1}{4}\right)^{\frac{2}{3}}$. Likewise, for $\frac{dW}{dm} = 0$,

$$m = 1.0845; 3.4669; 5.1446; 6.5782; 7.8685; \ldots$$

where the qth root converges to $3\left(q - \frac{3}{4}\right)^{\frac{2}{3}}$.

The different derivatives of Q with respect to ε, which enter into the expression (49), would all easily be expressible by Q and $\frac{dQ}{d\varepsilon}$, as it is noticed that one has

$$\frac{d^2Q}{d\varepsilon^2} = -\frac{\varepsilon}{3} Q\,,$$

whereof yet higher derivatives could be derived, for example

$$\frac{d^4Q}{d\varepsilon^4} = \frac{\varepsilon^2}{9} Q - \frac{2}{3}\frac{dQ}{d\varepsilon}\,, \text{ etc.}$$

Hence, the maximum and minimum points of $\frac{dQ}{d\varepsilon}$ correspond to $Q = 0$, whereof it is seen that the first maximum does not set in here until $m = 2.4955\ldots$. The modulus (or the amplitude) of the expression given in (49) changes with growing ε in a way corresponding to the integral W if one only has to take the first term, which is of highest order, into consideration. But if the subsequent terms in the expression are also of significance, the modulus will contain both Q and $\frac{dQ}{d\varepsilon}$, whereof it follows that the maximum points will be displaced, and that the modulus in general cannot become 0 as a consequence of the periodic changes. The periodicity will in this way become more blurred.

By comparison of the two expressions given in (43) and (49) for the integral (42), it is seen that the former is of the magnitude $\alpha^{\frac{1}{2}}$, the latter is of the order $\alpha^{\frac{2}{3}}$. How the transition happens from the one expression to the other can be seen if one imagines H decreasing to a very small quantity while one keeps $G - 2p\pi = 0$. One will then in the integral (42) be able

[1]In the original text, the constant before W is the reciprocal of the one given here. The correctness of the constant used here is confirmed by an errata page following the original article. (Translator's note.)

[2]Trans. of the Cambr. Phil. Soc. t. 9. p. 166.

to set $z = z' + \delta$ and determine δ such that the coefficient for z'^2 in the exponent becomes 0. Hereby we arrive at the form assumed in (45), where $G - 2p\pi$ becomes equal to $-\dfrac{H^2}{3I}$, and consequently

$$3\varepsilon = H^2 \sqrt[3]{\frac{\alpha^2}{I^4}}\,.$$

It is seen hereof that ε necessarily remains *positive* at this transition from the integral (42) to the integral (45). The transition from (43) to (49) thus happens through the periodic motion described above with positive decreasing m or ε, such that the last and largest maximum is reached before ε becomes 0, while modulus quickly decreases from here to 0 as ε at the same time goes through 0 to lower and lower negative values.

Lastly, we will in the following sections also meet sums that may be rewritten as an integral of the form

$$\int_0^{z_1} dz \left(A\frac{z}{\alpha} + B\frac{z^3}{\alpha^3} + \dots\right) e^{\left(F\alpha + G\frac{z^2}{\alpha} + H\frac{z^4}{\alpha^3} + I\frac{z^6}{\alpha^5} + \dots\right)i}\,. \tag{51}$$

When setting herein $Gz^2 = \alpha x$ and G is not 0 or very small, the upper bound of x will belong to the kind of quantities denoted ω above, and since the terms of lower order than the unit are discarded, the result of the integration becomes

$$\frac{A}{2G} e^{\left(F\alpha + \frac{\pi}{2}\right)i}\,. \tag{52}$$

On the other hand, if G is very small, we set $Hz^4 = \alpha^3 x^2$, the upper bound of x is as before denoted by ω, and for brevity we set

$$G = \pm\varepsilon\sqrt{\frac{H}{\alpha}}\,, \tag{53}$$

where the uppermost sign corresponds to G being positive, the bottommost to G being negative. The integral then becomes

$$\frac{1}{2H}\int_0^{\omega} dx \left((\alpha H)^{\frac{1}{2}} A + Bx + \frac{AIi}{H}x^3\right) e^{(F\alpha \pm \varepsilon x + x^2)i}\,. \tag{54}$$

For $\varepsilon = 0$, we obtain from this by integration

$$\frac{A}{4}\sqrt{\frac{\alpha\pi}{H}} e^{\left(F\alpha + \frac{\pi}{4}\right)i} + \frac{1}{4H^2}(BH - AI) e^{\left(F\alpha + \frac{\pi}{2}\right)i}\,, \tag{55}$$

while the ordinary integral (54) may be expressed by

$$\frac{e^{F\alpha i}}{2H}\left((\alpha H)^{\frac{1}{2}} AQ \mp iB\frac{dQ}{d\varepsilon} \mp \frac{AI}{H}\frac{d^3Q}{d\varepsilon^3}\right)\,, \tag{56}$$

as

$$Q = \int_0^{\omega} dx\, e^{(\pm\varepsilon x + x^2)i}\,. \tag{57}$$

From this last integral, we obtain by taking the derivative with respect to ε and by partial integration

$$\frac{dQ}{d\varepsilon} = \mp\frac{1}{2} - \frac{\varepsilon\, i}{2} Q\,. \tag{58}$$

whereof we further find

$$\frac{d^3Q}{d\varepsilon^3} = \pm\left(\frac{i}{2} + \frac{\varepsilon^2}{8}\right) + \left(-\frac{3\varepsilon}{4} + \frac{\varepsilon^3 i}{8}\right) Q\,. \tag{59}$$

By insertion of these values in (56), this expression for the integral that we seek will be determined by known qunatities and by the integral Q.

This lastter integral has often been dealt with in different forms, in particular in the computation of diffraction phenomena by Fresnel, Cauchy, Knochenhauer, Quet, and others. Ph. Gilbert[1] has computed a larger table of the two functions N and M determined by

$$\sqrt{\frac{\pi}{2}}\int_0^\omega dx\, e^{(\varepsilon x + x^2)i} = N + Mi\,, \quad \varepsilon = \sqrt{2\pi}\mu\,,$$

The table comprises all values from $\mu^2 = 0.00$ to $\mu^2 = 30.00$.

Thus, with the upper sign in the integral Q, the integral can be computed directly from the table. With the lower sign, and setting

$$\sqrt{\frac{\pi}{2}}\int_0^\omega dx\, e^{(-\varepsilon x + x^2)i} = N_1 + M_1 i\,,$$

one will have

$$N + N_1 + (M + M_1)i = \sqrt{\frac{\pi}{2}}\int_{-\omega}^\omega dx\, e^{(\varepsilon x + x^2)i} = \sqrt{2}\left(\cos\frac{\pi - \varepsilon^2}{4} + i\sin\frac{\pi - \varepsilon^2}{4}\right),$$

whereby N_1 and M_1 are determined by

$$N_1 = \sqrt{2}\cos\frac{\pi - \varepsilon^2}{4} - N\,, \quad M_1 = \sqrt{2}\sin\frac{\pi - \varepsilon^2}{4} - M\,.$$

Both quantities N and M decline rapidly and continually with increasing ε, from which follows that N_1 and M_1 are periodic functions. From (58), with the lower sign, follows

$$\frac{dN_1}{d\varepsilon} = \frac{1}{\sqrt{2\pi}} + \frac{\varepsilon}{2}M_1\,, \quad \frac{dM_1}{d\varepsilon} = -\frac{\varepsilon}{2}N_1\,,$$

and then

$$N_1\frac{dN_1}{d\varepsilon} + M_1\frac{dM_1}{d\varepsilon} = \frac{N_1}{\sqrt{2\pi}}\,.$$

From which we see that maximum and minimum of $N_1^2 + M_1^2$ correspond to $N_1 = 0$, which in turn, for large values of ε, correspond approximately to $\cos\frac{\pi-\varepsilon^2}{4}$, that is, to $\varepsilon^2 = (4p-1)\pi$ or $\mu = \sqrt{\frac{4p-1}{2}}$, since p is an integer.

[1] Recherches anal. sur la diffraction de la lumière. Mém. cour. de l'Acad. de Bruxelles, t. XXXI, p. 1, 1862–63.

According to Gilbert, we have

$$\begin{aligned} N_1^2+M_1^2 = 2.7407 \text{ at } \mu &= 1.2172\,, &\left(\sqrt{\frac{3}{2}}=1.2247\right), &\quad \text{1st max.}\\ 1.5562 \text{ at } \mu &= 1.8725\,, &\left(\sqrt{\frac{7}{2}}=1.8708\right), &\quad \text{1st min.}\\ 2.3985 \text{ at } \mu &= 2.3445\,, &\left(\sqrt{\frac{11}{2}}=2.3452\right), &\quad \text{2nd max.}\\ 1.6864 \text{ at } \mu &= 2.7390\,, &\left(\sqrt{\frac{15}{2}}=2.7386\right), &\quad \text{2nd min.} \end{aligned}$$

With $\mu = 0$ corresponding to $N_1^2 + M_1^2 = \dfrac{1}{2}$ and $\mu = \infty$ to $N_1^2 + M_1^2 = 2$.

If we only consider the term of the highest order ($\alpha^{\frac{1}{2}}$) in (56), it will appear from what we developed above that the modulus of this expression increases from 0 at $G = +\infty$ up to $\frac{A}{4}\sqrt{\frac{\alpha\pi}{H}}$ at $G = 0$, further increases with decreasing G up to $2.3412 \cdot \frac{A}{4}\sqrt{\frac{\alpha\pi}{H}}$ at $G = -1.2172\sqrt{\frac{2\pi H}{\alpha}}$, and, finally, through a series of decreasing oscillations, it reaches twice the value of the one corresponding to $G = 0$.

4 α very large. Propagation along the main axis.

Like in the previous section, α is here considered a very large number, and we should seek to determine the light propagation so that only quantities of order lower than unity are discarded.

We first seek to determine the propagation *close to the centre of the sphere*, since a', which is the distance from the centre to the observed point, measured with $\frac{\lambda'}{2\pi}$ as unit of length, is considered a *very small* number as compared with α and α'. Under this condition $v_n(a')$, as given by the series (22), becomes very small when the magnitude of n approaches that of α, why the terms in the series (31) for K' and S' become significant only for the lower values of n. In the expressions for k'_n and s'_n given in (34), one will by means of (23) and (25) be able to set

$$v_n(\alpha) = \sin\left(\alpha - \frac{n\pi}{2}\right)\,, \quad v_n(\alpha') = \sin\left(\alpha - \frac{n\pi}{2}\right)\,, \quad w_n(\alpha) = \cos\left(\alpha - \frac{n\pi}{2}\right)\,.$$

Thus, we obtain

$$\left.\begin{aligned} s'_{2n+1} = k'_{2n} = k'_0 = e^{\alpha i}\frac{N}{\cos\alpha' + i\,N\sin\alpha'}\,,\\ s'_{2n} = k'_{2n+1} = s'_0 = e^{\alpha i}\frac{N}{N\cos\alpha' + i\,\sin\alpha'}\,. \end{aligned}\right\} \tag{60}$$

The series (31) hereby become

$$K' = -i\,\frac{\cos\phi}{2a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)}e^{\left(kt-\frac{n\pi}{2}\right)i}\left[(P_n(\cos\varphi)+P_n(-\cos\varphi))k_0' + (P_n(\cos\varphi)-P_n(-\cos\varphi))s_0'\right]v_n(a'),$$
$$S' = -\;\frac{\sin\phi}{2a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)}e^{\left(kt-\frac{n\pi}{2}\right)i}\left[(P_n(\cos\varphi)+P_n(-\cos\varphi))s_0' + (P_n(\cos\varphi)-P_n(-\cos\varphi))k_0'\right]v_n(a').$$

These series can be summed by means of the equations (26) and (27), whereby we find

$$K' = -i\,\frac{\cos\phi}{a'\sin\varphi}e^{kti}\left[(-\sin a'\cos\varphi + \sin(a'\cos\varphi))\,k_0' + i\,(-\cos a' + \cos(a'\cos\varphi))\,s_0'\right],$$
$$S' = -\;\frac{\sin\phi}{a'\sin\varphi}e^{kti}\left[(-\sin a'\cos\varphi + \sin(a'\cos\varphi))\,s_0' + i\,(-\cos a' + \cos(a'\cos\varphi))\,k_0'\right].$$

Now, inserting these values in the equations (18) and abbreviating

$$e^{kti}\,(-i\sin(a'\cos\varphi)k_0' + \cos(a'\cos\varphi)s_0') = Q\,,$$

we have

$$\overline{\xi'} = \sin\varphi\cos\phi\,Q\,,\quad \overline{\eta'} = \cos\varphi\cos\phi\,Q\,,^{1}\quad \overline{\zeta'} = -\sin\phi\,Q\,.$$

From this, we further find the components with respect to the fixed axes

$$\xi' = 0\,,\quad \eta' = 0\,,\quad \zeta' = 0\,.$$

Inserting the value of Q given above, we obtain by a minor rearrangement

$$\eta' = e^{(kt-a'\cos\varphi)i}\frac{k_0'+s_0'}{2} - e^{(kt+a'\cos\varphi)i}\frac{k_0'-s_0'}{2}\,. \tag{61}$$

The physical meaning of this result is most easily seen when the values of k_0' and s_0' are expanded in series, with $\cos\alpha'$ and $\sin\alpha'$ expressed in exponential form, namely

$$k_0' = \frac{2N}{N+1}\sum_0^\infty\left(\frac{N-1}{N+1}\right)^m e^{(\alpha-(2m+1)\alpha')i},\quad s_0' = \frac{2N}{N+1}\sum_0^\infty\left(\frac{1-N}{1+N}\right)^m e^{(\alpha-(2m+1)\alpha')i},$$

where m runs through the numbers from 0 to ∞. Thus, we obtain

$$\eta' = \frac{2N}{N+1}\sum_0^\infty\left(\frac{N-1}{N+1}\right)^{2m} e^{(kt-a'\cos\varphi+\alpha-(4m+1)\alpha')i} - \frac{2N}{N+1}\sum_0^\infty\left(\frac{N-1}{N+1}\right)^{2m+1} e^{(kt+a'\cos\varphi+\alpha-(4m+3)\alpha')i}\,. \tag{62}$$

In this way, the light propagation in the vicinity of the centre is represented by a sum of oscillations that are parallel with the oscillations of the incident rays. These belong to two sets of rays: one set going in the direction of the incident rays, reflected an even number of times or not at all from the inner spherical surfaces; the other set going in the opposite direction after an odd number of reflections. When the rays enter the sphere, the out-scattering changes according to the ratio $1+N$ to $2N$ and at each reflection according to the ratio $1+N$ to $1-N$, while the phase corresponds to the optical distance travelled. This is all in agreement with the results that one would reach by using the more elementary approach, as long as the two refractive surfaces are considered planar and orthogonal to the incident rays.

[1] In this equation, $\cos\varphi\cos\varphi$ has been corrected to $\cos\varphi\cos\phi$. (Translator's note.)

When the considered point *is not very close to the centre*, one must take into account those terms of the series that correspond to very large values of n. For this case, it will therefore be necessary first to look for appropriate expansions for the functions v_n and w_n.

We have identically

$$v_n = \sqrt{v_n^2 + w_n^2}\sin\arctan\frac{v_n}{w_n}\,, \quad w_n = \sqrt{v_n^2 + w_n^2}\cos\arctan\frac{v_n}{w_n}\,,$$ [1]

or, when we set

$$v_n^2 + w_n^2 = q_n\,, \quad \arctan\frac{v_n}{w_n} = \lambda_n\,,$$

$$v_n = \sqrt{q_n}\sin\lambda_n\,, \quad w_n = \sqrt{q_n}\cos\lambda_n\,. \tag{63}$$

Moreover, by means of the equation

$$w_n v_n' - w_n' v_n = 1$$

we have, when a denotes the variable,

$$\frac{d\lambda_n}{da} = \frac{1}{q_n}\,, \tag{64}$$

whereof we have by integration, as $a = \infty$ corresponds to $\lambda_n = a - \frac{n\pi}{2}$,

$$\lambda_n = a - \frac{n\pi}{2} - \int_a^\infty da\left(\frac{1}{q_n} - 1\right). \tag{65}$$

From the series for v_n and w_n given in (23) and (25), we find

$$q_n = 1 + \frac{n(n+1)}{a^2}\cdot\frac{1}{2} + \frac{(n-1)n(n+1)(n+2)}{a^4}\cdot\frac{1\,.\,3}{2\,.\,4} + \ldots\,. \tag{66}$$

Now, assume that a is a very large number with the order of magnitude α and that all quantities of the order α^{-1} or lower order can be neglected as compared with the order of unity. Then, for all values of n until a certain limit, which is lower than a, and where the difference $a - \left(n + \frac{1}{2}\right)$ can still be considered of the order of magnitude α, one obtains from the summation of the series (66) that

$$q_n = \frac{a}{\sqrt{a^2 - \left(n + \frac{1}{2}\right)^2}}\,, \quad a > n + \tfrac{1}{2}\,. \tag{67}$$

Inserting this expression for q_n in (65), where it must be valid for all the elements of the integral, we have by integration

$$\lambda_n = \sqrt{a^2 - \left(n + \tfrac{1}{2}\right)^2} - \frac{n\pi}{2} + \left(n + \tfrac{1}{2}\right)\arcsin\frac{n + \frac{1}{2}}{a}\,. \tag{68}$$

The functional expressions q_n and λ_n will in the following include the variable that has here been omitted for the sake of brevity.

Insofar as Equation (68) is valid, the derivatives of $q_n(\alpha)$ and $q_n(\alpha')$ with respect to α and α' can be neglected as compared with magnitudes of the order α^0,

[1] In this equation, we use the more modern notation of $\arctan$ instead of the $\mathrm{arc}\,tg$. (translator's note)

such as $q_n(\alpha)$ and $q_n(\alpha')$. Let us return to the coefficient equations (33) and (34) and for the sake of brevity introduce the following notation

$$\frac{Nq_n(\alpha')-q_n(\alpha)}{Nq_n(\alpha')+q_n(\alpha)}=b_n\,,\quad \frac{2\,Nq_n(\alpha)\,q_n(\alpha')\,(Nq_n(\alpha')-q_n(\alpha))^m}{(Nq_n(\alpha')+q_n(\alpha))^{m+2}}=b_{n,m}\,,$$

$$\frac{q_n(\alpha')-Nq_n(\alpha)}{q_n(\alpha')+Nq_n(\alpha)}=c_n\,,\quad \frac{2\,Nq_n(\alpha)\,q_n(\alpha')\,(q_n(\alpha')-Nq_n(\alpha))^m}{(q_n(\alpha')+Nq_n(\alpha))^{m+2}}=c_{n,m}\,,$$

$$\frac{2N\sqrt{q_n(\alpha)q_n(\alpha')}\,(Nq_n(\alpha')-q_n(\alpha))^m}{(Nq_n(\alpha')+q_n(\alpha))^{m+1}}=\beta_{n,m}\,,\quad \frac{2N\sqrt{q_n(\alpha)q_n(\alpha')}\,(q_n(\alpha')-Nq_n(\alpha))^m}{(q_n(\alpha')+Nq_n(\alpha))^{m+1}}=\gamma_{n,m}\,,$$

then the coefficients can be expressed by fractions to be expanded in the following convergent series:

$$\left.\begin{aligned}
2\,k_n &= -1-b_n e^{2\,\lambda_n(\alpha)\,i}+\sum_{m=0}^{m=\infty}2\,b_{n,m}e^{2(\lambda_n(\alpha)-(m+1)\lambda(\alpha'))\,i}\,,\\
2\,s_n &= -1-c_n e^{2\,\lambda_n(\alpha)\,i}+\sum_{m=0}^{m=\infty}2\,c_{n,m}e^{2(\lambda_n(\alpha)-(m+1)\lambda(\alpha'))\,i}\,,\\
k_n' &= \sum_{m=0}^{m=\infty}\beta_{n,m}e^{2(\lambda_n(\alpha)-(2m+1)\lambda(\alpha'))\,i}\,,\quad s_n'=\sum_{m=0}^{m=\infty}\gamma_{n,m}e^{2(\lambda_n(\alpha)-(2m+1)\lambda(\alpha'))\,i}\,.
\end{aligned}\right\}\qquad(69)$$

Since we next proceed to the summation of the series (31), we will in this section restrict ourselves to the case where the considered point lies on the x-axis (the main axis). It should be noticed that one has

$$\text{for }\cos\varphi=1\,,\quad \frac{dP_n(\cos\varphi)}{\sin\varphi\,d\varphi}=\frac{d^2P_n(\cos\varphi)}{d\varphi^2}=-\frac{n(n+1)}{2}\,,$$

$$\text{for }\cos\varphi=-1\,,\quad \frac{dP_n(\cos\varphi)}{\sin\varphi\,d\varphi}=-\frac{d^2P_n(\cos\varphi)}{d\varphi^2}=(-1)^n\,\frac{n(n+1)}{2}\,.$$

The given series for K and S are now inserted in (17), and those for K' and S' in (18). When we then determine the components with respect to the fixed axes by means of the equations

$$\begin{aligned}
\xi &= \cos\varphi\,\overline{\xi}-\sin\varphi\,\overline{\eta}\,,\\
\eta &= \sin\varphi\cos\phi\,\overline{\xi}+\cos\varphi\cos\phi\,\overline{\eta}-\sin\phi\,\overline{\zeta}\,,\\
\zeta &= \sin\varphi\sin\phi\,\overline{\xi}+\cos\varphi\sin\phi\,\overline{\eta}+\cos\phi\,\overline{\zeta}\,,
\end{aligned}$$

and the corresponding equations for an interior point, we will find that the oscillations everywhere in the main axis are in the direction of the y-axis. This also follows immediately from the fact that the entire light propagation is symmetrical with respect to the xy-plane, and that the oscillatory deflections outside and inside the sphere would be determined by

$$\left.\begin{aligned}
\eta &= e^{(kt\mp a)i}+\sum_1^\infty\frac{n+\frac{1}{2}}{a}e^{\left(kt\mp\frac{n\pi}{2}\right)i}\left(\pm i\,k_n(v_n'(a)+w_n'(a)\,i)+s_n(v_n(a)+w_n(a)\,i)\right),\\
\eta' &= \sum_1^\infty\frac{n+\frac{1}{2}}{a'}e^{\left(kt\mp\frac{n\pi}{2}\right)i}\left(\pm i\,k_n'v_n'(a')+s_n'v_n(a')\right),
\end{aligned}\right\}\qquad(70)$$

upper sign applies to the positive side of the x-axis, lower sign to its negative side.

The functions of n that enter here can be expanded in powers of $n+\frac{1}{2}$. The series remain convergent up to a certain limit $n = n_1$. We will then first perform the mentioned summations up to this limit. Thus, the expression given in (68) for $\lambda_n(a)$ can be expanded in the following series

$$\lambda_n(a) = a - \frac{n\pi}{2} + \frac{\left(n+\frac{1}{2}\right)^2}{a}\cdot\frac{1}{2} + \frac{\left(n+\frac{1}{2}\right)^4}{3a^3}\cdot\frac{1}{2\,.\,4} + \frac{\left(n+\frac{1}{2}\right)^6}{5a^5}\cdot\frac{1\,.\,3}{2\,.\,4\,.\,6} + \dots \qquad (71)$$

For q_n, we have the series expansion (66), and from the equations (63), we obtain

$$v_n(a) + w_n(a)\, i = i\sqrt{q_n(a)}e^{-\lambda_n(a)\, i}\,,$$

and according to (64) if discarding $q_n'(a)$

$$v_n'(a) + w_n'(a)\, i = \frac{1}{\sqrt{q_n(a)}}e^{-\lambda_n(a)\, i}\,.$$

We will now single out the individual terms that constitute the equations (69) for the coefficients, and we first set

$$2\, k_n = -1\,, \quad 2\, s_n = -1\,.$$

With these prerequisites, the first of the equations (70) gives

$$\eta = e^{(kt\mp a)\, i} - i\sum_1^{n_1}\frac{n+\frac{1}{2}}{2\, a}\left(\frac{\pm 1}{\sqrt{q_n(a)}} + \sqrt{q_n(a)}\right)e^{\left(kt\mp\frac{n\pi}{2}-\lambda_n(a)\right)\, i}\,.$$

When inserting herein the series for $\lambda_n(a)$ given in (71), it is seen that the exponent will include the term $\frac{n\pi}{2}(\mp 1+1)$. When reading the lower sign, the term becomes $n\pi$, and according to the expansion of the previous section, the sum will be 0. Consequently, for the negative side of the x-axis, we have

$$\eta = e^{(kt+a)i}\,.$$

On the other hand, when reading the upper sign and setting $n+\frac{1}{2} = z$, the sum changes to an integral of the form (51). By comparison, we obtain

$$A = \frac{\alpha}{a}\,, \quad F\alpha = kt - a\,, \quad G = -\frac{\alpha}{2a}\,,$$

while according to (52) the integral becomes equal to

$$-e^{\left(kt-a+\frac{\pi}{2}\right)i}\,.$$

Consequently, for the positive side of the x-axis, we have

$$\eta = e^{(kt-a)\, i} + i\, e^{\left(kt-a+\frac{\pi}{2}\right)i} = 0\,.$$

The part of the propagation described here is thus nothing but the incident central ray up to the point where it hits the sphere.

Further singling out the second term of the first two equations (69) and setting

$$2\,k_n = -b_n e^{2\,\lambda_n(\alpha)\,i}\,, \quad 2\,s_n = -c_n e^{2\,\lambda_n(\alpha)\,i}\,,$$

the sum of the expression (70) for η will come to include the exponent

$$\left(kt - a + 2\,\alpha + \frac{n\pi}{2}(\mp 1 - 1) + \frac{(n+\frac{1}{2})^2}{2}\left(-\frac{1}{a} + \frac{2}{\alpha}\right) + \ldots\right)i\,.$$

Here, when reading the upper sign, the sum must be 0. On the other hand, the sum with the lower sign can be converted, as before, to an integral of the form (51), and by comparison we obtain

$$A = \frac{N-1}{N+1}\frac{\alpha}{a}i\,, \quad F\alpha = kt - a + 2\,\alpha\,, \quad G = \frac{\alpha}{2}\left(-\frac{1}{a} + \frac{2}{\alpha}\right)\,.$$

Thus, according to (52), the integral will become equal to

$$-\frac{N-1}{N+1} \cdot \frac{1}{a\left(-\frac{1}{a} + \frac{2}{\alpha}\right)} e^{(kt-a+2\,\alpha)i}\,. \tag{72}$$

This part of the propagation corresponds to the central ray reflected from the front part of the spherical surface. The result is the same as what one could derive from an elementary approach, since the phase is determined by the optical distance travelled, and the amplitude after reflection is $-\frac{N-1}{N+1}$ in the spherical surface itself, that is, in the distance $\frac{1}{2}\alpha$ (distances measured with $\frac{\lambda}{2\pi}$ as unit of length) from the virtual focal point of the central rays, and must then decrease with the same ratio as the considered point moves away from this focal point.

Finally, picking out the term of the equations (69) corresponding to

$$k_n = b_{n,m}\, e^{2(\lambda_n(\alpha)-(m+1)\lambda_n(\alpha'))\,i}\,, \quad s_n = c_{n,m}\, e^{2(\lambda_n(\alpha)-(m+1)\lambda_n(\alpha'))\,i}\,,$$

the deflection will be determined by

$$\sum_1^{n_1} i\,\frac{n+\frac{1}{2}}{a\sqrt{q_n(a)}}\left(\pm b_{n,m} + c_{n,m}q_n(a)\right) e^{\left(kt\mp\frac{n\pi}{2}-\lambda_n(a)+2\lambda_n(\alpha)-(2m+2)\lambda_n(\alpha')\right)\,i}\,.$$

Expanded in powers of $n+\frac{1}{2}$, the exponent becomes

$$\left(kt - a + 2\alpha - (2m+2)\alpha' + \frac{n\pi}{2}(\mp 1 + 2m + 1) + \frac{\left(n+\frac{1}{2}\right)^2}{2}\left(-\frac{1}{a} + \frac{2}{\alpha} - \frac{2m+2}{\alpha'}\right) + \ldots\right)i\,.$$

The sum vanishes unless we have

$$\mp 1 + 2m + 1 = 4p\,,$$

that is, unless m is an even number when the point is on the positive side of the x-axis (upper sign), or m is odd when the point is on the negative side. Assuming this, the sum can be transformed to an integral of the form (51), which yields by comparison

$$
\begin{aligned}
A &= i\,\frac{\alpha}{a}\,\frac{4N(1-N)^m}{(1+N)^{m+2}}\,, & B &= 0\,,\\
F\alpha &= kt-a+2\alpha-(2m+2)\alpha'\,, & G &= \frac{\alpha}{2}\left(-\frac{1}{a}+\frac{2}{\alpha}-\frac{2m+2}{\alpha'}\right),\\
H &= \frac{\alpha^3}{24}\left(-\frac{1}{a^3}+\frac{2}{\alpha^3}-\frac{2m+2}{\alpha'^3}\right), & I &= \frac{\alpha^5}{80}\left(-\frac{1}{a^5}+\frac{2}{\alpha^5}-\frac{2m+2}{\alpha'^5}\right).
\end{aligned}
$$

According to (52), which assumes that G is not very small, the result of the integration becomes

$$
-\frac{4N(1-N)^m}{(1+N)^{m+2}}\cdot\frac{1}{a\left(-\frac{1}{a}+\frac{2}{\alpha}-\frac{2m+2}{\alpha'}\right)}e^{(kt-a+2\alpha-(2m+2)\alpha')i}\,. \tag{73}
$$

This result can also be deduced through an elementary approach. Imagine a cylindrical bundle of central rays with the diameter 1 entering the sphere. After m internal reflections, this bundle will leave the sphere with the diameter $\frac{(2m+2)\alpha-\alpha'}{\alpha'}$ and subsequently unite in a real or virtual focal point. If the distance of this point from the centre is a_1, then the diameter of the ray bundle will be $\frac{a_1-a}{a_1-\alpha}\cdot\frac{(2m+2)\alpha-\alpha'}{\alpha'}$. Now, the focal length a_1 is determined by $-\frac{1}{a_1}+\frac{2}{\alpha}-\frac{2m+2}{\alpha'}=0$, and after m reflections and two refractions the amplitude of oscillation has changed to $\left(\frac{1-N}{1+N}\right)^m\frac{4N}{(1+N)^2}$ and will increase at the same ratio as the diameter of the ray bundle decreases. Furthermore, when the phase is determined by the optical distance travelled, it is seen that the result will be exactly the same as the one found above.

On the other hand, one cannot in this way determine the propagation in the focal points themselves. These are determined by the equation $G=0$, and tied to this is $0<\frac{1}{a}\le\frac{1}{\alpha}$ corresponding to the condition $2N>2m+2\ge N$, whereof we see that for $N\le 1$ there is no corresponding real focal point, for $1<N<2$ only one focal point, etc. The expression (55) corresponds to $G=0$, which with the values for A, B, H, and I given above, results in out-scattering in the considered focal point determined by

$$
-\frac{2N(1-N)^m}{(1+N)^{m+2}}\left(\sqrt{\frac{6\pi}{a^2\left(-\frac{1}{a^3}+\frac{2}{\alpha^3}-\frac{2m+2}{\alpha'^3}\right)}}\,e^{\left(F\alpha-\frac{\pi}{4}\right)i}-\frac{18}{5}\,\frac{-\frac{1}{a^5}+\frac{2}{\alpha^5}-\frac{2m+2}{\alpha'^5}}{a\left(-\frac{1}{a^3}+\frac{2}{\alpha^3}-\frac{2m+2}{\alpha'^3}\right)^2}e^{F\alpha i}\right). \tag{74}
$$

It is apparent from the expression for G that, when we along the main axis approach the sphere from an external point and pass a focal point, then G will go from a positive value through 0 and onto a negative value. From this, we see that, according to what was pointed out at the end of the previous section, the amplitude during this propagation rapidly grows from a very small quantity in the vicinity of the focal point to the value of magnitude $\alpha^{\frac{1}{2}}$ determined above for the focal point and then grows yet further to reach, through oscillations, twice the amplitude of the focal point. After this, the axis is hit by other rays that lie outside the central rays and whose effect will be determined in the following. A closer determination of the light propagation in the vicinity of a focal point appears from (56) and the thereupon given overview of the value of the integral Q (57).

As an example, I will assume $m = 0$, the radius of the sphere equal to 1 cm, the refractive index 1.5, and the wavelength of the incident light equal to 0.0005 mm. We will then have

$$\alpha = 40000\,\pi\,, \quad \alpha' = 1.5\,\alpha\,, \quad a = 1.5\,\alpha\,, \quad N = 1.5\,.$$

Inserted in (74), these numerical values give the result

$$-467.23\,e^{\left(F\alpha - \frac{\pi}{4}\right)i} + 1.50\,e^{F\alpha\,i}\,.$$

From this, we see that the second term is of little significance, and that the intensity, which we take to be proportional to the square of the amplitude, is very considerable in this focal point, namely 217311 times the intensity of the incident light. For a sphere with the same refractive index and twice the radius, the intensity would very nearly be doubled.

Within a small distance δ (measured with $\frac{\lambda}{2\pi}$ as the unit of length) from the focal point, we have $G = -\frac{\alpha\delta}{2a^2}$, and if the intensity has reached its first maximum in this point, one will find from the value of G given at the end of the previous section that $d = 1047$, corresponding to 0.0833 mm. In this point, the intensity will have increased to 1191200 as it is 5.4814 times larger than in the focal point.

The computation of the part of the light propagation in the axis within the sphere, which is due to the central rays, can be done in quite the same way by proceeding from the second equation (70). The sum that we are to compute, when extracting the general term of the sums given in (69) for k'_n and s'_n, will be

$$\sum_{1}^{n_1} \frac{n+\frac{1}{2}}{a'\sqrt{q_n(a')}}\left(\pm i\cos\lambda_n(a')\beta_{n,m} + \sin\lambda_n(a')\gamma_{n,m}\right) e^{\left(kt \mp \frac{n\pi}{2} + \lambda_n(a) - (2m+1)\lambda_n(a')\right)i}\,.$$

When we now herein give $\cos\lambda_n(a')$ and $\sin\lambda_n(a')$ their exponential form and then expand all the functions λ_n, according to the formula (71), the coefficients to $\frac{n\pi}{2}\,i$ will in the exponents be

$$\mp 1 + 2m + 1 \quad \text{and} \quad \mp 1 + 2m - 1\,.$$

Only when these coefficients are 0 or a multiple of 4, the sum will not vanish, and this will only be the case when they can be written in the form

$$\mp(1 - (-1)^m) + 2m\,.$$

In this case, we can give the sum the form of the integral (51), and by comparison we get

$$
\begin{aligned}
A &= \pm i\frac{\alpha}{a'}\frac{2N(N-1)^m}{(N+1)^{m+1}}\,, & B &= 0\,,\\
F\alpha &= kt \mp (-1)^m a' + \alpha - (2m+1)\alpha'\,, & G &= \frac{\alpha}{2}\left(\mp\frac{(-1)^m}{a'} + \frac{1}{\alpha} - \frac{2m+1}{\alpha'}\right),\\
H &= \frac{\alpha^3}{24}\left(\mp\frac{(-1)^m}{a'^3} + \frac{1}{\alpha^3} - \frac{2m+1}{\alpha'^3}\right), & I &= \frac{\alpha^5}{80}\left(\mp\frac{(-1)^m}{a'^5} + \frac{1}{\alpha^5} - \frac{2m+1}{\alpha'^5}\right).
\end{aligned}
$$

For G not very small, it follows from (52) that the result of the integration becomes

$$
\mp\frac{2N(N-1)^m}{(N+1)^{m+1}}\cdot\frac{1}{a'\left(\mp\frac{(-1)^m}{a'} + \frac{1}{\alpha} - \frac{2m+1}{\alpha'}\right)}e^{(kt\mp(-1)^m a' + \alpha - (2m+1)\alpha')\,i}\,. \tag{75}
$$

If we on the other hand have $G = 0$, it follows from (55) that we have

$$
\mp\frac{N(N-1)^m}{(N+1)^{m+1}}\left(\sqrt{\frac{6\pi}{a'^2\left(\mp\frac{(-1)^m}{a'^3} + \frac{1}{\alpha^3} - \frac{2m+1}{\alpha'^3}\right)}}\,e^{\left(F\alpha - \frac{\pi}{4}\right)i} - \frac{18}{5}\frac{\mp\frac{(-1)^m}{a'^5} + \frac{1}{\alpha^5} - \frac{2m+1}{\alpha'^5}}{a'\left(\mp\frac{(-1)^m}{a'^3} + \frac{1}{\alpha^3} - \frac{2m+1}{\alpha'^3}\right)^2}\,e^{F\alpha\, i}\right). \tag{76}
$$

As we need to have $a' < \alpha'$, we see that the equation $G = 0$ is not possible for $N - 1 < 2m + 1 < N + 1$, while conversely for all other values of m the equation can be satisfied either by one or the other of the two signs entering into G.

If we in (75) consider a' infinitely small and then replace m by $2m$ and $2m + 1$, the result will immediately fit the one found in (62) where the out-scattering near the centre is determined in another way.

We now continue the summations of the series (70) from $n = n_1$ to $n = n_2$, where n_2 is the highest possible limit for n when the functions q_n and λ_n are to be expressed by the formulae (67) and (68). The series then take the form given in (35), and as we also here insert $n = \nu + z$, where ν and z are considered to be integers, we introduce the following notation

$$
\nu + \frac{1}{2} = \alpha\sin\theta = \alpha'\sin\theta' = a\sin\vartheta = a'\sin\vartheta'\,, \tag{77}
$$

where the four angles θ, θ', ϑ, and ϑ' are lying between 0 and $\frac{\pi}{2}$ and for the moment are assumed not to be very close to these two limits.

It follows from (67) that

$$
1 = \cos\theta\, q_\nu(\alpha) = \cos\theta'\, q_\nu(\alpha') = \cos\vartheta\, q_\nu(a) = \cos\vartheta'\, q_\nu(a')\,, \tag{78}
$$

after which the coefficients b_ν, $b_{\nu,m}$ etc. are determined by

$$
\begin{aligned}
b_\nu &= \frac{N\cos\theta - \cos\theta'}{N\cos\theta + \cos\theta'}\,, & b_{\nu,m} &= 2N\cos\theta\cos\theta'\frac{(N\cos\theta - \cos\theta')^m}{(N\cos\theta + \cos\theta')^{m+2}}\,,\\
c_\nu &= \frac{\cos\theta - N\cos\theta'}{\cos\theta + N\cos\theta'}\,, & c_{\nu,m} &= 2N\cos\theta\cos\theta'\frac{(\cos\theta - N\cos\theta')^m}{(\cos\theta + N\cos\theta')^{m+2}}\,,\\
\beta_{\nu,m} &= 2N\sqrt{\cos\theta\cos\theta'}\frac{(N\cos\theta - \cos\theta')^m}{(N\cos\theta + \cos\theta')^{m+1}}\,, & \gamma_{\nu,m} &= 2N\sqrt{\cos\theta\cos\theta'}\frac{(\cos\theta - N\cos\theta')^m}{(\cos\theta + N\cos\theta')^{m+1}}\,.
\end{aligned}
$$

The corresponding coefficients b_n, $b_{n,m}$, etc. could be expanded in series in powers of z, such as for example

$$b_n = b_\nu + \left(\frac{1}{\alpha\cos\theta}\frac{db_\nu}{d\theta} + \frac{1}{\alpha'\cos\theta'}\frac{db_\nu}{d\theta'}\right) z + \dots .$$

In the same way, it follows from (68) that

$$\begin{aligned}
\lambda_\nu(\alpha) &= \alpha\cos\theta - \frac{\nu\pi}{2} + (\nu+\tfrac{1}{2})\theta\,, \\
\lambda_n(\alpha) &= \lambda_\nu(\alpha) + \left(\theta - \frac{\pi}{2}\right) z + \frac{z^2}{2\alpha\cos\theta} + \frac{\sin\theta\, z^3}{6\alpha^2\cos^3\theta} + \frac{(1+2\sin^2\theta)z^4}{24\,\alpha^3\cos^5\theta} + \dots\,,
\end{aligned}$$

and we have corresponding expansions for $\lambda_n(\alpha')\,,\ \ \lambda_n(a)\,,\ \ \lambda_n(a')\,.$

As we did earlier, we now pick out the individual terms from the series (69) for k_n and s_n and start with the assumption that

$$2k_n = -1\,,\quad 2s_n = -1\,.$$

With this prerequisite, the sum for η given in (70) and taken from $n = n_1$ to $n = n_2$ will include the exponential argument

$$\left(kt \mp \frac{n\pi}{2} - \lambda_n(a)\right) i = \left(kt \mp \frac{\nu\pi}{2} - \lambda_\nu(a) + \left(\mp\frac{\pi}{2} - \vartheta + \frac{\pi}{2}\right) z + \dots\right) i\,.$$

Since the coefficient for z here cannot become 0 or very small, the sum will in this case vanish.

Next, assuming that

$$2k_n = -b_n e^{2\,\lambda_n(\alpha)\,i}\,,\quad 2s_n = -c_n e^{2\,\lambda_n(\alpha)\,i}\,,$$

the sum will include the exponent

$$\left(kt \mp \frac{n\pi}{2} - \lambda_n(a) + 2\,\lambda_n(\alpha)\right) i\,,$$

wherein the coefficient for zi will be $\mp\frac{\pi}{2} - \left(\vartheta - \frac{\pi}{2}\right) + 2\left(\theta - \frac{\pi}{2}\right)$, and this coefficient also cannot be 0 or very small since $2\theta - \vartheta$ must be both smaller than π and larger than 0 as one must have $\theta \geq \vartheta$. Thus, in this case too, the sum must become 0.

Finally, setting

$$k_n = b_{n,m}\, e^{2\,(\lambda_n(\alpha) - (m+1)\lambda_n(\alpha'))\,i}\,,\quad s_n = c_{n,m}\, e^{2\,(\lambda_n(\alpha) - (m+1)\lambda_n(\alpha'))\,i}\,,$$

the sum will include the exponent

$$\left(kt \mp \frac{n\pi}{2} - \lambda_n(a) + 2\,\lambda_n(\alpha) - (2m+2)\lambda_n(\alpha')\right) i\,,$$

wherein the coefficient for zi will be

$$\frac{\pi}{2}(2m + 1 \mp 1) - \vartheta + 2\theta - (2m+2)\theta' = G\,.$$

Assuming now as in (41) that $G = 2p\pi$, the sum passes to an integral of the form (42), where the coefficients will be

$$
\begin{aligned}
A &= i\frac{\sin\vartheta}{\sqrt{\cos\vartheta}}(\pm\cos\vartheta\, b_{\nu,m} + c_{\nu,m})\,, \quad B = \alpha\frac{dA}{d\nu}\,,\\
F\alpha &= kt + (\nu+\tfrac{1}{2})G - \frac{\pi}{4}(2m+1\mp 1) - a\cos\vartheta + 2\alpha\cos\theta - (2m+2)\alpha'\cos\theta'\,,\\
H &= \frac{\alpha}{2}\left(-\frac{1}{a\cos\vartheta} + \frac{2}{\alpha\cos\theta} - \frac{2m+2}{\alpha'\cos\theta'}\right) = \frac{1}{2\sin\theta}(-\tan\vartheta + \tan\theta - (2m+2)\tan\theta')\,,\\
I &= \frac{1}{6\sin^2\theta}(-\tan^3\vartheta + 2\tan^3\theta - (2m+2)\tan^3\theta')\,,\\
K &= \frac{I}{4\sin\theta} + \frac{1}{8\sin^3\theta}(-\tan^5\vartheta + 2\tan^5\theta - (2m+2)\tan^5\theta')\,.
\end{aligned}
$$

Instead of using the term $(\nu + \frac{1}{2})G$ entering into $F\alpha$, one can also, since ν is an integer, use $p\pi$ when the condition $G = 2p\pi$ is satisfied.

The result of the integration will then be given by the formula (43) and by (50) if one has $H = 0$, or more commonly by (49) when $G - 2p\pi$ is not 0 but very small.

The results with respect to an interior point can also be determined from the same formulae, as we would then proceed from the second equation (70), which leads to the following values for the coefficients

$$
\begin{aligned}
A &= i\frac{\sin\vartheta'}{\sqrt{\cos\vartheta'}}(\pm\cos\vartheta'\,\beta_{\nu,m} - (\pm)\gamma_{\nu,m})\,, \quad B = \alpha\frac{dA}{d\nu}\,,\\
G &= \frac{\pi}{2}(2m - (\pm)1 \mp 1) + (\pm)\vartheta' + \theta - (2m+1)\theta'\,,\\
F\alpha &= kt + \left(\nu + \frac{1}{2}\right)G - \frac{\pi}{4}(2m - (\pm)1 \mp 1) + (\pm)a'\cos\vartheta' + \alpha\cos\theta - (2m+1)\alpha'\cos\theta'\,,\\
H &= \frac{1}{2\sin\theta}((\pm)\tan\vartheta' + \tan\theta - (2m+1)\tan\theta')\,,\\
I &= \frac{1}{6\sin^2\theta}((\pm)\tan^3\vartheta' + \tan^3\theta - (2m+1)\tan^3\theta')\,,\\
K &= \frac{I}{4\sin\theta} + \frac{1}{8\sin^3\theta}((\pm)\tan^5\vartheta' + \tan^5\theta - (2m+1)\tan^5\theta')\,.
\end{aligned}
$$

The sign in parenthesis $(\pm)$ is taken to be everywhere the same and *either* $+$ *or* $-$, and it is determined more specifically by the condition that $G - 2p\pi$ must be 0 or very small.

If we consider the thus computed light propagation in the main axis generated by refraction and internal reflections of light rays, these correspond to all the light rays hitting the sphere in the distance $\nu + \frac{1}{2}$ from the main axis. The angle of incidence corresponds to θ, the angle of refraction to θ', while ϑ and ϑ' become the acute angles under which the rays meet the main axis in the point a outside the sphere or in the point a' inside the sphere. After m internal reflections an incident ray has rotated the angle

$$\Delta_m = m\pi + 2\theta - (2m+2)\theta' ,$$

when the ray leaves the sphere, and the angle

$$\Delta'_m = m\pi + \theta - (2m+2)\theta' ,$$

when the ray has not left the sphere.

Thus, for an exterior point, the condition $G = 2p\pi$ can be expressed, according to the value of G given above, by

$$\Delta_m = \vartheta + (2p - \tfrac{1}{2} \pm \tfrac{1}{2})\pi ,$$

which is an equation expressing that the rays have rotated the angle ϑ and either an integral number of revolutions, when reading the upper sign for which the intersection with the x-axis takes place on the positive side, or an odd number of half revolutions, when reading the lower sign for which the intersection happens on the negative side of the x-axis.

For an interior point, the condition $G = 2p\pi$ corresponds to either

$$\Delta'_m = -\vartheta' + (2p + \tfrac{1}{2} \pm \tfrac{1}{2})\pi \quad \text{or} \quad \Delta'_m = \vartheta' + (2p - \tfrac{1}{2} \pm \tfrac{1}{2})\pi .$$

The latter case corresponds to the previous one where the intersection of the rays with the main axis was outside the sphere. The former case appears when the rays intersect the positive side of the axis after rotating an integral number of revolutions and the obtuse angle $\pi - \vartheta'$, or by having rotated an odd number of half revolutions and the angle $\vartheta' - \pi$ and intersecting the negative side of the axis, which cannot happen for intersections with the axis outside the sphere.

It is thus seen that each and every case in which a point on the axis can be hit by some of the rays that apart from the central rays fall into the sphere and suffer m reflections are included under the condition $G = 2p\pi$.

When $G - 2p\pi$ for a point cannot be 0 but is a very small quantity, then the point is not hit directly by the straight-lined refracted rays but only by the interfering, diffracted rays.

As mentioned above, when we approach the sphere from an exterior point along the main axis, then shortly after having passed one of the focal points of the central rays we meet an amplitude twice as great as the amplitude in the focal point. From this, the light propagation can be further determined by means of the results found above for an exterior point. Assuming in these results that the angles are very small, we have that

$$-\vartheta + 2\theta - (2m+2)\theta' = 0 , \quad \text{and} \quad 2m + 1 \mp 1 = \text{a multiple of } 4 ,$$

meaning that m is even for the upper sign, odd for the lower sign.

Moreover, we find that

$$A = i\vartheta 4N\frac{(1-N)^m}{(1+N)^{m+2}}\,, \quad F\alpha = kt - a + 2\alpha - (2m+2)\alpha'\,,$$

and by series expansion

$$H = \frac{1}{6\theta}(-\vartheta^3 + 2\theta^3 - (2m+2)\theta'^3)\,.$$

The out-scattering, as determined by (43), will consequently be

$$A\sqrt{\frac{\alpha\pi}{H}}e^{\left(F\alpha+\frac{\pi}{4}\right)i} = i\vartheta 4N\frac{(1-N)^m}{(1+N)^{m+2}}\sqrt{\frac{6\alpha\theta\pi}{-\vartheta^3+2\theta^3-(2m+2)\theta'^3}}e^{\left(F\alpha+\frac{\pi}{4}\right)i}\,.$$

Note now that when, as assumed, the angles are very small, one will according to (77) have $\alpha\theta = \alpha'\theta' = a\vartheta$, from which it follows that the obtained expression becomes exactly twice the out-scattering in the focal point, just as determined in (74). The other term in this last formula is of little significance and is here disregarded. From this, it is seen that the obtained results are valid also for angles so small that they readily follow the previous formulae derived for the central rays. Quite the same is valid for interior points.

When θ or θ' approach the upper limit $\frac{\pi}{2}$, H will approach plus or minus ∞ for both an exterior and an interior point, and the out-scattering determined by (43) will thus converge to 0. When ϑ' approaches $\frac{\pi}{2}$ for an interior point, A will converge to $-(\pm)i\frac{\gamma_{\nu,m}}{2\sqrt{\cos\vartheta'}}$, H to $(\pm)\frac{1}{2\sin\theta\cos\vartheta'}$, and $F\alpha$ to $C + (\pm)\frac{\pi}{4}$, since

$$C = kt + p\pi - \frac{\pi}{4}(2m \mp 1) + \alpha\cos\theta - (2m+1)\alpha'\cos\theta'\,.$$

The formula (43) thus becomes

$$A\sqrt{\frac{\alpha\pi}{H}}e^{\left(F\alpha+\frac{\pi}{4}\right)i} = -(\pm)i\frac{\gamma_{\nu,m}}{2\sqrt{\cos\vartheta'}}\sqrt{\frac{\alpha\pi\ .\ 2\sin\theta\cos\vartheta'}{(\pm)1}}e^{\left(C+\frac{\pi}{4}(1+(\pm)1)\right)i}\,,$$

which equals, both when reading the upper and the lower sign,

$$\tfrac{1}{2}\gamma_{\nu,m}\sqrt{2\pi\alpha\sin\theta}\,e^{Ci}\,.$$

When now a' is assumed to be a point for which ϑ' becomes exactly equal to $\frac{\pi}{2}$, and when one of the two signs $(\pm)$ is associated with a very close point $a' + h$, then the opposite sign will be associated with another point $a' - h$. However, from the result obtained above, it is seen that for both of these very close points the calculated out-scattering becomes the same independently of their distance from the point a'. We infer from this that the obtained formulae remain valid also in the case where ϑ' reaches the actual limit $\frac{\pi}{2}$.

The results produced in this section thus include all the cases where the light rays, after being reflected and refracted an arbitrary number of times, hit the main axis either directly or, in the vicinity of the focal points, by interference. Apart from these cases, we may also question the effect of the diffraction of the rays passing outside the sphere, but these diffraction phenomena only appear in the vicinity of the geometrical shadow edge of the sphere and will be subjected to further examination in the following section.

A general result emerges from what we here developed, namely that the light intensity corresponding to the squared amplitude appears very differently in the various points of the main axis. Sometimes it appears as a quantity of the same order as unity, that is, as the intensity of the incident light, sometimes as a quantity of the order α, namely in the focal points of the central rays and in the axial focal lines of the other rays, and finally as a quantity of the order $\alpha^{\frac{4}{3}}$ in some of the end points of the focal lines. In these last focal points, the intensity would be greater than in any other point on the axis (as well as outside the axis) for an *infinitely* large sphere, but in reality, when we stay within the limits of what is practically possible, the intensity in these points is always considerably less than in the first focal point of the central rays, which corresponds to $m = 0$. Taking $N = 1.5$ as an example, such an exterior focal point only appears after three internal reflections. Setting now $m = 3$, we find that

$$\theta = 73°39'16.6'' , \quad \theta' = 39°46'15.8'' , \quad \vartheta = 9°8'26.8'' ,$$

corresponding to $G = 2\pi$ and $H = 0$. Further assuming that $\alpha = 40000\,\pi$, we find the amplitude 24.681, the intensity 609.14, from the formula (50) when only including the term of the highest order, while the intensity in the first focal point is 217311 and thus multiple times greater as shown earlier.

5 α very large. Propagation outside the main axis.

For the spherical function $P_n(\cos\varphi)$, we have the known expansion

$$\begin{aligned} P_n(\cos\varphi) \quad &= \quad 2\,\frac{1\,.\,3\ldots 2n-1}{2\,.\,4\ldots 2n}\left(\cos n\varphi + \frac{2n}{2n-1}\cdot\frac{1}{2}\cos(n-2)\varphi + \right. \\ &\qquad \left. \frac{2n(2n-1)}{(2n-1)(2n-3)}\cdot\frac{1\,.\,3}{2\,.\,4}\cos(n-4)\varphi + \ldots\right) , \end{aligned}$$

When n is odd, this series ends with the term involving $\cos\varphi$, and when n is even, it ends with a constant term of which we take half.

We now assume here that φ is not 0 or very small, and also that n is a very large number. For the sum of the series one will then obtain, as is well known, the expression already found by Laplace

$$P_n(\cos\varphi) = \sqrt{\frac{2}{\pi n \sin\varphi}} \cos\left((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\right).$$

From this, by discarding quantities of lower order, we further form

$$\frac{dP_n(\cos\varphi)}{d\varphi} = -\sqrt{\frac{2n}{\pi \sin\varphi}} \sin\left((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\right).$$

This value is inserted in the series (31). Since the point under consideration is assumed to lie outside the main axis, it cannot be hit by the central rays, which correspond to $n < n_1$, why the summations here need only be carried out from $n = n_1$ to $n = \infty$. The series can thus be expressed by

$$\left.\begin{aligned}
K &= -\frac{\cos\phi}{a} \sum_{n_1}^{\infty} \frac{2q_n(a)}{\pi n \sin\varphi} \sin\left((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\right) e^{\left(kt - \frac{n\pi}{2} - \lambda_n(a)\right)i}\, 2k_n\,,\\
S &= i\frac{\sin\phi}{a} \sum_{n_1}^{\infty} \frac{2q_n(a)}{\pi n \sin\varphi} \sin\left((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\right) e^{\left(kt - \frac{n\pi}{2} - \lambda_n(a)\right)i}\, 2s_n\,,\\
K' &= i\frac{\cos\phi}{a'} \sum_{n_1}^{\infty} \frac{2q_n(a')}{\pi n \sin\varphi} \sin\left((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\right) e^{\left(kt - \frac{n\pi}{2}\right)i} \sin\lambda_n(a') 2k'_n\,,\\
S' &= \frac{\sin\phi}{a'} \sum_{n_1}^{\infty} \frac{2q_n(a')}{\pi n \sin\varphi} \sin\left((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\right) e^{\left(kt - \frac{n\pi}{2}\right)i} \sin\lambda_n(a') 2s'_n\,.
\end{aligned}\right\} \tag{79}$$

In this section, we limit ourselves to perform the summations until $n = n_2$, that is, until the highest limit for n within which the functions q_n and λ_n can be expressed by the formulae given in (67) and (68).

Using the same procedure as in the previous section, we single out from the series K and S the part corresponding to

$$2k_n = -1\,, \quad 2s_n = -1\,.$$

The terms herein will contain the two exponents

$$\left(kt - \frac{\pi n}{2} - \lambda_n(a) \pm \left((n+\tfrac{1}{2})\,\varphi - \frac{\pi}{4}\right)\right) i\,.$$

Setting $n = \nu + z$ in these, then by expansion in powers of z the coefficients of zi become

$$G = -\vartheta \pm \varphi\,,$$

where the angle ϑ lies between 0 and $\frac{\pi}{2}$, the angle φ between 0 and π, without reaching these limits. For this reason the condition $G = 2p\pi$ will be satisfied only for $p = 0$ and $\vartheta = \varphi$. Assuming this, the sum can be changed into an integral of the form (42), after which by comparison one has for the series K

$$\begin{aligned}
A &= \frac{\cos\phi}{2ai}\sqrt{\frac{2}{\pi a \cos\vartheta \sin\vartheta \sin\varphi}} = -i\frac{\cos\phi}{a\sin\varphi\sqrt{2\pi a\cos\varphi}}\,,\\
F\alpha &= kt - a\cos\varphi - \frac{\pi}{4}\,, \quad H = -\frac{\alpha}{2a\cos\varphi}\,.
\end{aligned}$$

The result of the integration given by (43) is

$$-\frac{\cos\phi}{a\sin\varphi}e^{(kt-a\cos\varphi)i}\,.$$

Due to the equation $\vartheta=\varphi$, it is hereby assumed that we have $a\sin\varphi<\alpha$ and $0<\varphi<\frac{\pi}{2}$. If this is not the case, the result is 0.

Correspondingly, for the series S, we find

$$i\frac{\sin\phi}{a\sin\varphi}e^{(kt-a\cos\varphi)i}\,.$$

By insertion of these two expressions for K and S in the equations (17), discarding the terms of order lower than unity, we have the corresponding part of the components $\overline{\xi}_e$, $\overline{\eta}_e$, $\overline{\zeta}_e$ determined by

$$\overline{\xi}_e=-\sin\varphi\cos\phi e^{(kt-a\cos\varphi)i}\,,^1\quad \overline{\eta}_e=-\cos\varphi\cos\phi e^{(kt-a\cos\varphi)i}\,,\quad \overline{\zeta}=\sin\phi e^{(kt-a\cos\varphi)i}\,,$$

what values are seen to be equal in size to expressions given in the equations (13) for the components of the incident light with opposite sign. Thus, this result merely states that when the reflected and refracted rays are disregarded and the sphere is thus considered entirely black and opaque we will then have complete darkness behind the illuminated sphere outside the main axis and up to a certain distance from it. In the previous section, it was proved that this is also the case on the main axis.

We then single out the term in the first two equations (69) that corresponds to

$$2k_n=-b_ne^{2\lambda_n(\alpha)i}\,,\quad 2s_n=-c_ne^{2\lambda_n(\alpha)i}\,.$$

Inserting these values in the series K and S gives us terms with the two exponents

$$\left(kt-\frac{n\pi}{2}-\lambda_n(a)+2\lambda_n(\alpha)\pm\left((n+\tfrac{1}{2})\varphi-\frac{\pi}{4}\right)\right)i\,,$$

where, by expansion in powers of z, the coefficient of zi becomes

$$G=-\pi-\vartheta+2\theta\pm\varphi\,.$$

Since we must have $\theta\geq\vartheta$, corresponding to $a\geq\alpha$, the condition $G=2p\pi$ can only be satisfied for $p=0$ and when reading the upper sign. Thus,

$$G=-\pi-\vartheta+2\theta+\varphi=0\,.$$

For the sum K, we next get, by comparison with the integral (42), the coefficients

$$A=-i\frac{\cos\phi b_\nu}{a\sqrt{2\pi\alpha\cos\vartheta\sin\theta\sin\varphi}}\,,\quad F\alpha=kt-a\cos\vartheta+2\alpha\cos\theta+\frac{\pi}{4}\,,$$

$$H=\frac{\tan\vartheta+2\tan\theta}{2\sin\theta}\,,$$

[1] In this equation, $\overline{\zeta}_e$ has been corrected to $\overline{\xi}_e$. (Translator's note.)

after which the value of the integral determined by (43) becomes

$$K = \frac{\cos\phi\, b_\nu}{a\sqrt{\cos\vartheta\sin\varphi(-\tan\vartheta + 2\tan\theta)}} e^{(kt - a\cos\vartheta + 2\alpha\cos\theta)i}\,.$$

Correspondingly, we find

$$S = \frac{-i\sin\phi\, c_\nu}{a\sqrt{\cos\vartheta\sin\varphi(-\tan\vartheta + 2\tan\theta)}} e^{(kt - a\cos\vartheta + 2\alpha\cos\theta)i}\,.$$ [1]

As these values are to be inserted in the equations (17) to determine the oscillation components, we first make the following generally valid remarks. When the series (79) for K and S are transformed into integrals, only the exponential arguments come into consideration during differentiation with respect to a and φ when discarding all lower order quantities. These exponents are denoted $F\alpha i$, and we have

$$\frac{d\,F\alpha}{d\nu} = G = 2p\pi\,.$$

Since any multiple of $2\pi i$ may be pulled out of the exponent, one would have $\frac{d\,F\alpha}{d\theta} = 0$ when choosing θ instead of ν as independent variable, whereof it follows that when a is also variable,

$$\frac{d\,F\alpha}{da} = -\cos\vartheta\,.$$

Moreover, φ must enter into $F\alpha$ in such a way that we get

$$\frac{d\,F\alpha}{d\varphi} = \pm(\nu + \tfrac{1}{2}) = \pm a\sin\vartheta\,,$$

the sign corresponding to the sign with which φ enters into $F\alpha$.

One would thus obtain ordinary

$$\overline{\xi}_e = \sin^2\vartheta\, aK\,, \quad \overline{\eta}_e = \pm\sin\vartheta\cos\vartheta\, aK\,, \quad \overline{\zeta}_e = \mp i\sin\vartheta\, aS\,. \tag{80}$$

Applying this to the case computed above gives

$$\begin{aligned}
&\overline{\xi}_e\cos\vartheta - \overline{\eta}_e\sin\vartheta = 0\,,\\
&\overline{\xi}_e\sin\vartheta + \overline{\eta}_e\cos\vartheta = \frac{\cos\phi\, b_\nu\sin\vartheta}{\sqrt{\cos\vartheta\sin\varphi(-\tan\vartheta + 2\tan\theta)}} e^{(kt - a\cos\vartheta + 2\alpha\cos\theta)i}\,,\\
&\overline{\zeta}_e = -\frac{\sin\phi\, c_\nu\sin\vartheta}{\sqrt{\cos\vartheta\sin\varphi(-\tan\vartheta + 2\tan\theta)}} e^{(kt - a\cos\vartheta + 2\alpha\cos\theta)i}\,.
\end{aligned}$$

This part of the light propagation corresponds to the propagation of the light rays reflected from the front surface of the sphere, and the same results can easily be derived by an elementary approach. As θ is the angle of incidence, ϑ the acute angle that the reflected ray forms with the radial vector, the law of reflection yields $-\pi - \vartheta + 2\theta + \varphi = 0$. The reflected light ray has a virtual focal point at the distance $\frac{\alpha}{2}\cos\theta$ (distance measured with

[1] In this equation, $2\alpha\cos\vartheta$ has been corrected to $2\alpha\cos\theta$. (Translator's note.)

$\frac{\lambda}{2\pi}$ as unit of length) from the reflecting surface element. The distance of the considered point from this element is $a\cos\vartheta - \alpha\cos\theta$, and its distance from the focal point is $a\cos\vartheta - \frac{1}{2}\alpha\cos\theta$.

If the considered point is located on the surface of the sphere itself, one has $\vartheta = \theta = \pi - \varphi$, and with the chosen system of axes the components of the incident light are here

$$\overline{\xi}_0 = \sin\varphi\cos\phi\, C\,, \quad \overline{\eta}_0 = \cos\varphi\cos\phi\, C\,, \quad \overline{\zeta}_0 = -\sin\phi\, C\,, \quad C = e^{(kt+\alpha\cos\theta)i}\,.$$

In the plane of incidence, the oscillatory deflection is thus

$$\overline{\eta}_0\cos\theta - \overline{\xi}_0\sin\theta = -\cos\phi\, C\,,$$

which according to Fresnel's laws of reflection can be changed into

$$\frac{\tan(\theta-\theta')}{\tan(\theta+\theta')}\cos\phi\, C = b_\nu\cos\phi\, C\,,$$

while the oscillatory deflection perpendicular to the plane of incidence after reflection becomes

$$\frac{\sin(\theta-\theta')}{\sin(\theta+\theta')}\sin\phi\, C = -c_\nu\sin\phi\, C\,.$$

In the reflected light ray, the intensity must decrease at the same rate as when light spreads over a larger surface element, and the amplitude at the rate of the square root of this surface element.

This surface element is in the considered point determined by

$$\Big(a\cos\vartheta - \frac{\alpha}{2}\cos\theta\Big)\, 2d\theta\, .\, a\sin\varphi\, d\phi\,,$$

which, for $a = \alpha$, corresponding to $\vartheta = \theta = \pi - \varphi$, yields

$$\alpha\cos\theta\, d\theta\, .\, \alpha\sin\theta\, d\phi\,.$$

The ratio between these two elements is

$$\frac{\alpha^2\sin\theta\cos\theta}{(2a\cos\vartheta - \alpha\cos\theta)a\sin\varphi} = \frac{\sin^2\vartheta}{\cos\vartheta\sin\varphi(-\tan\vartheta + 2\tan\theta)}\,,$$

since α and a are eliminated by the equation $a\sin\vartheta = \alpha\sin\theta$.

It is seen that one will in this way arrive at exactly the same result as the one found above.

Finally, inserting the general term of the first two series (69), namely

$$k_n = b_{n,m}e^{2(\lambda_n(\alpha)-(m+1)\lambda_n(\alpha'))i}\,, \quad s_n = c_{n,m}e^{2(\lambda(\alpha)-(m+1)\lambda_n(\alpha'))i}\,,$$

into the series (79) for K and S, the terms will contain the exponents

$$\Big(kt - \frac{n\pi}{2} - \lambda_n(a) + 2\lambda_n(\alpha) - (2m+2)\lambda_n(\alpha') \pm \Big((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\Big)\Big)\, i\,.$$

By expansion of this in powers of z, the coefficients to zi become

$$G = m\pi - \vartheta + 2\theta - (2m+2)\theta' \pm \varphi\,.$$

Herein, $m\pi + 2\theta - (2m+2)\theta' = \Delta_m$ is the angle with which the incident ray has rotated after m internal reflections (p. 31), meaning that the equation can also be written $G = \Delta_m - \vartheta \pm \varphi$. From this, it is seen that the condition $G = 2p\pi$ is satisfied when the angle of incidence θ is chosen so that the ray after m internal reflections hits the considered point, and the upper sign must be read when this point and the incident ray lie on the same side of the main axis. The lower sign when they lie on opposite sides of the main axis.

For the sum K, we next obtain by comparison with the integral (42) the coefficient

$$A = \pm i \frac{2\cos\phi\, b_{\nu,m}}{a\sqrt{2\pi\alpha\cos\vartheta\sin\theta\sin\varphi}}\,,$$

for the sum S, the coefficient

$$A = \pm \frac{2\sin\phi\, c_{\nu,m}}{a\sqrt{2\pi\alpha\cos\vartheta\sin\theta\sin\varphi}}\,,$$

and, for both sums, the coefficients

$$\begin{aligned}
F\alpha &= kt - a\cos\vartheta + 2\alpha\cos\theta - (2m+2)\alpha'\cos\theta' + (p - \tfrac{1}{2}m \mp \tfrac{1}{4})\pi\,,\\
H &= \frac{1}{2\sin\theta}(-\tan\vartheta + 2\tan\theta - (2m+2)\tan\theta')\,,\\
I &= \frac{1}{6\sin^2\theta}(-\tan^3\vartheta + 2\tan^3\theta - (2m+2)\tan^3\theta')\,.
\end{aligned}$$

The result is given in the formula (43) and, in case one has $H = 0$, by the formula (49). In the first case, the deflection with components determined by the equations (80) is of the same order as unity. In the second case ($H = 0$), which represents all the focal surfaces, the deflection will be of the order $\alpha^{\frac{1}{6}}$, and the intensity of the order $\alpha^{\frac{1}{3}}$. Since all quantities of order lower than unity are discarded throughout this calculation, we need only include the first term of the formula (49).

The nature of the light propagation in the vicinity of the focal surfaces appears from the calculations related to the formula (49) and from the following discussion. It is seen hereof that when H approaches 0, which happens when we approach the focal surface from the side reachable by the rectilinearly refracted and m times reflected light rays ($G = 2p\pi$), then the oscillation amplitude will grow through a periodic motion from being of the order α^0 to the order $\alpha^{\frac{1}{6}}$. The last and greatest maximum is reached just before we reach the focal surface itself, after which the amplitude decreases to the magnitude determined by the formula (50), corresponding to the focal surface itself ($H = 0$, $G = 2p\pi$). Afterwards, the amplitude decreases rapidly to 0. In the maximum point closest to the focal surface, the amplitude is 1.504, the intensity 2.262 times larger than in the focal surface.

Since the determination of the light intensity in and in the vicinity of the focal surface is of particular interest, especially with respect to *the theory of the rainbow*, I shall put the formulae for this on a form suitable for numerical computation.

Let $I_m(\varphi)$ denote the light intensity of the rays reflected m times from the inner surface of the sphere in the point determined by φ, ϕ, a. The amplitude is determined by the equations (80), after which we find that the intensity, the square of the amplitude, is expressed by

$$I_m(\varphi) = a^2 \sin^2 \vartheta \text{ Ampl. } (K^2 + S^2) .$$

According to the general formula (49), from which we only include the first term, we have

$$\text{Ampl. } K^2 = \frac{4\alpha^{\frac{4}{3}}}{9I^{\frac{2}{3}}} Q^2 A^2 , \quad \text{where } A^2 = \frac{2\cos^2\phi\, b_{\nu,m}^2}{a^2 \alpha\pi \cos\vartheta \sin\theta \sin\varphi} ,$$

$$\text{Ampl. } S^2 = \frac{4\alpha^{\frac{4}{3}}}{9I^{\frac{2}{3}}} Q^2 A^2 , \quad \text{where } A^2 = \frac{2\sin^2\phi\, c_{\nu,m}^2}{a^2 \alpha\pi \cos\vartheta \sin\theta \sin\varphi} .$$

If the incident light is unpolarised, which we would assume in the following, the intensity is the mean of all values corresponding to values of ϕ from 0 to 2π. We therefore set

$$\cos^2\phi\, b_{\nu,m}^2 + \sin^2\phi\, c_{\nu,m}^2 = \tfrac{1}{2}(b_{\nu,m}^2 + c_{\nu,m}^2) ,$$

after which, with the value of I given above, we get

$$I_m(\varphi) = \frac{4\alpha^{\frac{1}{3}} Q^2 \sin^2\vartheta}{9\pi \sin\varphi \cos\vartheta \sin\theta} \left(\frac{6\sin^2\theta}{-\tan^3\vartheta + 2\tan^3\theta - (2m+2)\tan^3\theta'} \right)^{\frac{2}{3}} (b_{\nu,m}^2 + c_{\nu,m}^2) .$$

Introducing two new terms p and p' given by

$$\tan\theta = p\tan\theta' , \quad N^2 p' = p ,$$

we have

$$b_{\nu,m} = 2N\cos\theta\cos\theta' \frac{(N\cos\theta - \cos\theta')^m}{(N\cos\theta + \cos\theta')^{m+2}} = 2p' \tfrac{(1-p')^m}{(1+p')^{m+2}} ,$$

$$c_{\nu,m} = 2N\cos\theta\cos\theta' \frac{(\cos\theta - N\cos\theta')^m}{(\cos\theta + N\cos\theta')^{m+2}} = 2p \tfrac{(1-p)^m}{(1+p)^{m+2}} .$$

Moreover, the angles θ, θ' and ϑ are determined by

$$\sin\theta = N\sin\theta' = \sqrt{\frac{p^2 - N^2}{p^2 - 1}} , \quad \tan\vartheta = 2(p - m - 1)\tan\theta' ,$$

just like one also has

$$a\sin\vartheta = \alpha\sin\theta , \quad \alpha\lambda = 2\pi R , \quad a\lambda = 2\pi r ,$$

where R is the radius of the sphere and r the distance of the point from the centre, both measured like λ with an arbitrary unit of length. In addition (see p. 17),

$$Q = 3\left(\frac{\pi}{2}\right)^{\frac{1}{3}} W .$$

By means of these substitutions, the intensity formula can be given the form

$$I(\varphi) = \frac{W^2}{\sin\varphi} C_m , \tag{a}$$

where C_m is independent of φ and determined by

$$C_m = \frac{R^2}{r^2}\cdot\frac{48p^2(N^2-1)}{\cos\vartheta\,(p-1)}\left(\frac{R(p^2-N^2)^{\frac{1}{2}}}{6\lambda(p^2-1)^{\frac{1}{2}}(p^3-4(p-m-1)^3-m-1)^2}\right)^{\frac{1}{3}}\left(p'^2\frac{(1-p')^{2m}}{(1+p')^{2m+4}}+p^2\frac{(1-p)^{2m}}{(1+p)^{2m+4}}\right), \tag{b}$$

$$\cos\vartheta = \frac{p\sqrt{N^2-1}}{\sqrt{p^2(N^2-1)+4(p-m-1)^2(p^2-N^2)}}\,.$$

The quantity W entering into the formula (a) is determined by

$$W = \int_0^\infty \cos\frac{\pi}{2}(\omega^3 - m'\omega)\,d\omega\,,$$

where m' depends on φ in the following way. We assume φ_0 to be the value of φ corresponding to the focal surface, and it is thus determined by

$$G = m\pi - \vartheta + 2\theta - (2m+2)\theta' \pm \varphi_0 = 2p_1\pi\,,$$

where p_1 is an integer. The sign for φ_0, which lies between 0 and π, is determined by the equation itself.

Now, setting $\varphi = \varphi_0 \mp \delta$, we have $G - 2p_1\pi = -\delta$, but according to (46)

$$G - 2p_1\pi = -\varepsilon\left(\frac{I}{\alpha^2}\right)^{\frac{1}{3}}\,,\quad \text{where } \varepsilon = \left(\frac{\pi}{2}\right)^{\frac{2}{3}} m'\,.$$

When also inserting the given value of I, we thus obtain

$$\delta = \left(\frac{\pi}{2}\right)^{\frac{2}{3}} m'\left(\frac{-\tan^3\vartheta + 2\tan^3\theta - (2m+2)\tan^3\theta'}{6\alpha^2\sin^2\theta}\right)^{\frac{1}{3}},$$ [1]

and, with the substitutions used above,

$$\delta = m'\left(\frac{\lambda^2(p^2-1)(p^3-4(p-m-1)^3-m-1)(p^2-N^2)^{\frac{1}{2}}}{48R^2p^3(N^2-1)^{\frac{3}{2}}}\right)^{\frac{1}{3}}. \tag{c}$$

In case a can be considered infinitely large (*the rainbow*), one has $\vartheta = 0$ and $p = m+1$, whereby the formulae (b) and (c) are reduced to

$$C_m = \frac{R^2}{r^2}\cdot\frac{48p^2(N^2-1)}{p^2-1}\left(\frac{R(p^2-N^2)^{\frac{1}{2}}}{6\lambda p^2(p^2-1)^{\frac{5}{2}}}\right)^{\frac{1}{3}}\left(p'^2\frac{(1-p')^{2m}}{(1+p')^{2m+4}}+p^2\frac{(1-p)^{2m}}{(1+p)^{2m+4}}\right), \tag{b'}$$

$$\delta = m'\left(\frac{\lambda^2(p^2-1)^2(p^2-N^2)^{\frac{1}{2}}}{48R^2p^2(N^2-1)^{\frac{3}{2}}}\right)^{\frac{1}{3}}. \tag{c'}$$

As mentioned on p. 17, the equation $W = 0$, which corresponds to $I_m(\varphi) = 0$ results in a series of values for m' of which the qth is determined by $m' = 3(q-\frac{1}{4})^{\frac{2}{3}}$ for sufficiently large values of q. This corresponds to

$$\delta = \frac{1}{4}\left(\frac{9}{4}\right)^{\frac{1}{3}}\left[\frac{(p^2-1)^2(p^2-N^2)^{\frac{1}{2}}}{p^2(N^2-1)^{\frac{3}{2}}}\right]^{\frac{1}{3}}\left(\frac{\lambda}{R}(4q-1)\right)^{\frac{2}{3}},$$

which is the result on a form recently derived in an elementary manner and presented by *M. Boitel*.[2]

[1] In this and the remaining equations of this page, the symbol ∂ was corrected to δ. (Translator's note.)

[2] Journ. de phys. S. II, t. 8, p. 282. 1889.

One difference, though, is that *Boitel* has $\tan\delta$ on the left-hand side of the equation instead of δ. On the other hand, *Mascart*[1] has used the formula $\delta = A(q - \frac{1}{4})^{\frac{2}{3}}$ in the calculation during experiments with a glass rod and found a good agreement between experiment and calculation even for fairly large values of $\delta(9°)$.

The intensity in the focal surface itself ($m' = 0$) is determined by

$$I_m(\varphi) = \frac{\Gamma(\frac{1}{3})^2}{12} \left(\frac{2}{\pi}\right)^{\frac{2}{3}} \frac{C_m}{\sin\varphi_0}\,,$$

of which the actual *maximum intensity* corresponding to $m' = 1.0845$ can be found, with sufficient approximation (since the value of φ corresponding to this value of m' generally differs only very slightly from φ_0), through multiplication with 2.262. In this way, I have calculated the maximum intensity for a couple of examples.

We assume $R = 10$ mm, $N = 1.5$, $\lambda = 0.0005$ mm, and $m = 1$. For an external point very close to the surface of the sphere, we have $r = R$, $\vartheta = \theta$, $\tan\theta = 4\tan\theta'$, and thus $p = 4$ and $p' = \frac{16}{9}$. Since we determine C_m by the formula (b), we find that for these numerical values the maximum intensity equals 4.5423. As this intensity is proportional to $R^{\frac{1}{3}}$, it is seen hereof that even for spheres much smaller, until almost 100 times smaller, the intensity will exceed 1. At a distance of half a radius from the surface of the sphere, we have $r = 1.5\,R$, $\vartheta = \theta'$, $p = \frac{5}{2}$, $p' = \frac{10}{9}$, to which corresponds a maximum intensity of 0.9423.

From these results, it appears that for almost all practical cases of transparent spheres there will be places outside the sphere that are illuminated *just as strongly* by the directly incident light as by the light from the other side, where this is strongest, after having reflected *once* off the inner surface of the sphere. As such places are presumably easily located experimentally, and as they can be determined theoretically too by the communicated formulae, good means have hereby been provided for checking the agreement between experiment and calculation.

To mention another example, consider a spherical drop of water with the refractive index $\frac{4}{3}$. For $m = 1$ and a infinitely large, we here find

$$\text{maxium intensity} = 0.06728\,\frac{R^2}{r^2}\left(\frac{R}{\lambda}\right)^{\frac{1}{3}}.$$

Taking for comparison another sphere of the same size but perfectly reflective, the intensity of the light reflected from the front surface will be $\frac{R^2}{4\,r^2}$ at the same distance. These two intensities will then be equal in magnitude when we have $R = 51.30\,\lambda$, which for $\lambda = 0.000585$ mm gives $R = 0.03$ mm. For a raindrop with a radius 8 times as

[1] Comptes rendus de l'Académie des Sciences, t. 106, p. 1575. 1888.

large, the maximum intensity of the light reflected *once* from the inner surface will be twice the intensity one would obtain if the raindrop were replaced by a perfectly reflective sphere of the same size.

Instead of using a single sphere, we now imagine a collection of equally sized separated spheres, all equally strongly illuminated by parallel incident and unpolarised light rays. We set their intensity to 1. The spheres are assumed to be positioned so closely or in a layer of such a large extent that the lines of sight of a distant observer will everywhere hit one of the spheres. The complete collection of spheres, lying within a cone with its apex in the eye of the observer, and which embraces the unit of solid angle, will then emit light whose intensity in the apex of the cone is $\frac{r^2}{R^2\pi}$ times larger than the intensity due to a single sphere. If we let *apparent brightness* denote the intensity of the light that within unit of solid angle hits the observer's eye, then for such a collection of spherical raindrops of refractive index $\frac{4}{3}$ we have

$$\text{Max. of apparent brightness} = 0.06728\,\frac{1}{\pi}\left(\frac{R}{\lambda}\right)^{\frac{1}{3}} .$$

For a similar collection of perfectly reflecting spheres, one would independently of the size of the spheres obtain an apparent brightness of $\frac{1}{4\pi}$. In the comparison, however, it must here be noted that all the light entering the system through a single reflection is returned by new reflections, why the apparent brightness should here be doubled or set equal to $\frac{1}{2\pi}$. Assuming this, the two systems would for monochromatic light or if observed through a solid coloured glass be seen with the same apparent brightness when the radius of the raindrops is 8 times as large as calculated above, that is, when it is 0.24 mm.

For the collection of raindrops here considered, the optical phenomena correspond to the *fully developed rainbows*. For the individual spectral colours, the computation of the apparent brightness of *these* and the *supernumerary rainbows* can now be carried out using the formulae (a), (b'), (c') in combination with a table of the integral W. We finally posit an example that allows an observational check. Based on calculation, the second rainbow appearing after two internal reflections has an apparent brightness 7.864 times smaller than that of the first rainbow, assuming of course that they are formed under the same conditions.

The propagation of light in the *interior* of a sphere can be calculated by means of the series for K' and S' (79), wherein we set

$$k'_n = \beta_{n,m} e^{(\lambda_n(\alpha)-(2m+1)\lambda_n(\alpha'))i}\,, \quad s'_n = \gamma_{n,m} e^{(\lambda_n(\alpha)-(2m+1)\lambda_n(\alpha'))i}\,.$$

The following four exponents will appear in the terms

$$\left(kt - \frac{n\pi}{2} + (\pm)\lambda_n(a') + \lambda_n(\alpha) - (2m+1)\lambda_n(\alpha') \pm \left((n + \tfrac{1}{2})\varphi - \frac{\pi}{4}\right)\right) i \,,$$

which by expansion in powers give as coefficient for zi

$$G = (2m-1)\frac{\pi}{2} + (\pm)\left(\vartheta' - \frac{\pi}{2}\right) + \theta - (2m+1)\theta' \pm \varphi \,.$$

Herein, $m\pi + \theta - (2m+1)\theta' = \Delta'_m$ is the angle that the incident ray has rotated after m internal reflections. The condition $G = 2p\pi$ determines the two double sign more closely. As is the case for an external point, the upper sign for φ corresponds to the case where the considered point and the incident ray lie on the same side of the main axis, and, moreover, ϑ' and φ have the same sign when the ray hitting the considered point cuts through the positive side of the main axis, but opposite signs when the intersection is at the negative side of the main axis.

By comparison with the integral (42), we next have for the series K' and S' the respective coefficients

$$A = \mp(\pm)\frac{i\cos\phi}{a'\sqrt{2\pi\alpha\cos\vartheta'\sin\theta\sin\varphi}} \quad \text{and} \quad A = \mp(\pm)\frac{\sin\phi}{a'\sqrt{2\pi\alpha\cos\vartheta'\sin\theta\sin\varphi}} \,.$$

and for both series

$$\begin{aligned}
F\alpha &= kt + (\pm)\left(a'\cos\vartheta' + \frac{\pi}{4}\right) + \alpha\cos\theta - (2m+1)\alpha'\cos\theta' + \left(p - \frac{1}{2}m + \frac{1}{4} \mp \frac{1}{4}\right)\pi \,, \\
H &= \frac{1}{2\sin\theta}\left((\pm)\tan\vartheta' + \tan\theta - (2m+1)\tan\theta'\right) \,, \\
I &= \frac{1}{6\sin^2\theta}\left((\pm)\tan^3\vartheta' + \tan^3\theta - (2m+1)\tan^3\theta'\right) \,.
\end{aligned}$$

In order to determine the oscillation components $\overline{\xi}'$, $\overline{\eta}'$, $\overline{\zeta}'$, we make use of the equations analogous to (80)

$$\overline{\xi}' = \sin^2\vartheta'\, a'K' \,, \quad \overline{\eta}' = \mp(\pm)\sin\vartheta'\cos\vartheta'\, a'K' \,, \quad \overline{\zeta}' = \mp i\, a'\sin\vartheta'\, S' \,. \tag{81}$$

The light propagation is thus determined everywhere, as to the extent that it suffices to carry out the summations with respect to n without crossing the limit $n = n_2$. It is hereby assumed that we can use the formulae (67) and (68) for q_n and λ_n, which in turn determine the functions v_n and w_n. To exceed this limit for n, it will be necessary to find other expansions for these functions, which I shall cover in the following section.

It should be noted that when ϑ' reaches the limit $\frac{\pi}{2}$ in *isolated* internal points, the propagation can also in this case be computed by means of the formulae given here, why the proof can be done in the same way as in the corresponding case treated previously (p. 32), where the point was located on the main axis.

6 Continuation. Total reflection, Diffraction.

The functions v_n and w_n can also be derived in a way different from the one previously used (p. 22), by means of an otherwise quite similar expansion. We have identically that

$$v_n = \sqrt{v_n w_n} e^{\frac{1}{2}\log\frac{v_n}{w_n}}\,, \quad w_n = \sqrt{v_n w_n} e^{-\frac{1}{2}\log\frac{v_n}{w_n}}\,.$$

Setting

$$v_n w_n = r_n\,, \quad \frac{1}{2}\log\frac{v_n}{w_n} = \mu_n\,,$$

we will thus have

$$v_n = \sqrt{r_n} e^{\mu_n}\,, \quad w_n = \sqrt{r_n} e^{-\mu_n}\,. \tag{82}$$

By applying the equation $w_n v_n' - w_n v_n = 1$, one would further obtain, when a denotes the variable,

$$\frac{d\mu_n}{da} = \frac{1}{2r_n}\,, \tag{83}$$

from which, by integration and with introduction of the value of μ_n corresponding to $a = 0$,

$$\mu_n = \frac{1}{2}\log\frac{a^{2n+1}}{1^2.3^2\ldots(2n-1)^2(2n+1)} + \int_0^a da\left(\frac{1}{2r_n} - \frac{2n+1}{2a}\right)\,. \tag{84}$$

Furthermore, by multiplication, the series (22) and (24) for v_n and w_n give

$$2r_n = \frac{2a}{2n+1} + \frac{(2a)^3}{(2n-1)(2n+1)(2n+3)}\cdot\frac{1}{2} + \frac{(2a)^5}{(2n-3)(2n-1)\ldots(2n+5)}\cdot\frac{1\,.\,3}{2\,.\,4} + \ldots\,. \tag{85}$$

The correctness of the law here suggested for the series can also be proven by forming the differential equation for r_n. Assuming that u_n satisfies the differential equation (21), one could in general set

$$u_n = \sqrt{p_n} e^{c\int\frac{da}{p_n}}\,, \tag{86}$$

which inserted into (21) leads to the equation

$$p_n\frac{d^2p_n}{da^2} - \frac{1}{2}\left(\frac{dp_n}{da}\right)^2 + \left(1 - \frac{n(n+1)}{a^2}\right)2p_n^2 + 2c^2 = 0\,, \tag{87}$$

and, by another differentiation, we obtain the linear equation

$$\frac{d^3p_n}{da^3} + 4\left(1 - \frac{n(n+1)}{a^2}\right)\frac{dp_n}{da} + \frac{4n(n+1)}{a^3}p_n = 0\,. \tag{88}$$

The equation (86) corresponds to the equations (82) for $p_n = r_n$ and $c = \pm\frac{1}{2}$, and it also corresponds to the equations (63) for $p_n = q_n$ and $c = \pm i$. Thus, the last equation (88) must be satisfied both for $p_n = q_n$ and for $p_n = r_n$, and it is then not difficult to verify the correctness of the laws in the series for q_n and r_n by means of this equation.

We consider both n and a to be large numbers, both of the order of magnitude of α. Furthermore, when we as previously in the summation of the series q_n disregard all quantities of order lower than unity, the summation of the series (85) can then under certain conditions be performed by

$$2r_n = \frac{a}{\sqrt{(n+\frac{1}{2})^2 - a^2}} \,. \tag{89}$$

The condition must consist in a not exceeding a certain limit, but by closer inspection of the series, one soon discovers that determination of this limit poses certain difficulties. The terms of the series will for $a < n$ first decrease, reach a minimum and then grow, get an alternating sign and reach a maximum, and then finally descent to 0. Thus, the term before the first negative term is already of the magnitude

$$\frac{(2a)^{2n+1}}{1\,.\,3 \ldots 4n+1} \cdot \frac{1\,.\,3 \ldots 2n-1}{2\,.\,4 \ldots 2n} \,,$$

which for $ea > 2n+1$, e.g. $a = 0.75\,n$, grows indefinitely with growing n.

It is therefore necessary to put the series for r_n on a different form. By means of the equation

$$\frac{1\,.\,2\,.\,3 \ldots 2m}{(2n-2m+1)(2n-2m+3)\ldots(2n+2m+1)} = (-1)^m \int_0^{\frac{\pi}{2}} dx\, \sin(2n+1)x \sin^{2m} x \,,$$

the series (85) can be given the form

$$2r_n = 2a \int_0^{\frac{\pi}{2}} dx\, \sin(2n+1)x \left(1 - \frac{a^2}{1^2} \sin^2 x + \frac{a^4}{1^2\,.\,2^2} \sin^4 x - \ldots\right),$$

and with the use of Bessel function J_0

$$2r_n = 2a \int_0^{\frac{\pi}{2}} dx\, \sin(2n+1)x J_0(2a \sin x) \,. \tag{90}$$

We first perform this integration from $x = 0$ to $x = h$, assuming h so small that we can insert x in place of $\sin x$ without noticeable error as long as x is smaller than h. By introducing a new variable $y = (2n+1)x$, this part of the integral thus becomes

$$\frac{a}{n+\frac{1}{2}} \int_0^{(2n+1)h} dy\, \sin y\, J_0\!\left(\frac{ay}{n+\frac{1}{2}}\right) = \frac{a}{n+\frac{1}{2}} \int_0^{(2n+1)h} dy\, \sin y \left(1 - \left(\frac{ay}{n+\frac{1}{2}}\right)^2 \cdot \frac{1}{2^2} + \left(\frac{ay}{n+\frac{1}{2}}\right)^4 \frac{1}{2^2\,.\,4^2} - \ldots\right). \tag{91}$$

The upper limit of this integral can be regarded just as the kind of indefinite, arbitrary quantities that we have denoted by the common mark ω, and the integration can therefore be done using the formula (39). The result is the series

$$\frac{a}{n+\frac{1}{2}} + \left(\frac{a}{n+\frac{1}{2}}\right)^3 \frac{1}{2} + \left(\frac{a}{n+\frac{1}{2}}\right)^5 \frac{1\,.\,3}{2\,.\,4} + \cdots = \frac{a}{\sqrt{(n+\frac{1}{2})^2 - a^2}} \,,$$

where the only convergence condition is $a < n + \frac{1}{2}$.

In the second part of the integral (90), we can expand the Bessel function in decreasing powers of a according to the well-known semi-convergent series

$$J_0(2a\sin x) = \frac{1}{\sqrt{\pi a \sin x}} \cos\Big(2a\sin x - \frac{\pi}{4}\Big) + \dots ,$$

where the orders of magnitude of the terms are those of $\alpha^{-\frac{1}{2}}, \alpha^{-\frac{3}{2}}, \dots$

Leaving out the subsequent terms of the series for J_0, this part of the integral becomes

$$\frac{a}{n+\frac{1}{2}} \int_{(2n+1)h}^{(2n+1)\frac{\pi}{2}} dy \; \frac{\sin y \cos\big(2a\sin\frac{y}{2n+1} - \frac{\pi}{4}\big)}{\sqrt{\pi a \sin\frac{y}{2n+1}}} =$$

$$\sqrt{\frac{a}{(2n+1)\pi}} \int_{(2n+1)h}^{(2n+1)\frac{\pi}{2}} dy \; \frac{\sin\Big(\Big(1+\frac{a}{n+\frac{1}{2}}\Big)y - \frac{ay^3}{24(n+\frac{1}{2})^3} + \dots - \frac{\pi}{4}\Big) + \sin\Big(\Big(1-\frac{a}{n+\frac{1}{2}}\Big)y + \frac{ay^3}{24(n+\frac{1}{2})^3} - \dots + \frac{\pi}{4}\Big)}{\sqrt{y - \frac{y^3}{24(n+\frac{1}{2})^2} + \dots}} . \tag{92}$$

From this, it is seen that as long as the difference $n+\frac{1}{2}-a$ is of the order of α, then this part of the integral is of order lower than unity, and since an assumption for the equation (89) is that we omit these quantities, the last equation remains valid when the difference $n+\frac{1}{2}-a$ is just positive and of the order of α. This condition thus, when interchanging a and $n+\frac{1}{2}$, quite corresponds to the one applying to q_n in equation (67).

Setting in equation (84), as n is assumed very large,

$$1^2 \,.\, 3^2 \dots (2n-1)^2(2n+1) = 2(2n+1)^{2n+1} e^{-(2n+1)} ,$$

we obtain by means of (89)

$$\mu_n = -\tfrac{1}{2}\log 2 + (n+\tfrac{1}{2}) \log \frac{n+\frac{1}{2} - \sqrt{(n+\frac{1}{2})^2 - a^2}}{a} + \sqrt{(n+\tfrac{1}{2})^2 - a^2} . \tag{93}$$

We are thus able to determine the functions v_n and w_n both for $n+\frac{1}{2} > a$ and for $n+\frac{1}{2} < a$, in the first case by means of r_n and μ_n and in the second case by q_n and λ_n. However, there is still an area left where these functions cannot be determined by the established formulae, namely when the difference $n+\frac{1}{2}-a$, whether it is positive or negative, is of an order lower than α.

While we have so far sought summation of all occurring series with such accuracy that only the quantities of an order lower than unity are discarded, we will in the following reduce the accuracy so that only the terms of the highest order are included. Assuming this, one will when $n+\frac{1}{2}-a$ is of an order lower than α be able to discard all the quantities of the same order as unity when determining r_n, since r_n will itself turn out to be a quantity of higher order. Thus, when we consider the chosen limit $(2n+1)h$ as a quantity of the order of α^0, the entire integral (91) can be discarded, as the two functions appearing in the integral, $\sin$ and J_0, cannot for any value of the variable become numerically larger than 1. Moreover, the second part of the integral, determined by (92), reduces to

$$\sqrt{\frac{a}{(2n+1)\pi}}\int_{(2n+1)h}^{(2n+1)\frac{\pi}{2}} dy\, \frac{\sin\left((1-\frac{a}{n+\frac{1}{2}})y+\frac{ay^3}{24(n+\frac{1}{2})^3}-\cdots+\frac{\pi}{4}\right)}{\sqrt{y-\frac{y^3}{24(n+\frac{1}{2})^2}+\ldots}}\,, \tag{94}$$

where again the lower limit can be changed to 0, since here too the integral from 0 to $(2n+1)h$ cannot give a result of order higher than unity, while the upper limit for x, after the substitution $ay^3 = 24(n+\frac{1}{2})^3x$, just like before can be denoted ω. When discarding all terms leading to results of a lower order, we thus obtain

$$2r_n(a) = \frac{a^{\frac{1}{3}}}{3^{\frac{5}{6}}\sqrt{\pi}}\int_0^{\omega} dx\, x^{-\frac{5}{6}} \sin\left((n+\tfrac{1}{2}-a)\left(\frac{24}{a}\right)^{!\frac{1}{3}} x^{\frac{1}{3}} + x + \frac{\pi}{4}\right). \tag{95}$$

From this, by expansion in powers of $n+\frac{1}{2}-a$ and integration by means of the equation (39), we obtain

$$2r_n(a)=\frac{a^{\frac{1}{3}}}{3^{\frac{5}{6}}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{6}\right)\sin\frac{\pi}{3}+\Gamma\left(\frac{3}{6}\right)\sin\frac{3\pi}{3}\cdot(n+\tfrac{1}{2}-a)\left(\frac{24}{a}\right)^{\frac{1}{3}}\frac{1}{1}+\Gamma\left(\frac{5}{6}\right)\sin\frac{5\pi}{3}\cdot(n+\tfrac{1}{2}-a)^2\left(\frac{24}{a}\right)^{\frac{2}{3}}\frac{1}{1\,.\,2}+\ldots\right]. \tag{96}$$

In this series, we the 2nd, 5th, 8th, … terms become equal to 0.

Setting $a = n+\frac{1}{2}$, for example, we obtain

$$2r_n(n+\tfrac{1}{2}) = c(n+\tfrac{1}{2})^{\frac{1}{3}}\,, \quad c = \frac{\Gamma(\frac{1}{6})}{3^{\frac{5}{6}}\sqrt{\pi}}\cdot\frac{\sqrt{3}}{2} = 1.08874\,, \quad \operatorname{Log} c = 0.0369226\,. \tag{97}$$

By inserting the series (23) and (25) for v_n and w_n in $r_n = v_n w_n$, I have computed the table below, which exhibits a surprisingly good agreement between the true values of $r_n(n+\frac{1}{2})$ and those calculated using the formulas (97), already at the lowest values of n.

$n =$		0,	1,	2,	3,	4,	5,	6,
$2r_n(n+\frac{1}{2})$	=	0.8415,	1.2416,	1.4756,	1.6518,	1.7967,	1.9212,	2.0314,
$c(n+\frac{1}{2})^{\frac{1}{3}}$	=	0.8641,	1.2463,	1.4776,	1.6530,	1.7975,	1.9218,	2.0319.

Notice that when $n+\frac{1}{2}-a$ is of higher order than $\alpha^{\frac{1}{3}}$, the order of magnitude of the terms will be growing. However, with this precondition and with the substitution $\left(1-\frac{a}{n+\frac{1}{2}}\right)y = x$, the integral (94) reduces to

$$2r_n = \sqrt{\frac{a}{(2n+1-2a)\pi}}\int_0^{\omega}\frac{dx}{\sqrt{x}}\sin\left(x+\frac{\pi}{4}\right) = \frac{a}{\sqrt{2a(n+\frac{1}{2}-a)}}\,, \tag{98}$$

which shows that we could now again proceed with the simpler formula (89) for $2r_n$ as this leads to the same result when only taking the quantities of the highest order into consideration. With this reduction in accuracy, the formula (89) remains valid as long as the difference $n + \frac{1}{2} - a$ is of a higher order than $\alpha^{\frac{1}{3}}$. When the difference $n + \frac{1}{2} - a$ is not of lower order than α, then $r_n(a)$ is never of higher order than unity. Since, if the difference is positive, this is evident from equation (89), and if the difference is negative, the same result is seen by using equations (23) and (25) to express v_n and w_n in the equation $2r_n = v_n w_n$. On the other hand, if the difference $n + \frac{1}{2} - a$ is of lower order than α, then $r_n(a)$ can be of higher order than unity, and by variation of n this function will according to (96) reach its highest value for $n + \frac{1}{2} = a$.

When $n + \frac{1}{2} - a$ is of lower order than α, the general term of the series (66) can be determined by

$$\frac{(n-m+1)(n-m+2)\dots(n+m)}{a^{2m}} \cdot \frac{1\,.\,3\dots 2m-1}{2\,.\,4\dots 2m} = \frac{e^{-2m}(n+\frac{1}{2}+m)^{n+\frac{1}{2}+m}}{\sqrt{\pi m}\, a^{2m}(n+\frac{1}{2}-m)^{n+\frac{1}{2}-m}}\,.$$

By changing from summation to integration, one obtains

$$q_n(a) = \int_0^n \frac{dm}{\sqrt{\pi m}} e^{F(m)}\,, \quad F(m) = -2m + m\log\frac{(n+\frac{1}{2})^2 - m^2}{a^2} + (n+\frac{1}{2})\log\frac{n+\frac{1}{2}+m}{n+\frac{1}{2}-m}\,,$$

or, by expansion in powers of m,

$$F(m) = -2m\log\frac{a}{n+\frac{1}{2}} - 2\left(\frac{m^3}{(n+\frac{1}{2})^2}\cdot\frac{1}{2\,.\,3} + \frac{m^5}{(n+\frac{1}{2})^4}\cdot\frac{1}{4\,.\,5} + \dots\right).$$

Next, setting $m^3 = 3(n+\frac{1}{2})^2 x$ the integral can with sufficient accuracy be reduced to

$$q_n(a) = \frac{(n+\frac{1}{2})^{\frac{1}{3}}}{3^{\frac{5}{6}}\sqrt{\pi}} \int_0^\infty dx\, x^{-\frac{5}{6}} e^{-(24)^{\frac{1}{3}}(n+\frac{1}{3})^{\frac{2}{3}} \log\frac{a}{n+\frac{1}{2}}\cdot x^{\frac{1}{3}} - x}\,.$$

Herein, again with sufficient accuracy, we set $\log\frac{a}{n+\frac{1}{2}} = \frac{a-n-\frac{1}{2}}{n+\frac{1}{2}}$ after which the integration leads to the result

$$q_n(a) = \frac{(n+\frac{1}{2})^{\frac{1}{3}}}{3^{\frac{5}{6}}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{6}\right) + \Gamma\left(\frac{3}{6}\right)(n+\tfrac{1}{2}-a)\left(\frac{24}{n+\frac{1}{2}}\right)^{\frac{1}{3}}\frac{1}{1} + \Gamma\left(\frac{5}{6}\right)(n+\tfrac{1}{2}-a)^2\left(\frac{24}{n+\frac{1}{2}}\right)^{\frac{2}{3}}\frac{1}{1\,.\,2} + \dots\right]. \tag{99}$$

Inserting here $a = n + \frac{1}{2}$, we get with the same meaning of c as above

$$q_n(n+\frac{1}{2}) = \frac{2}{\sqrt{3}}c(n+\tfrac{1}{2})^{\frac{1}{3}}\,, \quad \mathrm{Log}\,\frac{2}{\sqrt{3}}c = 0.0993920\,. \tag{100}$$

Here too a good agreement is found with the exact values of $q_n(n+\frac{1}{2})$ calculated directly from the series (66) already at the lowest values of n, as shown by the following table.

$$
\begin{array}{lcccccccc}
n = & & 0\,, & 1\,, & 2\,, & 3\,, & 4\,, & 5\,, & 6\,, \\
q_n(n+\frac{1}{2}) & = & 1.0000\,, & 1.4444\,, & 1.7104\,, & 1.9121\,, & 2.0783\,, & 2.2215\,, & 2.3482\,, \\
\frac{2c}{\sqrt{3}}(n+\frac{1}{2})^{\frac{1}{3}} & = & 0.9978\,, & 1.4391\,, & 1.7062\,, & 1.9087\,, & 2.0755\,, & 2.2191\,, & 2.3462\,.
\end{array}
$$

In analogy with r_n, we can express q_n with limited accuracy by equation (**??**) as long as the difference $a - (n + \frac{1}{2})$ is of higher order than $\alpha^{\frac{1}{3}}$. However, contrary to r_n, q_n with growing n has a continually growing value.

Based on the values of r_n and q_n found in this way, we can calculate λ_n and μ_n as well as v_n and w_n. From the equations $2r_n = 2v_n w_n = q_n \sin 2\lambda_n$, we find $\sin 2\lambda_n(n+\frac{1}{2}) = \sin\frac{\pi}{3}$, out of which the values $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}, \ldots$ appear for $\lambda_n(n+\frac{1}{2})$, but if $\lambda_n(n+\frac{1}{2})$ is further determined by the equations $v_n = \sqrt{q_n}\sin\lambda_n$, $w_n = \sqrt{q_n}\cos\lambda_n$ for $n = 0, 1, 2, 3\ldots$, we respectively find

$$\lambda_n(n+\tfrac{1}{2}) = 0.5\,, \quad 0.5165\,, \quad 0.5203\,, \quad 0.5215\ldots\,.$$

Apparently this series converges to the lowest of the values stated above, namely to

$$\lambda_n(n+\tfrac{1}{2}) = \frac{\pi}{6} = 0.5236\ldots\,. \tag{101}$$

From this, we find again by means of the equations $v_n^2 = r_n e^{2\mu_n} = q_n \sin^2\lambda_n$

$$\mu_n(n+\tfrac{1}{2}) = -\tfrac{1}{4}\log 3\,. \tag{102}$$

As one has $\lambda_n'(a) = \frac{1}{q_n(a)}$ and $\mu_n'(a) = \frac{1}{2r_n(a)}$, the series expansions for $\lambda_n(a)$ and $\mu_n(a)$ become the following, where we for the sake of brevity let $q, r, q', \ldots$ denote $q_n(n+\frac{1}{2})$, $r_n(n+\frac{1}{2})$, $q_n'(n+\frac{1}{2})$, etc.,

$$\lambda_n(a) = \frac{\pi}{6} + \frac{1}{q}\frac{a-n-\frac{1}{2}}{1} - \frac{q'}{q^2}\cdot\frac{(a-n-\frac{1}{2})^2}{1\,.\,2} + \ldots\,, \tag{103}$$

$$\mu_n(a) = -\frac{1}{4}\log 3 + \frac{1}{2r}\frac{a-n-\frac{1}{2}}{1} - \frac{r'}{2r^2}\cdot\frac{(a-n-\frac{1}{2})^2}{1\,.\,2} + \ldots\,. \tag{104}$$

Herein, q', r', and the higher derivatives of $q_n(a)$ and $r_n(a)$ with respect to a for $a = n+\frac{1}{2}$ can be calculated from the equations (99) and (96). In this way we find $q' = -\frac{2}{\sqrt{3}}$, $r' = \frac{r}{3(n+\frac{1}{2})}$, where the latter value is only of the order $\alpha^{-\frac{2}{3}}$ and may therefore be considered to be 0.

The functions v_n and w_n themselves can be determined using the equations $(v_n \pm w_n)^2 = q_n \pm 2r_n$, since the signs for v_n and w_n that are here undecided are further determined by $v_n = \sqrt{q_n}\sin\lambda_n$, $w_n = \sqrt{q_n}\cos\lambda_n$, where $\sqrt{q_n}$ is positive. The series expansions I have found in this way by means of the series expansions (96) and (99), wherein the two quantities $n+\frac{1}{2}$ and a can be considered equal when outside the difference $n+\frac{1}{2}-a$, are

$$v_n(a) = C\left(\Gamma\left(\frac{1}{3}\right)\cos\frac{\pi}{6} + \Gamma\left(\frac{2}{3}\right)\cos\frac{5\pi}{6}\cdot\frac{\varepsilon}{1} + \Gamma\left(\frac{3}{3}\right)\cos\frac{9\pi}{6}\cdot\frac{\varepsilon^2}{1\,.\,2} + \ldots\right), \tag{105}$$

$$w_n(a) = C\left(\Gamma\left(\frac{1}{3}\right)\left(1+\sin\frac{\pi}{6}\right) + \Gamma\left(\frac{2}{3}\right)\left(1+\sin\frac{5\pi}{6}\right)\frac{\varepsilon}{1} + \Gamma\left(\frac{3}{3}\right)\left(1+\sin\frac{9\pi}{6}\right)\frac{\varepsilon^2}{1\,.\,2} + \ldots\right), \tag{106}$$

where

$$C = \left(\frac{a}{6}\right)^{\frac{1}{6}}\frac{1}{\sqrt{3}\pi}, \quad \varepsilon = \left(\frac{6}{a}\right)^{\frac{1}{3}}(n+\tfrac{1}{2}-a)\,.$$

These series can also easily be led back to the definite integrals

$$v_n(a) = C\int_0^\omega dx\, x^{-\frac{2}{3}}\cos(\varepsilon x^{\frac{1}{3}}+x)\,, \tag{107}$$

$$w_n(a) = C\left[\int_0^\infty dx\, x^{-\frac{2}{3}}e^{\varepsilon x^{\frac{1}{3}}-x} + \int_0^\omega dx\, x^{-\frac{2}{3}}\sin(\varepsilon x^{\frac{1}{3}}+x)\right]. \tag{108}$$

By inserting the series (105) and (106) in $(v_n \pm w_n)^2 = q_n \pm 2r_n$, one will without difficulties be able to convince oneself of the correctness of these expansions. For use in this calculation, I shall here state the equations

$$\Gamma\left(\frac{1}{3}\right)^2 = 2^{\frac{1}{3}}\sqrt{\frac{\pi}{3}}\,\Gamma\left(\frac{1}{6}\right), \quad \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = 2\sqrt{\frac{\pi}{3}}\,\Gamma\left(\frac{1}{2}\right), \quad \Gamma\left(\frac{2}{3}\right)^2 = 2^{\frac{2}{3}}\sqrt{\frac{\pi}{3}}\,\Gamma\left(\frac{5}{6}\right).$$

We now proceed to the calculation which was interrupted in the previous section and consider first case where the sphere has a refractive index smaller than the one of the surrounding medium. We thus assume $N < 1$ from which follows that the equation $\alpha\sin\theta = \alpha'\sin\theta'$ is impossible for $\sin\theta > N$.

In the equations (33) and (34), we now set $v_n(\alpha') = \sqrt{r_n(\alpha')}e^{\mu_n(\alpha')}$, while $v_n(\alpha)$ and $w_n(\alpha)$ as previously are expressed by $q_n(\alpha)$ and $\lambda_n(\alpha)$, and just like q_n' can be discarded in comparison with q_n, r_n' can in the same way also be discarded in comparison with r_n. Since $q_n(\alpha)$ is determined by (67), $r_n(\alpha)$ by (89), we obtain

$$2k_n = -1 + e^{2\lambda_n(\alpha)i}\frac{q_n - 2r_n(\alpha')Ni}{q_n + 2r_n(\alpha')Ni} = -1 + e^{2\lambda_n(\alpha)i}\frac{\sqrt{(n+\frac{1}{2})^2-\alpha'^2} - \sqrt{\alpha^2-(n+\frac{1}{2})^2}N^2 i}{\sqrt{(n+\frac{1}{2})^2-\alpha'^2} + \sqrt{\alpha^2-(n+\frac{1}{2})^2}N^2 i},$$

and setting

$$\frac{\sqrt{\alpha^2-(n+\frac{1}{2})^2}}{\sqrt{(n+\frac{1}{2})^2-\alpha'^2}}N^2 = \tan\delta\,,$$

the expression gets the simpler form

$$2k_n = -1 + e^{2(\lambda_n(\alpha)-\delta)i}\,.$$

In a similar way, we obtain

$$2s_n = -1 + e^{2(\lambda_n(\alpha)-\Delta)i}, \quad \tan\Delta = \frac{\sqrt{\alpha^2-(n+\frac{1}{2})^2}}{\sqrt{(n+\frac{1}{2})^2-\alpha'^2}}\,.$$

The case of only having $2k_n = -1$, $2s_n = -1$, has already been dealt with in the previous section (p. 34). There, it was generally assumed that the functions q_n and λ_n could be expressed for all occurring variables by the formulae given in the equations (67) and (68). However, it should be noted that, as far as this particular case is concerned where k_n and s_n do not contain the variables α and α' at all, we only have to deal with the functions $q_n(a)$ and $\lambda_n(a)$, and the condition for them to be expressed by (67) and (68) is solely $\nu + \frac{1}{2} = a \sin\vartheta < a$. The results found are thus valid until the distance a from the main axis, and, as we recall, the propagation of light outside the sphere that is generated in this way consists of the incident light itself on the negative side of the yz-plane and total darkness on the positive side of the yz-plane.

Next assuming

$$2k_n = e^{2(\lambda_n(\alpha)-\delta)i}\,, \quad 2s_n = e^{2(\lambda_n(\alpha)-\Delta)i}\,,$$

and setting here in the usual way $n = \nu + z$, we note that expanding $\lambda_{\nu+z}(\alpha)$ in powers of z yields coefficients to the various powers of z of a higher order of magnitude than those obtained by the corresponding expansion of δ and Δ. As we set $\nu + \frac{1}{2} = \alpha \sin\theta$, we can express δ and Δ by the constant values

$$\tan\delta = \frac{\cos\theta}{\sqrt{\sin^2\theta - N^2}}\,, \quad \tan\Delta = \frac{\cos\theta}{\sqrt{\sin^2\theta - N^2}}\,.$$

The expressions for k_n and s_n now correspond closely to the case dealt with previously (p. 35), by which we determined the reflection from the outer surface. The only differences are that the factors b_ν and c_ν have now become -1 and that the phase has diminished in K with 2δ and in S with 2Δ, and the results found previously would thus with these changes still find application.

The limiting case $\sin\theta = N$ does not form a special exception since δ and Δ go to $\frac{\pi}{2}$ for θ decreasing toward this limit, and the factors $e^{-2\delta i}$ and $e^{-2\Delta i}$ thus become equal to -1. In this way, K and S will take on the same values as those presented by the previous formulae for θ increasing toward the same limit.

The coefficients k'_n and s'_n are determined by

$$k'_n = e^{\lambda_n(\alpha)i - \mu_n(\alpha')}\frac{2N\sqrt{q_n(\alpha)r_n(\alpha')}}{q_n(\alpha) + 2r_n(\alpha')Ni}\,, \quad s'_n = e^{\lambda_n(\alpha)i - \mu_n(\alpha')}\frac{2N\sqrt{q_n(\alpha)r_n(\alpha')}}{Nq_n(\alpha) + 2r_n(\alpha')i}\,.$$

In addition, as $n > \alpha'$ for an interior point must correspond to also $n > a'$, one must set $\sqrt{q_n(a')}\sin\lambda_n(a') = \sqrt{r_n(a')}e^{\mu_n(a')}$[1] in the series for K' and S' (79). It is thus seen that these series will contain the factor $e^{\mu_n(a')-\mu_n(\alpha')}$, which becomes a vanishingly small quantity when a' and α' are not very nearly equal. This is clear from the expression for μ_n given in (93), which is seen to be, when the variable is not very close to n,

[1]In this inline equation, $\mu_n(\alpha')$ has been replaced with $\mu_n(a')$. (Translator's note.)

a very large negative quantity and even larger the smaller the variable. The light propagation within the totally reflecting part of the sphere thus only happens noticeably in a thin layer immediately below the surface of the sphere.

Setting $a' = \alpha' - Nh$ and assuming h very small, one has

$$\mu_n(\alpha') - \mu_n(a') = \frac{Nh}{2r_n(\alpha')} = \frac{h}{\alpha}\sqrt{(n+\tfrac{1}{2})^2 - \alpha'^2}\,.$$

One will then in the usual way find

$$\begin{aligned} K' &= \frac{2N\cos\phi}{\alpha\sqrt{1-N^2}\tan\theta\,\sqrt{\sin^2\theta - N^2\cos^2\theta}} e^{\left(kt+\alpha\cos\theta+\frac{\pi}{2}-\delta\right)i - h\sqrt{\sin^2\theta - N^2}}\,, \\ S' &= -i\frac{2\sin\phi}{\alpha\sqrt{1-N^2}\tan\theta} e^{\left(kt+\alpha\cos\theta+\frac{\pi}{2}-\Delta\right)i - h\sqrt{\sin^2\theta - N^2}}\,, \end{aligned}$$

and $\varphi + \theta = \pi$. To determine the components $\overline{\xi}', \overline{\eta}', \overline{\zeta}'$, we must return to the equations (18), from which we notice that, since K' and S' originally contained the factor $e^{\mu_n(a')}$, by discarding the quantities of lower order we have

$$\frac{dK'}{da'} = \frac{d\mu_n(a')}{da'}K' = \frac{\sqrt{(n+\frac{1}{2})^2 - a'}}{a'}K' = \frac{\sqrt{\sin^2\theta - N^2}}{N}K'\,.$$

Moreover, we obtain

$$\frac{dK'}{d\varphi'} = (n+\tfrac{1}{2})K'i = \alpha\sin\theta K'i\,,$$

and the same equations are valid also when K' is replaced by S'. For this case, the equations (18) yield

$$\overline{\xi}' = \frac{\sin^2\theta}{N}aK'\,,\quad \overline{\eta}' = i\frac{\sin\theta\sqrt{\sin^2\theta - N^2}}{N}\alpha K'\,,\quad \overline{\zeta}' = -i\sin\theta\,\alpha S'\,,$$

where the values of K' and S' found above can be inserted.

The results of this calculation of the total reflection, with respect to the external as well as the internal points, turn out to agree with what is known from the theory of total reflection for planar surfaces, and the calculation thus does not go beyond what one could also derive by elementary means.

What is left is only to continue the summations of the series K and S (79) from the limit of n at which the equations (67) and (68) are no longer valid for the variable α. The value of k_n given in (33) can in all cases be transformed to

$$2k_n = -1 + Ae^{2\lambda_n(\alpha)i}\,,\quad A = \frac{q_n(\alpha)(1+r_n'(\alpha')) - N(i+\frac{1}{2}q_n'(\alpha))2r_n(\alpha')}{q_n(\alpha)(1+r_n'(\alpha')) - N(-i+\frac{1}{2}q_n'(\alpha))2r_n(\alpha')}\,.$$

When n exceeds the mentioned limit, this fraction denoted A turns out to be equal to 1, assuming that N is different from 1. We shall here omit the case where $N-1$ is so small that this difference can be regarded as a quantity of order lower than unity.

The equation $A = 1$ always holds true when $q'_n(\alpha)$ is of a higher order than unity, which according to (99) is the case when $n-\alpha$ is positive and of a higher order than $\alpha^{\frac{1}{3}}$. Moreover, n is so large in the considered sum that $q_n(\alpha)$ is of higher order than unity, while $r_n(\alpha')$ and $r'_n(\alpha')$ cannot be of higher order than unity when the difference $n-\alpha$, whether positive or negative, is of lower order than α. The latter is evident from what was previously stated (p. 48), as one has $n - a' = n - \alpha - (N-1)\alpha$, where the last term cannot be lower order than α. We thus see that in the current case we always have $A = 1$. Since the very same considerations apply to the value of s_n given in (33), we thus have

$$2k_n = -1 + e^{2\lambda_n(\alpha)i}\,, \quad 2s_n = -1 + e^{2\lambda_n(\alpha)i}\,.$$

Both these coefficients converge rapidly to 0 for $n > \alpha$ with growing n.

Since we with respect to the case $2k_n = -1\,, \quad s_n = -1$ can refer to the foregoing, we have for our consideration the series

$$Q = \frac{aK}{\cos\phi} = \frac{iaS}{\sin\phi} = -\sum_{n_2}^{n_3}\sqrt{\frac{2q_n(a)}{\pi n \sin\varphi}}\sin\Big((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\Big)\, e^{\left(kt - \frac{n\pi}{2} + 2\lambda_n(\alpha) - \lambda_n(a)\right)i}\,,$$

where n_3 is the upper limit of n within which $q_n(a)$ and $\lambda_n(a)$ can be determined using (67) and (68).

The exponential argument in this sum is

$$\left(kt - \frac{n\pi}{2} + 2\lambda_n(\alpha) - \lambda_n(a) \pm \Big((n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\Big)\right)i\,,$$

and setting herein $n = \nu + z$ and $\nu + \frac{1}{2} = a\sin\vartheta$, the coefficient of z, when omitting quantities smaller than unity, becomes only $-\vartheta \pm \varphi$. For this coefficient to be 0 or very small, we must use the upper sign and $\varphi - \vartheta$ must be 0 or very small. From this, we see that the oscillation components according to (80) can be determined by

$$\bar{\xi}_e = \sin^2\varphi\cos\phi\, Q\,, \quad \bar{\eta}_e = \sin\varphi\cos\varphi\cos\phi\, Q\,, \quad \bar{\zeta}_e = -\sin\varphi\sin\phi\, Q\,,$$

from which we obtain the components with respect to the fixed axes

$$\xi_e = 0\,, \quad \eta_e = \sin\varphi\, Q\,, \quad \zeta_e = 0\,.$$

Since $\varphi - \vartheta$ is very small and one can therefore outside the exponent set $q_n = \frac{1}{\cos\vartheta} = \frac{1}{\cos\varphi}$ and $n = \alpha\sin\vartheta = a\sin\varphi$, the quantity $\sin\varphi\, Q$ itself reduces to

$$\sin\varphi\, Q = \eta_e = \frac{i}{\sqrt{2\pi a\cos\varphi}}\sum_{n_2}^{n_3} e^{F_n i}\,, \quad F_n = kt - \frac{n\pi}{2} + 2\lambda_n(\alpha) - \lambda_n(a) + (n+\tfrac{1}{2})\varphi - \frac{\pi}{4}\,,$$

whereby the composite phenomenon comprising the diffraction of parallel rays of light by a reflecting sphere is presented in a simple form.

We first consider the part of the sum where n is greater than α and see that $\lambda_n(\alpha)$ decreases from $\frac{\pi}{6}$ to 0 for growing n. A closer determination of this is obtained from the equations

$$e^{2\lambda_n(\alpha)i} = \frac{1+\tan\lambda_n(\alpha)i}{1-\tan\lambda_n(\alpha)i} = \frac{1+e^{2\mu_n(\alpha)i}}{1-e^{2\mu_n(\alpha)i}} = 1 + 2\sum_0^\infty e^{2m\mu_n(\alpha)}i^m ,$$

where $\mu_n(\alpha)$ for $n = \alpha$ has the value $-\frac{1}{4}\log 3$ and rapidly decreases with growing n.

Setting thus in the considered sum $e^{2\lambda_n(\alpha)i} = 1$ and inserting in the exponent in the usual way $n = \nu + z$, $\nu + \frac{1}{2} = a\sin\vartheta$, the coefficients of zi in the exponent become $\varphi - \vartheta$ through expansion in powers of z. In this way, for $\varphi = \vartheta$, the sum becomes the integral

$$\int_{\alpha - a\sin\vartheta}^{n_3 - a\sin\vartheta} dz\, e^{\left(kt - a\cos\varphi - \frac{\pi}{4} - \frac{z^2}{2a\cos\varphi}\right)i} = -i\sqrt{2\pi a\cos\varphi}\,\eta_e ,$$

which by the substitution

$$z = \left(x - \frac{\varepsilon}{2}\right)\sqrt{2a\cos\varphi}, \quad \frac{\varepsilon}{2} = \frac{a\sin\varphi - \alpha}{\sqrt{2a\cos\varphi}} ,$$

yields

$$\eta_e = \frac{i}{\sqrt{\pi}} e^{\left(kt - a\cos\varphi - \frac{\pi+\varepsilon^2}{4}\right)i} \int_0^\omega dx\, e^{(\varepsilon x - x^2)i} ,$$

where the integral corresponds to the integral (57) when the sign for i is flipped. It emerges from the investigation of this last integral that for $\varepsilon > 0$, that is, the point located beyond the geometrical shadow boundary of the sphere ($a\sin\varphi > \alpha$), the integral is a periodic function. Conversely, it becomes aperiodic within the shadow boundary ($\varepsilon < 0$). Along the shadow boundary ($\varepsilon = 0$), we have

$$\eta_e = \tfrac{1}{2} e^{(kt - a\cos\varphi)i} .$$

The result is in any case the same as the one for the diffraction of light at a planar circular disk inserted in place of the sphere in the great circle to which the incident rays are tangent.

The other part of the sum considered above is

$$2\sum_{m=0}^{m=\infty}\sum_{\alpha}^{n_3} e^{\left(kt - \lambda_n(a) + (n+\frac{1}{2})\varphi - (2n-2m+1)\frac{\pi}{4}\right)i + 2m\mu_n(\alpha)} .$$

Setting in this $n = \nu + z$, $\nu + \frac{1}{2} = \alpha = a\sin\vartheta$, and using the expansion (104) for $\mu_n(\alpha)$, the coefficient of z in the exponent becomes $(\varphi - \vartheta)i - \frac{m}{r}$ through expansion in powers of z, where $r = r_\nu(\nu + \frac{1}{2})$ is determined by (97) and is of the order of $\alpha^{\frac{1}{3}}$.

If now $\varphi - \vartheta$ is of a higher order than $\alpha^{-\frac{1}{3}}$, the considered sum, when including only the terms of the highest order, can be expressed by

$$\sum_{m=0}^{m=\infty} \frac{1}{\varphi - \vartheta} e^{\left(kt - a\cos\vartheta + \alpha(\varphi-\vartheta) + (2m+1)\frac{\pi}{4}\right)i - \frac{m}{2}\log 3},$$

which is of a lower order than $\alpha^{\frac{1}{3}}$.

On the other hand, if the considered point lies so close to the geometric shadow boundary of the sphere that $\varphi - \vartheta$ becomes of the same order as $\alpha^{-\frac{1}{3}}$ or even lower, then all terms in the expansion of the exponent in powers of z need to be considered. However, by the substitution $z = rx$ they would all become of the order of α^0, and the entire integral becomes of the same order as r, meaning the order of $\alpha^{\frac{1}{3}}$. The oscillation amplitude corresponding to this case can then be expressed by

$$C \frac{\alpha^{\frac{1}{3}}}{\sqrt{a\cos\vartheta}},$$

where C is a numerical constant. A more detailed calculation of this constant is hardly of sufficient interest as it is quickly seen that this part of the light propagation can be very minute, and since it coincides with the other diffracted light it will hardly be the subject of the observation. The formula shows that the intensity of this light is proportional to the radius of the sphere lifted to the power of $\frac{2}{3}$ and to the wavelength lifted to the power of $\frac{1}{3}$. In addition, it is inversely proportional to the distance of the considered point from the great circle to which the incident rays are tangent, assuming though that this distance does not become very small.

Finally, the oscillatory deflection corresponding to $n < \alpha$ is determined by

$$\eta_e = \frac{i}{\sqrt{2\pi a\cos\varphi}} \sum_{n_2}^{\alpha} e^{Fni},$$

and during this summation $\lambda_n(\alpha)$ decreases with growing n from an indefinite large value to $\frac{\pi}{6}$. Setting $n = \nu - z$, $\nu + \frac{1}{2} = \alpha = a\sin\vartheta$, we obtain

$$\eta_e = \frac{i}{\sqrt{2\pi a\cos\varphi}} \int_0^{\omega} dz\, e^{\left(kt - a\cos\vartheta + \alpha(\varphi-\vartheta) + 2\lambda_{\nu-z}(\alpha) - \frac{\pi}{4} - (\varphi-\vartheta)z\right)i},$$

where $\lambda_{\nu-z}(\alpha)$ can be expanded according to (103). We now see that this case corresponds closely to the one treated above and that the result can be presented in the same form. This part of the light propagation corresponds to the diffraction of the rays of light totally reflected at grazing incidence. The intensity of these latter rays decreases as the angle of incidence increases. However, because of the diffraction, this intensity will not be zero in the geometric shadow boundary. It will rather be of a magnitude similar to the intensity of the diffracted rays considered above, and the intensity then rapidly decreases within the shadow boundary.

The summation with respect to n has only been carried out until the upper limit $n = n_3$, but as mentioned above the coefficients k_n and s_n for $n > \alpha$ will rapidly converge to 0 for growing n. This part of the sums will therefore generally be vanishingly small.

7 Amount of out-scattered light. α very small. System of small spheres.

From the illuminated sphere, imagine all out-scattered light being collected on the inner side of a concentric spherical surface at an infinite distance from the sphere. Let L be the total amount of collected light, r the radius of the infinite sphere, and I the light intensity in the distance r as measured by the square of the amplitude. Then L is defined and determined by

$$L = r^2 \int_0^\pi \sin\varphi \, d\varphi \int_0^{2\pi} d\phi \, I \, . \tag{109}$$

According to the equations (17) and (31) the oscillation components for $a = \frac{2\pi r}{\lambda}$ and r infinite can be expressed by

$$\begin{aligned}
\overline{\xi}_e = 0\,, \quad \overline{\eta}_e &= -\frac{i\cos\phi}{a} e^{(kt-a)i} \sum_1^\infty \frac{2n+1}{n(n+1)} \left(k_n \frac{d^2 P_n}{d\varphi^2} + s_n \frac{dP_n}{\sin\varphi \, d\varphi} \right), \\
\overline{\zeta}_e &= \frac{i\sin\phi}{a} e^{(kt-a)i} \sum_1^\infty \frac{2n+1}{n(n+1)} \left(k_n \frac{dP_n}{\sin\varphi \, d\varphi} + s_n \frac{d^2 P_n}{d\varphi^2} \right).
\end{aligned}$$

Herein, k_n and s_n are complex quantities whose moduli we denote $\overline{k_n}$ and $\overline{s_n}$. Now, determining I by the sum of the squares of the amplitudes of these components, equation (109) yields, once the integration with respect to ϕ is done,

$$L = \frac{\lambda^2}{4\pi} \int_0^\pi \sin\varphi \, d\varphi \left[\left(\sum_1^\infty \frac{2n+1}{n(n+1)} \left(\overline{k_n} \frac{d^2 P_n}{d\varphi^2} + \overline{s_n} \frac{dP_n}{\sin\varphi \, d\varphi} \right) \right)^2 + \left(\sum_1^\infty \frac{2n+1}{n(n+1)} \left(\overline{k_n} \frac{dP_n}{\sin\varphi \, d\varphi} + \overline{s_n} \frac{d^2 P_n}{d\varphi^2} \right) \right)^2 \right].$$

Each of these squares can also be expressed as a product of two sums with the variables n and m, and noting that one has

$$\int_0^\pi \sin\varphi \, d\varphi \left(\frac{d^2 P_n}{d\varphi^2} \cdot \frac{d^2 P_m}{d\varphi^2} + \frac{1}{\sin^2\varphi} \frac{dP_n}{d\varphi} \frac{dP_m}{d\varphi} \right) = \begin{cases} 0 & \text{for} \quad m \neq n \\ \frac{2n^2(n+1)^2}{2n+1} & \text{for} \quad m = n\,, \end{cases}$$

$$\int_0^\pi d\varphi \left(\frac{d^2 P_n}{d\varphi^2} \cdot \frac{dP_m}{d\varphi} + \frac{dP_n}{d\varphi} \cdot \frac{d^2 P_m}{d\varphi^2} \right) = 0\,,$$

one finds the amount of light L to be determined by

$$L = \frac{\lambda^2}{2\pi} \sum_1^\infty (2n+1) \left(\overline{k_n^2} + \overline{s_n^2} \right). \tag{110}$$

The general expressions (33) for the coefficients k_n and s_n can also be written in the form

$$k_n = -\frac{1}{1+p_n i}\,, \qquad p_n = \frac{w_n(\alpha)v_n'(\alpha') - Nw_n'(\alpha)v_n(\alpha')}{v_n(\alpha)v_n'(\alpha') - Nv_n'(\alpha)v_n(\alpha')}\,, \tag{111}$$

$$s_n = -\frac{1}{1+q_n i}\,, \qquad q_n = \frac{Nw_n(\alpha)v_n'(\alpha') - w_n'(\alpha)v_n(\alpha')}{Nv_n(\alpha)v_n'(\alpha') - v_n'(\alpha)v_n(\alpha')}\,. \tag{112}$$

The moduli of these coefficients are less than 1 except in those cases where one has $p_n = 0$, which corresponds to $k_n = -1$, or where $q_n = 0$, which corresponds to $s_n = -1$.

We will now more closely examine the light propagation in the case where the diameter of the illuminated sphere is very small in comparison with the wavelength of the incident light, so that we may consider α such a small number that in series expansions in powers of α, as a rule, only the term with the lowest power of α is included. As far as α' is concerned, for the moment we make no restrictive assumption.

According to the series expansions (22) and (24) we have, when only the first term of the series are included,

$$\begin{aligned} v_n(\alpha) &= \frac{\alpha^{n+1}}{1\,.\,3\ldots 2n+1}\,, & v_n'(\alpha) &= \frac{(n+1)\alpha^n}{1\,.\,3\ldots 2n+1}\,, \\ w_n(\alpha) &= \frac{1\,.\,3\ldots 2n-1}{\alpha^n}\,, & w_n'(\alpha) &= -n\frac{1\,.\,3\ldots 2n-1}{\alpha^{n+1}}\,. \end{aligned}$$

Inserting these values in (111) and (112) it is seen that generally k_n and s_n become very small quantities of the order of α^{2n+1}. As one will have

$$\begin{aligned} p_n &= \frac{1^2\,.\,3^2\ldots(2n-1)^2(2n+1)}{\alpha^{2n+1}}\cdot\frac{\alpha'v_n'(\alpha') + N^2 n v_n(\alpha')}{\alpha'v_n'(\alpha') - N^2(n+1)v_n(\alpha')}\,, \\ q_n &= \frac{1^2\,.\,3^2\ldots(2n-1)^2(2n+1)}{\alpha^{2n+1}}\cdot\frac{\alpha'v_n'(\alpha') + n v_n(\alpha')}{\alpha'v_n(\alpha') - (n+1)v_n(\alpha')}\,, \end{aligned}$$

where, in the latter expression, one can also set

$$\alpha'v_n'(\alpha') + nv_n(\alpha') = \alpha'v_{n-1}(\alpha')\,, \quad \alpha'v_n'(\alpha') - (n+1)v_n(\alpha') = -\alpha'v_{n+1}(\alpha')\,.$$

Apart from the special cases, the series (31) for K and S thus confine themselves to the first term corresponding to $n = 1$, which includes

$$k_1 = i\frac{\alpha^3}{3}\cdot\frac{\alpha'v_1'(\alpha') - 2N^2v_1(\alpha')}{\alpha'v_1'(\alpha') + N^2v_1(\alpha')}\,, \quad s_1 = -i\frac{\alpha^3}{3}\frac{v_2(\alpha')}{v_0(\alpha')}\,,$$

after which the oscillation components $\overline{\xi}_e$, $\overline{\eta}_e$, $\overline{\zeta}_e$ can easily be determined by means of the equations (17).

If now α' like α is a very small quantity, k_1 can be reduced to the form

$$k_1 = -i\frac{2\alpha^3}{3}\cdot\frac{N^2-1}{N^2+2}\,,$$

while for α' very small, or when α' is a root of the equation $v_2(\alpha') = 0$, we obtain $s_1 = 0$.

In this last case, $\overline{\xi}_e$ will always according to the equations (17) be proportional to $\cos\varphi$, from which follows that the out-scattered light orthogonal to the incident rays oscillate perpendicularly to the plane of incidence and is thus completely polarised in the plane of incidence. Of course, this is also valid when the incident light is unpolarised.

Likewise, the same law must be valid under the same conditions when we, instead of a single sphere, imagine a collection of similar, mutually separated, and randomly ordered spheres. If we furthermore in the expression for k_1 set $\alpha = \frac{2\pi R}{\lambda}$, where R is the radius of the sphere, it is seen that the light propagation in an arbitrary point outside the sphere beyond the incident light and the coordinates of the point depends only on the quantity $\frac{N^2-1}{N^2+2}R^3$. We now imagine that the radii of these spheres, with their centre positions unchanged, grow until R_1, which must still be very small as compared with a wavelength, while their refractive indices change from N to N_1. If this change happens in a ways so that we keep

$$\frac{N^2-1}{N^2+2}R^3 = \frac{N_1^2-1}{N_1+2}R_1^3\,,$$ [1]

then the light propagation outside the spheres and everywhere outside the system remains unaffected by the change. If we let R_1 become as large as the smallest half mean distance between the sphere centres, which is assumed very small compared with the wavelength, then the system very closely resembles a homogeneous medium with refractive index N_1. From this, we in turn conclude that when the spheres in the system remain unchanged while the density d_1 of the system changes, then the refractive index N_1 will change in such a way that $\frac{N_1^2-1}{N_1^2+1}\frac{1}{d_1}$ remains constant (cf. “Farvespredningens Theori”).

The total amount of out-scattered light from a single sphere is according to (110) determined by

$$L = \frac{2\lambda^2\alpha^6}{3\pi}\left(\frac{N^2-1}{N^2+2}\right)^2,$$

and if A is the number of spheres within a unit of volume, then AL is the total amount of light out-scattered from each unit of volume of the system. This quantity is the absorption coefficient of the system, and denoting this by h when also expressing α by $\frac{2\lambda R}{\lambda}$, we have

$$h = AL = A\frac{128\pi^5R^6}{3\lambda^4}\left(\frac{N^2-1}{N^2+2}\right)^2\,,\quad A = \frac{3}{4\pi R_1^3}\,.$$

From which it is seen that the absorption coefficient is inversely proportional to the fourth power of the wavelength (Rayleigh’s law[2]). Conversely, if the absorption coefficient h of the system and its

[1] N^2+1 in the denominator on the left-hand side has been corrected to N^2+2. (Translator’s note.)

[2] J. W. Strutt: Phil. Mag. 41, Febr., Apr., Jun. 1871.

refractive index N_1 are given, the number of spheres per unit of volume and a lower limit on their sizes can be derived under the given assumptions, as we find from the given formulae

$$A = \frac{24\pi^3}{h\lambda^4}\left(\frac{N^2-1}{N^2+2}\right)^2, \quad R^3 = \frac{h\lambda^4}{32\pi^4}\frac{(N_1^2+2)(N^2+2)}{(N_1^2-1)(N^2-1)} > \frac{h\lambda^4}{32\pi^4}\frac{N_1^2+2}{N_1^2-1}\,.$$

As an example we take the refractive index and the absorption coefficient of atmospheric air at normal pressure, namely $N_1 = 1.00029$[1] and, with 10^{-6} mm as length unit, $h\lambda^4 = 0.0017$. With this latter coefficient, 11.3 per cent of light at wavelength 580 will be absorbed over a distance of 8 km, twice as much for $\lambda = 480$.

Inserting these numerical values above, we get

$$A = 0.0163\,, \quad R = 0.141\left(\frac{N^2+2}{N^2-1}\right)^{\frac{1}{3}} > 0.141\,,$$

so, in a cubic millimetre, there is a number of spheres $0.0163\cdot 10^{18}$ with a radius of at least 0.141 mm.[2] Corresponding to this, $\alpha = 0.00153$ for $\lambda = 580$ and $\alpha = 0.00185$ for $\lambda = 480$.

There is a widely different kind of light propagation occurring in the special cases where one has $p_n = 0$ or $q_n = 0$, which is possible for an entire series of wavelengths. According to the equations (111) and (112), this correspond to

$$w_n(\alpha)v_n'(\alpha') - Nw_n'(\alpha)v_n(\alpha') = 0\,, \quad Nw_n(\alpha)v_n'(\alpha') - w_n'(\alpha)v_n(\alpha') = 0\,.$$

The first of these equations corresponds approximately to $v_n(\alpha') = 0$, the other to $v_{n-1}(\alpha') = 0$. More precisely, setting $\alpha' = \beta + \varepsilon$ in the first equation, and if β is a root of the equation $v_n(\beta) = 0$, then by expansion in powers of ε and by discarding the terms containing powers of ε larger than the first, we obtain

$$\varepsilon = \frac{w_n(\alpha)}{Nw_n'(\alpha)} = -\frac{\alpha}{Nn}\,.$$

If $q_{n+1} = 0$ is the given equation, then corresponding to this, when including the first two terms in the expansion of $w_{n+1}(\alpha)$ and $w_{n+1}'(\alpha)$,

$$N\alpha\left(1+\frac{\alpha^2}{2(2n+1)}\right)v_{n+1}'(\alpha') + \left(n+1+\frac{(n-1)\alpha^2}{2(2n+1)}\right)v_{n+1}(\alpha') = 0\,,$$

where

$$v_{n+1}(\alpha') = -v_n'(\alpha') + \frac{n+1}{\alpha'}v_n(\alpha') \quad \text{and} \quad v_{n+1}'(\alpha') = -\left(\frac{(n+1)^2}{\alpha'^2}-1\right)v_n(\alpha') + \frac{n+1}{\alpha'}v_n'(\alpha')\,.$$

From this, with the chosen degree of approximation, we find

$$(2n+1)\alpha' v_n(\alpha') + \alpha^2 v_n'(\alpha') = 0\,.$$

Now, setting here $\alpha' = \beta + \varepsilon'$ while still having $v_n(\beta) = 0$, we obtain

$$\varepsilon' = -\frac{\alpha^2}{(2n+1)\alpha'} = -\frac{\alpha}{N(2n+1)}\,.$$

[1] $N_1 = 0.00029$ has been corrected to $N_1 = 1.00029$. (Translator's note.)
[2] 0.141^{-6} has been corrected to 0.141. (Translator's note.)

The roots of $p_n = 0$ and $q_{n+1} = 0$ are thus very close but not exactly equal, and the difference between two corresponding roots is

$$\varepsilon' - \varepsilon = \frac{\alpha(n+1)}{Nn(2n+1)}\,, \quad n > 0\,.$$

Let δ and δ' denote the corresponding changes of the wavelength, then

$$\frac{\varepsilon}{\beta} = -\frac{\delta}{\lambda}\,, \quad \frac{\varepsilon'}{\beta} = -\frac{\delta'}{\lambda}\,, \quad \text{and} \quad \delta - \delta' = \frac{\lambda\alpha(n+1)}{\beta Nn(2n+1)} = \frac{\pi^2}{\beta^2}\cdot\frac{4R^2}{\lambda}\cdot\frac{n+1}{n(2n+1)}\,.$$

The table below provides the first five values of $\frac{\pi}{\beta}$ for $n = 0, 1, 2, 3$, where β is root of $v_n(\beta) = 0$.

$n = 0$,	$n = 1$,	$n = 2$,	$n = 3$,	
1	0.6992	0.5451	0.4496	…
0.5000	0.4067	0.3454	0.3016	…
0.3333	0.2881	0.2549	0.2293	…
0.2500	0.2233	0.2025	0.1856	…
0.2000	0.1823	0.1681	0.1561	…
⋮	⋮	⋮	⋮	

It is now seen that the greatest difference in wavelength $\delta - \delta'$ corresponds to $\frac{\pi}{\beta} = 0.6992$, $n = 1$. Setting next $R = 0.141$ and $\lambda = 580$, we obtain $\delta - \delta' = 0.000045$, which is 13,000 times smaller than the difference (0.6) between the wavelengths of the two lines D_1 and D_2 in the solar spectrum.

In a system of spheres, absorption lines appear in the special cases considered here when transmitted white light is resolved into a spectrum. While the amount of out-scattered light from each sphere is generally a very small quantity proportional to R^6, just as we have seen, it will for $p_n = 0$ or $q_n = 0$ be $\frac{\lambda^2(2n+1)}{2\pi}$ or as great as the amount of incident light that, when the light rays pass undisturbed, would hit a sphere of radius $\frac{\lambda\sqrt{n+\frac{1}{2}}}{\pi}$. As the mean distances between neighbouring spheres are preconditioned to be much smaller in the assumed system, it is seen that the system can be said to be nearly impenetrable for this kind of rays. We further note that the absorption lines corresponding to $q_1 = 0$ or $v_0(\beta) = 0$ are single, while all the others are double.

If one has determined wavelengths of a series of absorption lines for a system, these can be ascribed to the reciprocals of the roots in $v_n(\beta) = 0$, $n = 0, 1, 2, \ldots$, through multiplication by a single constant factor. As this factor is equal to $\frac{N}{2\pi R}$, it will thus be possible from the refractive index of the system and from its ordinary absorption coefficient to determine all the constants of the system, namely the number of spheres in a unit of volume, the size of the spheres and their refractive index.

For this purpose, we can also use measurements of the widths of the lines, whereof the calculation can be carried out as follows.

When the wavelength λ corresponds to $p_n = 0$, the value of p_n for a nearby wavelength $\lambda + \delta$ is determined by

$$p_n = -\left[\frac{dp_n}{d\alpha} + \frac{dp_n}{d\alpha'}N\right]^{p=0} \cdot \frac{\alpha\delta}{\lambda} = \frac{1^2 \,.\, 3^2 \ldots (2n-1)^2}{\alpha^{2n+1}} \cdot \frac{N^2-1}{N^2}\left(nN^2 + n(n+1)\right)\frac{\delta}{\lambda}\,.$$

In the same way, when λ corresponds to $q_n = 0$, we have that q_n for the wavelength $\lambda + \delta$ is determined by

$$q_n = \frac{1^2 \,.\, 3^2 \ldots (2n-1)^2}{\alpha^{2n-1}}(N^2-1)\frac{\delta}{\lambda}\,.$$

Although δ is considered a small quantity, it can however always be assumed so large that p_n and q_n become very large as compared with unity, so that k_n and s_n can be determined by

$$k_n = \frac{i}{p_n}\,, \quad s_n = \frac{i}{q_n}\,.$$

For a system of spheres, the absorption coefficients corresponding to this will be

$$\frac{A}{p_n^2} \cdot \frac{\lambda^2(2n+1)}{2\pi} \quad \text{and} \quad \frac{A}{q_n^2} \cdot \frac{\lambda^2(2n+1)}{2\pi}\,.$$

In the spectrum of the transmitted light, we now consider the two limits of an absorption line as the points where the light intensity is reduced to a constant fraction e^{-c}, and the width of the line is then thought to be determined by the difference 2δ between the wavelengths in these two points. If x is the distance travelled through the system, we have

$$c = \frac{Ax}{p_n^2} \cdot \frac{\lambda^2(2n+1)}{2\pi} \quad \text{and} \quad c = \frac{Ax}{q_n^2} \cdot \frac{\lambda^2(2n+1)}{2\pi}\,.$$

Inserting herein the values of p_n and q_n calculated above, it is seen that the width of the lines is always proportional to the square root of the distance travelled, and also to the square root of the number of spheres in a unit of volume.

The widest line corresponds to a$\alpha' = \pi$, $q_1 = \frac{\alpha'^2 - \alpha^2}{\alpha^3} \cdot \frac{\delta}{\lambda}$, which gives

$$2\delta = \frac{8R^3}{\lambda}\sqrt{\frac{6\pi Ax}{c}}\,.$$

For $A = 0.0163$, $R = 0.141$, $\lambda = 580$, $x = 10^{10}$ or 10 metres, and $c = 0.693$, corresponding to a 50 per cent absorption in the limits of the line, we obtain

$$2\delta = 2.57\,,$$

which corresponds to a width 4.3 times larger than the distance between the two lines $D_1 D_2$. It is not without interest to notice that 2δ can also immediately be calculated from the general absorption coefficient h without knowledge of the other constants of the system, since N in the formula for h can here be considered a very large number.

The absorption lines can thus become very wide and rather get the characteristic of absorption bands when α' belongs to the smallest of the roots of $v_0(\alpha') = 0$. On the other hand, if α' belongs to the roots of $v_1(\alpha') = 0, v_2(\alpha') = 0, \ldots$ the stripes are reduced, with the numerical constants used as an example, to lines of a width which is scarcely measurable, even in the most fortunate case. Of course, this does not rule out that they can be made visible.

With this computation of the light propagation within a system of small spheres, it has not been my purpose to conduct an exact analysis, which would have required a larger apparatus. I have only sought to present the peculiarity of this light propagation, which in the case of a single sphere enables us to perform an exact calculation of it, and thereby largely also enables us to compute it for a collection of spheres. The purpose was thus in part to demonstrate the possibility of using the optical properties of a system to gain knowledge of the elements,which in their smallness escape direct observation, and in part to open our eyes to see the striking analogy, which here emerges in its own right, between the optical properties of the assumed system and those of the gases.