Light propagation in and outside a sphere illuminated by plane waves of light.

by

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As long as we consider light to be rays that mutually interfere, refract, and reflect in the surfaces of bodies according to certain laws, our understanding of light propagation is still only elementary and piecewise since we decompose the general fundamental law of the entire light propagation into individual laws and separate phenomena although they essentially belong together. This elementary approach has and will always have its own great importance, but as long as we were unable to go beyond it many problems in optics would be left unsolved and indissoluble.

The general fundamental law of light propagation is like the laws for transmission of electricity and electrical forces of simple form, since it is expressible by three concurrent partial linear differential equations of the second order in which the three oscillatory components are the dependent while the coordinates of space and time are the independent variables. All the problems in formal optics must be subject to integration of these equations.

In a treatise "Ueber die Reflexion an einer Kugelfläche", A. Clebsch[1] tried to determine the reflection of light from *perfectly* reflective spherical surfaces by taking the differential equations of the theory of elasticity as a starting point, but this skilled mathematician did not succeed in surmounting the actual main difficulty. The author expresses this in the introduction using the words: "Die Resultate der ganzen Untersuchung sind sehr verwickelt, und namentlich für den in der Optik wichtigen Fall einer sehr kleinen Wellenlänge scheint es sehr schwer dieselben einfach in passender Form darzustellen".[2] Whereas the following is added: "Der entgegengesetzte Fall eines gegen die Wellenlänge sehr kleinen Radius der reflectirenden Kugel ist dagegen für eine Annäherung sehr geeignet".[3]

The differential equations from which the present investigation takes its starting point have been presented and substantiated in several of my previous works. They differ from the theory of elasticity by the fact that they rule out the possibility of longitudinal oscillations, and,

[1] Crelles Journal, Vol. 61, p. 195. 1863. ["On the reflection from a spherical surface", translated title.]

[2] "The result of the entire investigation is very complicated, and especially the case most important for optics, where a very short wavelength is radiated, is very difficult to put into a suitable form" (translator's note).

[3] "The opposite case, where, as opposed to the wavelength, the reflecting sphere has a very small radius, is on the other hand very suitable for approximation" (translator's note).

since they are valid for every point in any transparent heterogeneous medium, the boundary conditions at the transition from one body to another can be derived from the differential equations themselves.

In a previous work “Farvespredningens Theori”,[1] I have from the same differential equations derived formulae which serve to compute the light propagation in a medium consisting of concentric spherical layers, and the computation was here applied to a system of *small* spheres mutually separated by *large* distances of “empty” space, with the goal in mind to determine the dependency of light refraction on the density of the system. Later I have employed the same series expansions to solve the problem I here have in mind, namely the computation of light propagation which appears when a *homogeneous, transparent, and isotropic sphere is illuminated by plane, parallel waves of light*, and I have in this way also succeeded in arriving at the same results which should be reported here. But, in the following, I have preferred a different and simpler way of presentation where I, also to ease the reading, shall avoid assuming knowledge of my previous work.

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1 Boundary Conditions

Let ξ, η, ζ denote the components of the light oscillations, corresponding to the time and space coordinates t, x, y, z. Moreover, introducing the notation[2]

$$\Delta_2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\,, \quad \theta = \frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\,,$$

the laws of light propagation in any transparent medium can be expressed by the three differential equations

$$\Delta_2\xi - \frac{d\theta}{dx} = \frac{1}{\omega^2}\frac{d^2\xi}{dt^2}\,, \quad \Delta_2\eta - \frac{d\theta}{dy} = \frac{1}{\omega^2}\frac{d^2\eta}{dt^2}\,, \quad \Delta_2\zeta - \frac{d\theta}{dz} = \frac{1}{\omega^2}\frac{d^2\zeta}{dt^2}\,, \tag{1}$$

since ω in general is a variable dependent of x, y, z which corresponds to the velocity of light in the point x, y, z, in so far as you can consider this constant within a very small volume.

The present task is to integrate these equations under the assumption that ω has a constant value inside the surface of a given sphere and a different constant value outside said surface with a discontinuous transition in the spherical surface itself. This discontinuous

[1] Vidensk. Selsk. Skr. 6. Række, p. 167. 1883. [“Theory of colour dispersion”, translated title.]

[2] In modern terminology, Δ_2 is written ∇^2 and is referred to as the Laplacian. (Translator’s note.)

transition is considered to be produced by a surface layer of finite thickness and continuous change of ω, considered to be a function of the distance r from the centre of the sphere, that goes to a layer of thickness zero. At this transition, the oscillatory components must here as everywhere stay finite, whereas the differential coefficients with respect to r might become infinite. The components and their differential coefficients therefore, in general in the boundary surface when the thickness of the boundary layer is reduced to 0, go discontinuously from one value to another, while some combinations of these might really keep their value unchanged.

Since I shall seek these out, I prefer instead of the components with respect to a fixed axial system to employ the projection of the oscillatory deflection on the radius, the projection orthogonal to this and positioned in the plane through radius and the x-axis, and the projection orthogonal to the two preceding and thus orthogonal to the x-axis.

Defining in polar[1] coordinates

$$x = r\,\cos\varphi\,, \quad y = r\,\sin\varphi\,\cos\psi\,, \quad z = r\,\sin\varphi\,\sin\psi\,,$$

and letting $\overline{\xi}$, $\overline{\eta}$, $\overline{\zeta}$ denote the new components, these are given by

$$\left.\begin{array}{rcr} \overline{\xi} & = & \cos\varphi\,\xi + \sin\varphi\,\cos\psi\,\eta + \sin\varphi\,\sin\psi\,\zeta\,, \\ \overline{\eta} & = & -\sin\varphi\,\xi + \cos\varphi\,\cos\psi\,\eta + \cos\varphi\,\sin\psi\,\zeta\,, \\ \overline{\zeta} & = & -\sin\psi\,\eta + \cos\psi\,\zeta\,. \end{array}\right\} \tag{2}$$

When the equations (2) are multiplied by x, y, and z, respectively, and added together, one obtains

$$\Delta_2 r\overline{\xi} - \frac{dr^2\theta}{rdr} = \frac{1}{\omega^2}\frac{d^2r\overline{\xi}}{dt^2}\,,$$

wherefrom it is seen, when Δ_2 is expressed in polar coordinates, that

$$\frac{d^2r^2\,\overline{\xi}}{dr^2} - \frac{dr^2\theta}{dr}$$

is expressible by quantities that remain finite, even when the thickness of the boundary layer is reduced to zero.

Hence, it follows that

$$\frac{dr^2\,\overline{\xi}}{dr} - r^2\theta$$

is a continuous function which thus also remains finite in the boundary layer as it is finite on both sides outside this layer. Consequently,

$$\frac{d\,\overline{\xi}}{dr} - \theta$$

is also everywhere a finite quantity.

Furthermore, multiplying the equations (1) by $-\sin\varphi$, $\cos\varphi\cos\psi$, $\cos\varphi\sin\psi$, respectively, and adding them together, one obtains

[1]In modern terminology, one would write *spherical polar* or simply *spherical* coordinates as the coordinate system is three-dimensional. (Translator's note.)

$$\frac{d^2 r\,\overline{\eta}}{dr^2} - \frac{d\theta}{d\varphi}$$

expressed by quantities that remain finite everywhere. Similarly, we find by multiplication of the equations (1) by 0, $-\sin\psi$, $\cos\psi$, and addition,

$$\frac{d^2 r\,\overline{\zeta}}{dr^2} - \frac{d\theta}{\sin\varphi\,d\varphi}$$

expressed by finite quantities everywhere.

In this way, we have found three combinations that are finite everywhere. Eliminating θ from these, it is seen that the quantities

$$\frac{d^2 r\,\overline{\eta}}{dr^2} - \frac{d^2\,\overline{\xi}}{d\varphi dr} \quad \text{and} \quad \frac{d^2 r\,\overline{\zeta}}{dr^2} - \frac{d^2\,\overline{\xi}}{\sin\varphi\,d\psi dr}$$

are finite everywhere, from which it follows that

$$\frac{dr\,\overline{\eta}}{dr^2} - \frac{d\,\overline{\xi}}{d\varphi} \quad \text{and} \quad \frac{dr\,\overline{\zeta}}{dr} - \frac{d\,\overline{\xi}}{\sin\varphi\,d\psi}$$

are continuous functions that thus remain unchanged at the transition from one side of the boundary surface of the sphere to the other. I will express this by

$$\left[\frac{dr\,\overline{\eta}}{dr^2} - \frac{d\,\overline{\xi}}{d\varphi}\right] = 0\,, \quad \left[\frac{dr\,\overline{\zeta}}{dr} - \frac{d\,\overline{\xi}}{\sin\varphi\,d\psi}\right] = 0\,. \tag{3}$$

In addition, note that the same quantities, as continuous functions and finite everywhere outside the boundary surface, must also be finite in the boundary surface. Hence, it follows that $r\,\overline{\eta}$ and $r\,\overline{\zeta}$ must be continuous such that one, using the same notation as above, would have

$$\left[\,\overline{\eta}\,\right] = 0\,, \quad \left[\,\overline{\zeta}\,\right] = 0\,. \tag{4}$$

The boundary conditions corresponding to $r = 0$ and $r = \infty$ are given by the fact that light propagation is finite everywhere, thus also for $r = 0$, and that at an infinite distance from the sphere we only find, besides the given incident light, light that is propagated from the sphere, but none that propagates *toward* it.

2 Expansion in Terms of Spherical Functions

The light incident on the sphere is taken to consist of plane, parallel waves of light. In general, these waves could contain a collection of oscillations, different with respect to amplitude, direction inside the wave plane, period of oscillation, and phase, but this general case is easily derived from the simple one wherein the vibrational components, which we shall denote by ξ_0, η_0, ζ_0, outside the sphere are given by

$$\xi_0 = 0\,, \quad \eta_0 = e^{(kt-lx)\,i}\,, \quad \zeta_0 = 0\,. \tag{5}$$

Here the exponential form is chosen as the simplest, the oscillations with amplitude 1 occur along the y-axis and propagate along the x-axis with the constant velocity $\frac{k}{l} = \Omega$, with wavelength $\frac{2\pi}{l} = \lambda$, and period of oscillation $\frac{2\pi}{k} = T$.

Since we, *outside* the sphere, in this way separate the incident light from the other light, generated by the change of velocity in the spherical surface, we here set

$$\xi = \xi_0 + \xi_e\,, \quad \eta = \eta_0 + \eta_e\,, \quad \zeta = \zeta_0 + \zeta_e\,, \tag{6}$$

while we, *inside* the spherical surface, set

$$\xi = \xi'\,, \quad \eta = \eta'\,, \quad \zeta = \zeta'\,, \tag{7}$$

where also l', Ω', λ' replace the corresponding unmarked quantities outside the sphere. Furthermore, letting N (the refraction ratio of the sphere) denote the ratio between the two velocities Ω and Ω', one has

$$\Omega = N\Omega'\,, \quad l' = Nl\,, \quad \lambda = N\lambda'\,. \tag{8}$$

The components ξ, η, ζ are mutually connected outside as well as inside the spherical surface by the equation $\theta = 0$, which for constant ω appears from the equations (1), and thus they could be represented as dependents of two quantities alone: Q and S outside the sphere, or Q' and S' inside the sphere. That is, one would be able to set

$$\left.\begin{aligned} \xi_e = \frac{dC}{dy} - \frac{dB}{dz}\,, \quad \eta_e = \frac{dA}{dz} - \frac{dC}{dx}\,, \quad \zeta_e = \frac{dB}{dx} - \frac{dA}{dy}\,, \\ A = z\frac{dQ}{dy} - y\frac{dQ}{dz} + xS\,, \quad B = x\frac{dQ}{dz} - z\frac{dQ}{dx} + yS\,, \quad C = y\frac{dQ}{dx} - x\frac{dQ}{dy} + zS\,, \end{aligned}\right\} \tag{9}$$

just as also ξ', η', ζ' could be expressed in the same way. The equations (1) would then be satisfied under the assumption that one has

$$\Delta_2 Q + l^2 Q = 0\,, \qquad \Delta_2 S + l^2 S = 0\,, \tag{10}$$

$$\Delta_2 Q' + l'^2 Q' = 0\,, \qquad \Delta_2 S' + l'^2 S' = 0\,. \tag{11}$$

It can be noticed here that the two radial projections

$$x\xi_e + y\eta_e + z\zeta_e$$

and
$$x\left(\frac{d\zeta_e}{dy} - \frac{d\eta_e}{dz}\right) + y\left(\frac{d\xi_e}{dz} - \frac{d\zeta_e}{dx}\right) + z\left(\frac{d\eta_e}{dx} - \frac{d\xi_e}{dy}\right)$$

by means of equations (9) could be transformed into

$$-r^2\Delta_2 Q + r\frac{d^2 rQ}{dr^2} = -\frac{1}{\sin\varphi}\frac{d}{d\varphi}\sin\varphi\frac{dQ}{d\varphi} - \frac{d^2Q}{\sin^2\varphi\, d\psi^2}$$

and
$$-r^2\Delta_2 S + r\frac{d^2 rS}{dr^2} = -\frac{1}{\sin\varphi}\frac{1}{d\varphi}\sin\varphi\frac{dS}{d\varphi} - \frac{d^2S}{\sin^2\varphi\, d\psi^2}\,.$$

Hence, it is seen that when Q and S are expanded in series in terms of spherical functions Q_n and S_n, namely

$$Q = \sum Q_n\,, \quad S = \sum S_n\,,$$

then the above-mentioned radial projections would be given, respectively, by

$$\sum n(n+1)Q_n \quad \text{and} \quad \sum n(n+1)S_n\,.$$

The same holds true in the space inside the sphere.

In analogy with (6) for points outside the sphere, we express the components $\overline{\xi}$, $\overline{\eta}$, $\overline{\zeta}$ introduced in the preceding section by

$$\overline{\xi}=\overline{\xi}_0+\overline{\xi}_e\,,\quad \overline{\eta}=\overline{\eta}_0+\overline{\eta}_e\,,\quad \overline{\zeta}=\overline{\zeta}_0+\overline{\zeta}_e\,, \tag{12}$$

as these new components are given by

$$\overline{\xi}_0=\sin\varphi\cos\psi\, e^{(kt-lx)\,i}\,,\quad \overline{\eta}_0=\cos\varphi\cos\psi\, e^{(kt-lx)\,i}\,,\quad \overline{\zeta}_0=-\sin\psi\, e^{(kt-lx)\,i}\,, \tag{13}$$

$$\left.\begin{array}{rcl} \overline{\xi}_e &=& \cos\varphi\,\xi_e+\sin\varphi\cos\psi\,\eta_e+\sin\varphi\sin\psi\,\zeta_e\,,\\ \overline{\eta}_e &=& -\sin\varphi\,\xi_e+\cos\varphi\cos\psi\,\eta_e+\cos\varphi\sin\psi\zeta_e\,,\\ \overline{\zeta}_e &=& -\sin\psi\,\eta_e+\cos\psi\,\zeta_e\,. \end{array}\right\} \tag{14}$$

Now, introducing for the sake of brevity the following notation

$$lr=a\,,\quad l'r=a'\,,\quad lQ=K\,,\quad l'Q'=K' \tag{15}$$

and, as R is the radius of the given sphere,

$$lR=\alpha\,,\quad l'R=\alpha'\,, \tag{16}$$

then by the equations (9) and by using the equations (10) one obtains

$$\left.\begin{array}{rcl} \overline{\xi}_e &=& \dfrac{d^2aK}{da^2}+aK\,,\\ \overline{\eta}_e &=& \dfrac{d^2aK}{a\,d\varphi\,da}+\dfrac{dS}{\sin\varphi\,d\psi}\,,\\ \overline{\zeta}_e &=& \dfrac{d^2aK}{a\sin\varphi\,d\psi\,da}-\dfrac{dS}{d\varphi}\,, \end{array}\right\} \tag{17}$$

just as one for an inner point correspondingly has

$$\left.\begin{array}{rcl} \overline{\xi'} &=& \dfrac{d^2a'K'}{da'^2}+a'K'\,,\\ \overline{\eta'} &=& \dfrac{d^2a'K'}{a'\,d\varphi\,da'}+\dfrac{dS'}{\sin\varphi\,d\psi}\,,\\ \overline{\zeta'} &=& \dfrac{d^2a'K'}{a'\sin\varphi\,d\psi\,da'}-\dfrac{dS'}{d\varphi}\,. \end{array}\right\} \tag{18}$$

The task is now to develop these components in series in terms of spherical functions. If it is at all possible to expand a function $f(x)$ in terms of spherical functions, the expansion is, as is well-known, the following:

$$f(x)=\sum_0^\infty \frac{2n+1}{2}P_n(x)\int_{-1}^{1} f(u)P_n(u)\,du\,,$$

as the sum is taken for all values of n from $n=0$ to $n=\infty$, and

$$P_n(x)=\frac{1\,.\,3\ldots 2n-1}{1\,.\,2\ldots n}\left(x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\,.\,4(2n-1)(2n-3)}x^{n-4}-\ldots\right)\,.$$

If we now first seek to expand the expressions for $\overline{\xi}_0$, $\overline{\eta}_0$, $\overline{\zeta}_0$ given in the equations (13), wherein we set $lx = a\cos\varphi$, then according to the above-mentioned we have

$$e^{-a\cos\varphi\, i} = \sum_0^\infty \frac{2n+1}{2} P_n(\cos\varphi) \int_{-1}^1 e^{-aui} P_n(u)\, du\ .$$

The definite integral involved herein is expressible by the Bessel function $J_{n+\frac{1}{2}}(a)$, or, as I shall here prefer, by another function denoted $v_n(a)$ which differs from the Bessel function by only a factor since we set

$$v_n(a) = \sqrt{\frac{\pi a}{2}} J_{n+\frac{1}{2}}(a)\, .$$

One will then, as is well-known from the theory of Bessel functions, be able to define $v_n(a)$ by

$$v_n(a) = \frac{a^{n+1}}{2^{n+1}[n]} \int_{-1}^1 e^{-aui}(1-u^2)^n\, du\ .$$

This integral transforms by n-fold integration by parts to

$$v_n(a) = \frac{a}{2^{n+1}[n] i^n} \int_{-1}^1 e^{-aui} \frac{d^n(1-u^2)^n}{du^n}\, du\, ,$$

which by using another familiar expression for P_n, namely

$$P_n(u) = \frac{(-1)^n}{2^n[n]} \frac{d^n(1-u^2)^n}{du^n}\, ,$$

can also be given the form

$$v_n(a) = \frac{a}{2} i^n \int_{-1}^1 e^{-aui} P_n(u)\, du\ . \tag{19}$$

In this way, we obtain

$$e^{-a\cos\varphi\, i} = \frac{1}{a} \sum_0^\infty (2n+1) P_n(\cos\varphi) e^{-\frac{n\pi}{2} i} v_n(a)\, . \tag{20}$$

It is noted that the function $v_n(a)$ satisfies the differential equation

$$\frac{d^2 v_n(a)}{da^2} = \left(\frac{n(n+1)}{a^2} - 1\right) v_n(a)\, , \tag{21}$$

and that it, expanded in powers of a, gives the series

$$v_n(a) = \frac{a^{n+1}}{1\,.\,3\ldots 2n+1}\left(1 - \frac{a^2}{2(2n+3)} + \frac{a^4}{2\,.\,4(2n+3)(2n+5)} - \ldots\right)\, . \tag{22}$$

Another series expansion well-known from the theory of Bessel functions, where the number of terms is finite, is

$$\left.\begin{aligned} v_n(a) &= g_n(a)\sin\left(a - \frac{n\pi}{2}\right) + h_n(a)\cos\left(a - \frac{n\pi}{2}\right)\, , \\ g_n(a) &= 1 - \frac{(n-1)n(n+1)(n+2)}{2\,.\,4\, a^2} + \frac{(n-3)(n-2)\ldots(n+4)}{2\,.\,4\,.\,6\,.\,8\, a^4} - \ldots\, , \\ h_n(a) &= \frac{n(n+1)}{2\, a} - \frac{(n-2)(n-1)\ldots(n+3)}{2\,.\,4\,.\,6\, a^3} + \ldots\ . \end{aligned}\right\} \tag{23}$$

Furthermore, denoting by $w_n(a)$ another particular integral for Equation (21), and defining this integral more specifically by the series expansion

$$w_n(a) = \frac{1\,.\,3\ldots 2n-1}{a^n}\left(1+\frac{a^2}{2(2n-1)}+\frac{a^4}{2\,.\,4(2n-1)(2n-3)}+\ldots\right), \tag{24}$$

this function will likewise differ only by a factor from a Bessel function, namely $J_{-n-\frac{1}{2}}(a)$, and with the series for g_n and h_n given above it will also be expressible by

$$w_n(a) = g_n(a)\cos\left(a-\frac{n\pi}{2}\right) - h_n(a)\sin\left(a-\frac{n\pi}{2}\right). \tag{25}$$

From the expansion (20), the expressions given in the equations (13) can now be determined in the following way. One extracts from the series (20) the term corresponding to $n = 0$ and sets

$$P_n(\cos\varphi) = -\frac{1}{n(n+1)}\cdot\frac{1}{\sin\varphi}\frac{d}{d\varphi}\sin\varphi\frac{dP_n(\cos\varphi)}{d\varphi},$$

whereby one obtains

$$e^{-a\cos\varphi\, i} = \frac{\sin a}{a} - \frac{1}{a}\sum_1^\infty \frac{2n+1}{n(n+1)}\frac{1}{\sin\varphi}\frac{d}{d\varphi}\sin\varphi\frac{P_n(\cos\varphi)}{d\varphi}e^{-\frac{n\pi}{2}i}\,v_n(a).$$

Introducing herein for brevity the notation

$$\left.\begin{aligned} K_0 &= -i\frac{\cos\psi}{a}\frac{d}{d\varphi}\sum_1^\infty\frac{2n+1}{n(n+1)}P_n(\cos\varphi)e^{\left(kt-\frac{n\pi}{2}\right)i}\,v_n(a), \\ S_0 &= -\frac{\sin\psi}{a}\frac{d}{d\varphi}\sum_1^\infty\frac{2n+1}{n(n+1)}P_n(\cos\varphi)e^{\left(kt-\frac{n\pi}{2}\right)i}\,v_n(a), \end{aligned}\right\} \tag{26}$$

one obtains by multiplication of the equation by $\cos\psi\, e^{kti}\sin\varphi\, d\varphi$ or by $-\sin\psi\, e^{kti}\sin\varphi\, d\varphi$, and by integration from $\varphi = 0$ to $\varphi = \varphi$ of the two equations obtained in this way,

$$\left.\begin{aligned} K_0 &= \frac{\cos\psi}{a\sin\varphi}(i\sin a\cos\varphi - \cos a + e^{-a\cos\varphi\, i})\,e^{kti}, \\ S_0 &= -\frac{\sin\psi}{a\sin\varphi}(-\sin a\cos\varphi - i\cos a + i\,e^{-a\cos\varphi\, i})\,e^{kti}. \end{aligned}\right\} \tag{27}$$

From this we find in conclusion

$$\left.\begin{aligned} &\frac{d^2aK_0}{da^2} + aK_0 = \sin\varphi\cos\psi\, e^{(kt-a\cos\varphi)\,i} &&= \overline{\xi}_0, \\ &\frac{d^2aK_0}{a\,d\varphi\,da} + \frac{dS_0}{\sin\varphi\, d\psi} = \cos\varphi\cos\psi\, e^{(kt-a\cos\varphi)\,i} &&= \overline{\eta}_0, \\ &\frac{d^2aK_0}{a\sin\varphi\, d\psi\, da} - \frac{dS_0}{d\varphi} = -\sin\psi\, e^{(kt-a\cos\varphi)\,i} &&= \overline{\zeta}_0. \end{aligned}\right\} \tag{28}$$

These expressions for the components $\overline{\xi}_0$, $\overline{\eta}_0$, $\overline{\zeta}_0$ correspond to the expressions presented in (17) for the components $\overline{\xi}_e$, $\overline{\eta}_e$, $\overline{\zeta}_e$, as K_0 and S_0 take the place of K and S in the equations (17). For K_0 and S_0, we have in (26) the expansions in terms of spherical functions, and these must, as one can also easily convince oneself, satisfy the same differential equations as K and S, namely according to (10) $\Delta_2 K_0 + l^2 K_0 = 0\,,\ \ \Delta_2 S_0 + l^2 S_0 = 0$. The expansions of K and S in terms of spherical functions must consequently be analogous with the expansions (26), as one would here instead of the particular integral $v_n(a)$ of the equation (21) insert the ordinary integral expressed linearly by $v_n(a)$ and $w_n(a)$. Thus, one obtains with the as yet undetermined constants k_n, x_n, s_n, σ_n,

$$\left.\begin{aligned} K &= -i\,\frac{\cos\psi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(k_n v_n(a) + x_n w_n(a)\right)\,, \\ S &= -\,\frac{\sin\psi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(s_n v_n(a) + \sigma_n w_n(a)\right)\,, \end{aligned}\right\} \tag{29}$$

and for a corresponding inner point

$$\left.\begin{aligned} K' &= -i\,\frac{\cos\psi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(k'_n v_n(a') + x'_n w_n(a')\right)\,, \\ S' &= -\,\frac{\sin\psi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i}\left(s'_n v_n(a') + \sigma'_n w_n(a')\right)\,. \end{aligned}\right\} \tag{30}$$

Herein $P_n(\cos\varphi)$ is shortened to P_n.

If we now first use the boundary condition corresponding to $a' = 0$, it is seen from (24) that $w_n(a')$ becomes ∞ for $a' = 0$ and $n > 0$, and that the finiteness condition therefore requires

$$x'_n = 0\,,\quad \sigma'_n = 0\,.$$

According to (23) and (25), $a = \infty$ corresponds to $v_n(a) = \sin\left(a - \frac{n\pi}{2}\right)\,,\quad w_n(a) = \cos\left(a - \frac{n\pi}{2}\right)$. At an infinite distance from the sphere one therefore has

$$2(k_n v_n(a) + x_n w_n(a))e^{\left(kt-\frac{n\pi}{2}\right)i} = (-k_n\,i + x_n)e^{(kt+a-n\pi)\,i} + (k_n\,i + x_n)e^{(kt-a)\,i}\,.$$

From this it is seen that the light propagation in general at this distance appears to be a periodic function of $kt + a$ and $kt - a$, corresponding to two opposite wave propagations, one propagating toward the centre of the sphere, the other in the direction away from the centre. Since now only the latter, according to the assumed conditions, is really permitted, one must have

$$-k_n\,i + x_n = 0\,,\quad \text{as also correspondingly}\quad - s_n\,i + \sigma_n = 0\,.$$

In this way, the series (29) and (30) are reduced to

$$\left.\begin{aligned} K &= -i\,\frac{\cos\psi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} k_n(v_n(a) + w_n(a)\,i)\,, \\ S &= -\,\frac{\sin\psi}{a}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} s_n(v_n(a) + w_n(a)\,i)\,, \\ K' &= -i\,\frac{\cos\psi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} k'_n v_n(a')\,, \\ S' &= -\,\frac{\sin\psi}{a'}\frac{d}{d\varphi}\sum_1^\infty \frac{2n+1}{n(n+1)} P_n e^{\left(kt-\frac{n\pi}{2}\right)i} s'_n v_n(a')\,. \end{aligned}\right\}{}^{1} \tag{31}$$

[1]The original reads s_n instead of s'_n in the equation for S'. This must be a typo. (Translator's note.)

Finally, we also have the boundary conditions presented in (3) and (4) which can be expressed by

$$\left.\begin{array}{rcl} \overline{\eta} = \overline{\eta'}\,, & & \overline{\zeta} = \overline{\zeta'} \\ \dfrac{da\overline{\eta}}{da} - \dfrac{d\overline{\xi}}{d\varphi} & = & \dfrac{da'\overline{\eta'}}{da'} - \dfrac{d\overline{\xi'}}{d\varphi} \\ \dfrac{da\overline{\zeta}}{da} - \dfrac{d\overline{\xi}}{\sin\varphi\, d\psi} & = & \dfrac{da'\overline{\zeta'}}{da'} - \dfrac{d\overline{\xi'}}{\sin\varphi\, d\psi} \end{array}\right\} \begin{array}{rcl} a & = & \alpha \\ a' & = & \alpha'\,. \end{array}$$

Inserting herein the expressions for $\overline{\xi}$, $\overline{\eta}$, $\overline{\zeta}$ given by the equations (12), (17), and (28), and for $\overline{\xi'}$, $\overline{\eta'}$, $\overline{\zeta'}$ the expressions (18), these conditions can be transformed into

$$\left.\begin{array}{c} a(K_0+K) = a'K'\,, \quad S_0+S = S' \\ \dfrac{da(K_0+K)}{a\,da} = \dfrac{da'K'}{a'\,da'}\,, \quad \dfrac{da(S_0+S)}{da} = \dfrac{da'S}{da'} \end{array}\right\} \begin{array}{rcl} a & = & \alpha \\ a' & = & \alpha'\,. \end{array} \tag{32}$$

Herein K_0, S_0, K, S, K', S' are expanded by the series given in (26) and (31), whereby 4 equations between the coefficients are obtained. Denoting for the sake of brevity the derived functions $\frac{dv_n(\alpha)}{da}$, $\frac{dw_n(\alpha)}{da}$, $\frac{dv_n(\alpha')}{da'}$ by $v_n'(\alpha)$, $w_n'(\alpha)$, $v_n'(\alpha')$, these equations become

$$\begin{array}{rcl} N\,(v_n'(\alpha) + k_n(v_n'(\alpha) + w_n'(\alpha)\,i)) & = & k_n'v_n'(\alpha') \\ N\,(v_n(\alpha) + s_n(v_n(\alpha) + w_n(\alpha)\,i)) & = & s_n'v_n(\alpha') \\ v_n(\alpha) + k_n(v_n(\alpha) + w_n(\alpha)\,i) & = & k_n'v_n(\alpha') \\ v_n'(\alpha) + s_n(v_n'(\alpha) + w_n'(\alpha)\,i) & = & s_n'v_n'(\alpha')\,. \end{array}$$

From this we can determine the four coefficients. By introducing a small reduction by means of the equation

$$w_n(\alpha)v_n'(\alpha) - w_n'(\alpha)v_n(\alpha) = 1\,,$$

one will thus obtain

$$\left.\begin{array}{rcl} 2k_n & = & -1 - \dfrac{(v_n(\alpha) - w_n(\alpha)\,i)\,v_n'(\alpha') - N\,(v_n'(\alpha) - w_n'(\alpha)\,i)\,v_n(\alpha')}{(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - N\,(v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,, \\ 2s_n & = & -1 - \dfrac{N\,(v_n(\alpha) - w_n(\alpha)\,i)\,v_n'(\alpha') - (v_n'(\alpha) - w_n'(\alpha)\,i)\,v_n(\alpha')}{N\,(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - (v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,, \end{array}\right\} \tag{33}$$

$$\left.\begin{array}{rcl} k_n' & = & \dfrac{Ni}{(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - N\,(v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,, \\ s_n' & = & \dfrac{Ni}{N\,(v_n(\alpha) + w_n(\alpha)\,i)\,v_n'(\alpha') - (v_n'(\alpha) + w_n'(\alpha)\,i)\,v_n(\alpha')}\,. \end{array}\right\} \tag{34}$$

The posed task is hereby solved in so far as the oscillatory components have been determined everywhere in space by infinite series with known coefficients. It shall be seen that the series in the given form are well-suited for the calculation when either α, which corresponds to the circumference of the sphere measured in units of the wavelength λ, is a small number or when the considered point is close to the centre, whereas when α is a very large number, which we can almost say is the case for all spheres that are visible to the naked eye, it will in general be necessary to transform the series such that the summations can be done with sufficient approximation. I shall now first produce the summation formulae that would here be brought into play.

3 Summation Formulae

In the following section, sums will be produced that can be referred to the form

$$\sum_{n_1}^{n_2} A_n e^{F_n\, i}\,, \tag{35}$$

where n runs through the sequence from $n = n_1$ to $n = n_2$.

The two functions A_n and F_n are constituted in such a way that when inserting $n = \nu + z$, where both the new variables are also considered to be integers, one will obtain the following series, convergent within the given bounds,

$$A_n = A + B\frac{z}{\alpha} + C\frac{z^2}{\alpha^2} + \ldots\,, \qquad F_n = F\alpha + Gz + H\frac{z^2}{\alpha} + I\frac{z^3}{\alpha^2} + \ldots\,. \tag{36}$$

The terms have here been ordered according to increasing powers of z and decreasing powers of the quantity α. The latter is considered to be a *very large*, however, not infinitely large, number, and all quantities will in the following be ordered according to powers of α such that the quantity which contains the larger power of α is considered to be a quantity of higher order. Here the coefficients $A, B, \ldots, F, G, \ldots$ *in the highest* are quantities of the same order as the unit (α^0). The calculation shall now aim at production of the results with such an accuracy that only quantities which are of lower order than the unit are considered small enough to be discarded.

The number of terms in the series (35) is itself a very large number, of the same order as α. The bounds n_1 and n_2 are *indeterminate* and *to a certain degree arbitrary*, that is, they are only constrained on the one hand by the convergence conditions for the series (36), on the other hand by the demand that $n_2 - n_1$ must be a very large number. This kind of indeterminate, arbitrary quantities which is here introduced, and for which I in the following shall use the common marker ω, are defined by the concept that a function of this quantity denotes the bound whereto the average of the same function of a particular quantity x converges when one lets x run through the gradually larger and larger sequence of values within the bounds delimited by ω.

Thus, taking our starting point in the two well-known integrals

$$\int_0^\infty e^{-x} x^{\mu-1}\, dx = \Gamma(\mu)\,, \ldots \quad (37) \qquad\qquad \int_0^\infty e^{xi} x^{\mu-1}\, dx = \Gamma(\mu) e^{\frac{\mu\pi}{2} i}\,, \ldots \quad (38)$$

the first one valid for all positive values of μ, the second one only for the positive values which are smaller than 1, then it is seen that one, also for $\mu < 1$, must have

$$\int_0^\omega e^{xi} x^{\mu-1}\, dx = \Gamma(\mu) e^{\frac{\mu\pi}{2} i}\,, \tag{39}$$

since

$$\int_0^\omega e^{xi} x^{\mu-1}\, dx = \int_0^\infty e^{xi} x^{\mu-1}\, dx - \int_\omega^\infty e^{xi} x^{\mu-1}\, dx\,,$$

where the latter integral can be expanded by integration by parts into a semi-convergent series whose average value, as corresponding to different values of ω, converges to 0 when the average value is taken between wider and wider bounds in the way stated above. Furthermore, if $\mu > 1$ in the integral (39), the exponent can be reduced by integration by parts to become smaller than 1, and the average value of the periodical terms appearing outside the integral will likewise converge to 0. Consequently, the equation (39) with the agreed meaning of the upper bound ω is valid for *all positive* values of μ.

As another example that will of use in the following, we take the sum (35) reduced to the simplest form

$$\sum_{n_1}^{n^2} e^{ani} = \frac{e^{an_1 i} - e^{a(n_2+1)i}}{1 - e^{ai}}$$

Here, the right-hand side must also disappear, assuming that a is not 0 or a multiple of 2π since in this case the sum becomes $n_2 - n_1 + 1$ which presumably is indeterminate, but in any case cannot be equal to zero. Furthermore, if a is very small or very close to a multiple of 2π, one dare not consider the sum to be zero since the number of terms is assumed very large, but not infinitely large.

If the sum is zero, it will continue to be so when differentiated an arbitrary number of times with respect to a. Hence, more generally, one has

$$\sum_{n_1}^{n_2} n^m e^{ani} = 0\,, \tag{40}$$

when m is an integer or 0, and when a is not equal to or lying very close to 0 or a multiple of 2π.

Now, considering the sum given by the expansions (35) and (36), it is seen that it can be changed into a convergent series with terms that, when leaving out constant factors, have the form

$$\sum_{n_1-\nu}^{n_2-\nu} z^m e^{Gzi}$$

That is, if one does *not* have

$$G = 2p\pi\,, \tag{41}$$

for $p = 0$ or an integer, and neither $G - 2p\pi$ very close to 0, then the entire sum (35) will disappear.

Conversely, if one is able to find a value of ν that enables satisfaction of the above-mentioned condition (41), then Gz can be omitted from the exponent, and the summation can now be changed to integration without appreciable error. Hence, the sum (35) can be given the form

$$\int_{-(\nu-n_1)}^{n_2-\nu} dz\left(A+B\frac{z}{\alpha}+\dots\right)e^{\left(F\alpha+H\frac{z^2}{\alpha}+I\frac{z^3}{\alpha^2}+\dots\right)i}, \tag{42}$$

where we restrict ourselves to the assumption that ν is situated between n_1 and n_2 and, thus, that both $\nu-n_1$ and $n_2-\nu$ will have to belong to the kind of indeterminate quantities defined above. Changing in this integral the sign of z for $z<0$, and afterwards setting $Hz^2=\alpha x$, the bounds of x, assuming that H is not 0 or very small, belongs to the kind of quantities denoted above by the common marker ω, and the integral will by series expansion become

$$\int_0^\omega\frac{dx}{2}\left(A\sqrt{\frac{\alpha}{Hx}}+\frac{B}{H}\dots+\frac{AIxi}{H^2}+\dots\right)e^{(F\alpha+x)i}+\int_0^\omega\frac{dx}{2}\left(A\sqrt{\frac{\alpha}{Hx}}-\frac{B}{H}\dots-\frac{AIxi}{H^2}+\dots\right)e^{(F\alpha+x)i}.$$

These integrals will according to (39), as $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, together become

$$A\sqrt{\frac{\alpha\pi}{H}}e^{\left(F\alpha+\frac{\pi}{4}\right)i}, \tag{43}$$

since the terms which are of the order $\alpha^{-\frac{1}{2}}$ and of lower order are discarded. This result is also valid for negative values of H when it is taken care of that one in this case must set

$$\frac{1}{\sqrt{-1}}=-i=e^{-\frac{\pi}{2}i}.$$

The result becomes invalid for

$$H=0. \tag{44}$$

To further generalise it, we can in this case assume that $G-2p\pi$ is a very small quantity. Also in this case, the summation will change to integration and instead of (42) one will obtain the integral

$$\int_{-(\nu-n_1)}^{n_2-\nu} dz\left(A+B\frac{z}{\alpha}+C\frac{z^2}{\alpha^2}+\dots\right)e^{\left(F\alpha+(G-2p\pi)z+I\frac{z^3}{\alpha^2}+K\frac{z^4}{\alpha^3}+L\frac{z^5}{\alpha^4}\dots\right)i}. \tag{45}$$

Changing herein too the sign of z for $z<0$, and afterwards setting $\pm Iz^3=\alpha^2x$, where the double sign is determined such that $\pm I$ becomes positive. Introducing for the sake of brevity the notation

$$G-2p\pi=-\varepsilon\sqrt[3]{\frac{I}{\alpha^2}}, \tag{46}$$ $$\int_0^\omega x^{-\frac{2}{3}}\cos(-\varepsilon\, x^{\frac{1}{3}}+x)\,dx=Q, \tag{47}$$

as well as

$$A = A_1 I \,, \quad B = B_1 I \,, \quad C = C_1 I \,, \quad K = K_1 I \,, \quad L = L_1 I \,, \tag{48}$$

one will without difficulty be able to give the integral (45) the form

$$\begin{aligned} \pm\tfrac{2}{3} e^{F\alpha i} \bigg[(\alpha I)^{\frac{2}{3}} A_1 Q + (\alpha I)^{\frac{1}{3}} i \left(B_1 \frac{dQ}{d\varepsilon} + A_1 K_1 \frac{d^4 Q}{d\varepsilon^4} \right) - C_1 \frac{d^2 Q}{d\varepsilon^2} \\ -(A_1 L_1 + B_1 K_1) \frac{d^5 Q}{d\varepsilon^5} - \tfrac{1}{2} A_1 K_1^2 \frac{d^8 Q}{d\varepsilon^8} \bigg] \,, \end{aligned} \tag{49}$$

as the terms of the order $\alpha^{-\frac{1}{2}}$ or less are discarded.

In case of one having $\varepsilon = 0$, one obtains by means of (39)

$$\Gamma\left(\frac{1}{3}\right) \cos\frac{\pi}{6} = Q = -3 \frac{d^3 Q}{d\varepsilon^3} \,, \quad \Gamma\left(\frac{2}{3}\right) \cos\frac{\pi}{6} = \frac{dQ}{d\varepsilon} = -\frac{2}{3} \frac{d^4 Q}{d\varepsilon^4} \,,$$
$$0 = \frac{d^2 Q}{d\varepsilon^2} = \frac{d^5 Q}{d\varepsilon^5} = \frac{d^8 Q}{d\varepsilon^8} \,,$$

where

$$\Gamma\left(\frac{1}{3}\right) = 2.67894\ldots \,, \quad \Gamma\left(\frac{2}{3}\right) = 1.35412\ldots \,,$$

or by the common logarithms

$$\mathrm{Log}\ \Gamma\left(\frac{1}{3}\right) = 0.4279627\ldots \,, \quad \mathrm{Log}\ \Gamma\left(\frac{2}{3}\right) = 0.1316565\ldots \,.$$

Here (49) thus becomes

$$\pm \frac{1}{\sqrt{3}} e^{F\alpha i} \left[(\alpha I)^{\frac{2}{3}} A_1 \, \Gamma\left(\frac{1}{3}\right) + (\alpha I)^{\frac{1}{3}} i \left(B_1 - \frac{2}{3} A_1 K_1 \right) \Gamma\left(\frac{2}{3}\right) \right] \,. \tag{50}$$

The integral Q (47) has under a somewhat different form been calculated numerically by *Airy*[1], who for the integral

$$\int_0^\infty d\omega \, \cos\frac{\pi}{2}(\omega^3 - m\omega) = W$$

has provided the following table

m	W	m	W
-5	0.00041	0	0.66527
-4	0.00298	1	1.00041
-3	0.01730	2	0.56490
-2	0.07908	3	-0.56322
-1	0.27283	4	-0.47446
		5	0.68182.

[1] On the intensity of Light in the neighbourhood of a Caustic. Trans. of the Cambr. Soc. t. VI, p. 379, t. VIII, p. 595.

By means hereof one can also calculate Q, as one has

$$\varepsilon = \left(\frac{\pi}{2}\right)^{\frac{2}{3}} m\,, \quad Q = \frac{1}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{3}} W\,.$$

Going from $m = 0$ to the negative side, W keeps decreasing until 0. Going to the positive side, W is first increasing, reaching a maximum at $m = 1.08$, and hereafter also approaches 0 through a periodic motion around the zero point. The first and largest maximum of W is 1.504 times larger than the value of W for $m = 0$.

Stokes[1] has extended the calculation of Airy to the first 50 roots of the equation $W = 0$ and the first 10 roots of $\frac{dW}{dm} = 0$. Thus, the sequence corresponding to $W = 0$ is

$$m = 2.4955; 4.3631; 5.8922; 7.2436; 8.4788; \ldots$$

in which the qth root for growing q converges to $3\left(q - \frac{1}{4}\right)^{\frac{2}{3}}$. Likewise, for $\frac{dW}{dm} = 0$,

$$m = 1.0845; 3.4669; 5.1446; 6.5782; 7.8685; \ldots$$

where the qth root converges to $3\left(q - \frac{3}{4}\right)^{\frac{2}{3}}$.

The different derivatives of Q with respect to ε, which enter into the expression (49), would all easily be expressible by Q and $\frac{dQ}{d\varepsilon}$, as it is noticed that one has

$$\frac{d^2Q}{d\varepsilon^2} = -\frac{\varepsilon}{3} Q\,,$$

whereof yet higher derivatives could be derived, for example

$$\frac{d^4Q}{d\varepsilon^4} = \frac{\varepsilon^2}{9} Q - \frac{2}{3}\frac{dQ}{d\varepsilon}\,, \text{ etc.}$$

Hence, the maximum and minimum points of $\frac{dQ}{d\varepsilon}$ correspond to $Q = 0$, whereof it is seen that the first maximum does not set in here until $m = 2.4955\ldots$. The modulus (or the amplitude) of the expression given in (49) changes with growing ε in a way corresponding to the integral W if one only has to take the first term, which is of highest order, into consideration. But if the subsequent terms in the expression are also of significance, the modulus will contain both Q and $\frac{dQ}{d\varepsilon}$, whereof it follows that the maximum points will be displaced, and that the modulus in general cannot become 0 as a consequence of the periodic changes. The periodicity will in this way become more blurred.

By comparison of the two expressions given in (43) and (49) for the integral (42), it is seen that the former is of the magnitude $\alpha^{\frac{1}{2}}$, the latter is of the order $\alpha^{\frac{2}{3}}$. How the transition happens from the one expression to the other can be seen if one imagines H decreasing to a very small quantity while one keeps $G - 2p\pi = 0$. One will then in the integral (42) be able to set $z = z' + \delta$ and determine δ such that the coefficient for z'^2 in the exponent becomes 0.

[1] Trans. of the Cambr. Phil. Soc. t. 9. p. 166.

Hereby we arrive at the form assumed in (45), where $G - 2p\pi$ becomes equal to $-\dfrac{H^2}{3I}$, and consequently

$$3\varepsilon = H^2 \sqrt[3]{\frac{\alpha^2}{I^4}}\,.$$

It is seen hereof that ε necessarily remains *positive* at this transition from the integral (42) to the integral (45). The transition from (43) to (49) thus happens through the periodic motion described above with positive decreasing m or ε, such that the last and largest maximum is reached before ε becomes 0, while modulus quickly decreases from here to 0 as ε at the same time goes through 0 to lower and lower negative values.

Lastly, we will in the following sections also meet sums that may be rewritten as an integral of the form

$$\int_0^{z_1} dz \left(A\frac{z}{\alpha} + B\frac{z^3}{\alpha^3} + \ldots \right) e^{\left(F\alpha + G\frac{z^2}{\alpha} + H\frac{z^4}{\alpha^3} + I\frac{z^6}{\alpha^5} + \ldots\right)i}\,. \tag{51}$$

When setting herein $Gz^2 = \alpha x$ and G is not 0 or very small, the upper bound of x will belong to the kind of quantities denoted ω above, and since the terms of lower order than the unit are discarded, the result of the integration becomes

$$\frac{A}{2G} e^{\left(F\alpha + \frac{\pi}{2}\right)i}\,. \tag{52}$$

On the other hand, if G is very small, we set $Hz^4 = \alpha^3 x^2$, the upper bound of x is as before denoted by ω, and for brevity we set

$$G = \pm\varepsilon\sqrt{\frac{H}{\alpha}}\,, \tag{53}$$

where the uppermost sign corresponds to G being positive, the bottommost to G being negative. The integral then becomes

$$\frac{1}{2H}\int_0^{\omega} dx \left((\alpha H)^{\frac{1}{2}} A + Bx + \frac{AIi}{H}x^3 \right) e^{(F\alpha \pm \varepsilon x + x^2)i}\,. \tag{54}$$

For $\varepsilon = 0$, we obtain from this by integration

$$\frac{A}{4}\sqrt{\frac{\alpha\pi}{H}} e^{\left(F\alpha + \frac{\pi}{4}\right)i} + \frac{1}{4H^2}(BH - AI) e^{\left(F\alpha + \frac{\pi}{2}\right)i}\,, \tag{55}$$

while the ordinary integral (54) may be expressed by

$$\frac{e^{F\alpha i}}{2H}\left((\alpha H)^{\frac{1}{2}} AQ \mp iB\frac{dQ}{d\varepsilon} \mp \frac{AI}{H}\frac{d^3Q}{d\varepsilon^3} \right)\,, \tag{56}$$

as

$$Q = \int_0^{\omega} dx\, e^{(\pm\varepsilon x + x^2)i} \tag{57}$$

From this last integral, we obtain by taking the derivative with respect to ε and by partial integration

$$\frac{dQ}{d\varepsilon} = \mp\frac{1}{2} - \frac{\varepsilon i}{2}Q\,, \tag{58}$$

of which we furthermore find

$$\frac{d^3Q}{d\varepsilon^3} = \pm\left(\frac{i}{2} + \frac{\varepsilon^2}{8}\right) + \left(-\frac{3\varepsilon}{4} + \frac{\varepsilon^3 i}{8}\right)Q\,. \tag{59}$$

By insertion of these values in (56), this expression for the integral that we seek will be determined by the integral Q.

to be continued…