On the Computation of the Principal Curvatures of Regular Surfaces
D R A F T

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March 17, 2000

1 Introduction

This is an attempt at presenting in very concise but also complete form the steps needed to compute the principal curvatures for a regular surface. Two methods are presented. The first method is based on a parametric representation of the surface, and this method is covered in detail. The second method is based on an implicit representation, it is somewhat simpler, but also less frequently used, and it is covered in less detail.

Probably the biggest problem in understanding this text is the sloppy notation. For instance, several different notations are used (depending on context) to indicate derivatives. f' or $\frac{df}{dt}$ is the derivative of a function. $\frac{d\mathbf{x}}{dt}$ is the derivative of a vector function, $\mathbf{x}_u$ is the partial derivative of a vector function w.r.t. u, but in a few places $\frac{\partial \mathbf{x}}{\partial u}$ is used for the same thing. Of course, all the derivatives and partial derivatives are "at a point". Usually, we are well aware of what point we are interested in, so instead of (e.g.) $\mathbf{x}_u(x, y)$ we simply write $\mathbf{x}_u$.

Finally, it should be mentioned that in the following presentation, almost everything that is not directly related to finding the principal curvatures has been shaved away, so if you want a more complete presentation of the topics try a book on differential geometry.

Do Carmo [1] is recommended as a good, exhaustive textbook.

For a gentle but very lucid introduction to the notion of curvature, read the chapter on differential geometry in David Hilbert [2].

Jan Koenderink [3] will make you unlearn clunky old stuff like the cross product and replace them with funky new stuff like the wedge product (which is essentially the same, but seen from a different view–point.) His book is neither easy nor really a "textbook", but it is interesting and insightful.

Finally, If all you need is a quick and concise math reference, try [4]

2 Definitions

Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be a parametric surface and

$$\mathbf{x}(\alpha(t)) = \mathbf{x}(u(t), v(t)) \tag{1}$$

be a curve in that surface, where $\alpha : \mathbb{R} \to \mathbb{R}^2$ is a 2D curve. In the following we will simplify the notation and write $\mathbf{x}(t)$ instead of $\mathbf{x}(\alpha(t))$.

The *arc length* of a curve from a point t_0 to t is

$$s(t) = \int_{t_0}^{t} \left| \frac{d\mathbf{x}}{dt} \right| dt \tag{2}$$

Differentiating the above equation, we obtain

$$\frac{ds}{dt} = \left| \frac{d\mathbf{x}}{dt} \right| \tag{3}$$

These relations will become useful in the following, because we know that for an arc length parameterized curve $\mathbf{x}(s) = \mathbf{x}(t(s))$, the unit tangent to the curve at a point $\mathbf{p}_0 = \mathbf{x}(u(t_0), v(t_0))$ is

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{d\mathbf{x}}{dt} \left| \frac{d\mathbf{x}}{dt} \right|^{-1} \tag{4}$$

In case you were wondering, a curve is parametrized by arc length if the length of the curve from $\mathbf{x}(t_0)$ to $\mathbf{x}(t_1)$ is equal to $\lvert t_1 - t_0 \rvert$, i.e. we move as far along the curve as we do in parameter space.

Furthermore, the *curvature vector* is

$$\mathbf{k} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\frac{dt}{ds} = \frac{d\mathbf{t}}{dt} \left| \frac{d\mathbf{x}}{dt} \right|^{-1} \tag{5}$$

The (unit) *normal* at a point on the surface is.

$$\mathbf{N} = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} \tag{6}$$

The *normal curvature* is the length of the curvature vector projected onto the normal.

$$\kappa_n = \mathbf{k} \cdot \mathbf{N} \tag{7}$$

All curves that lie in the surface and have the same tangent at a given point have the same normal curvatures at that point.

In general, the curvature vector of a curve $\mathbf{x}(\alpha(t))$ at a given point and the surface normal at the same point need not be parallel. However, if the curve is a *normal section* (i.e. the curve is the intersection of the surface and a plane which contains the normal – illustrated in figure 1) then the normal and the curvature vector are parallel, and

$$|\mathbf{k}| = |\mathbf{k} \cdot \mathbf{N}| = |\kappa_n| \tag{8}$$

If we rotate the normal plane about the normal, the value of the normal

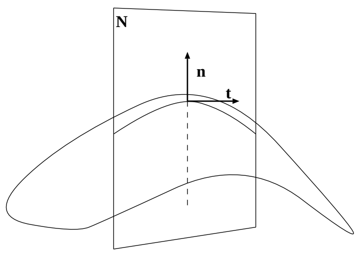

Figure 1: Normal Section

curvature will change smoothly as we turn the normal plane. During this rotation, the normal curvature attains maximum and minimum values. The maximum and minimum values correspond to two perpendicular directions of the tangent. Intuitively, if we have found the tangential direction which yields the maximum curvature, then we know that the perpendicular direction yields the minimum curvature. These two directions are known as the *principal directions* and the curvature values are, unsurprisingly, called the *principal curvatures*.

3 The First and Second Fundamental Forms

The *first fundamental form* is just the dot product of a tangent vector with itself, but if the tangent vector is expressed in the basis of the two first order partial derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ of $\mathbf{x}(u,v)$ things become slightly more interesting.

$$\begin{aligned}
I(\mathbf{w}) &= \mathbf{w}\cdot\mathbf{w} && (9)\\
&= (\mathbf{x}_u u' + \mathbf{x}_v v')\cdot(\mathbf{x}_u u' + \mathbf{x}_v v') && (10)\\
&= \mathbf{x}_u\cdot\mathbf{x}_u(u')^2 + 2\mathbf{x}_u\cdot\mathbf{x}_v u'v' + \mathbf{x}_v\cdot\mathbf{x}_v(v')^2 && (11)\\
&= E(u')^2 + 2Fu'v' + G(v')^2 && (12)
\end{aligned}$$

where the coefficients of I are

$$\begin{aligned}
E &= \mathbf{x}_u\cdot\mathbf{x}_u && (13)\\
F &= \mathbf{x}_u\cdot\mathbf{x}_v && (14)\\
G &= \mathbf{x}_v\cdot\mathbf{x}_v && (15)
\end{aligned}$$

and $\mathbf{w} = u'\mathbf{x}_u + v'\mathbf{x}_v$ is a tangent vector. If the length of the tangent vector is 1, then obviously the value of I is also 1.

$d\mathbf{N}(\mathbf{w})$ is a linear map that maps a tangent vector to a vector which expresses the change of the normal in the direction of the tangent. The *second fundamental form* is minus the dot product of $d\mathbf{N}(\mathbf{w})$ and $\mathbf{w}$.

$$\begin{aligned}
II(\mathbf{w}) &= -d\mathbf{N}(\mathbf{w})\cdot\mathbf{w} && (16)\\
&= -(\mathbf{N}_u u' + \mathbf{N}_v v')\cdot(\mathbf{x}_u u' + \mathbf{x}_v v') && (17)\\
&= -(\mathbf{N}_u\cdot\mathbf{x}_u)(u')^2 - ((\mathbf{N}_u\cdot\mathbf{x}_v) + (\mathbf{N}_v\cdot\mathbf{x}_u))u'v' - (\mathbf{N}_u\cdot\mathbf{x}_u)(u')^2 && (18)\\
&= (\mathbf{N}\cdot\mathbf{x}_{uu})(u')^2 + 2(\mathbf{N}\cdot\mathbf{x}_{uv})u'v' + (\mathbf{N}\cdot\mathbf{x}_{vv})(u')^2 && (19)\\
&= e(u')^2 + 2fu'v' + g(u')^2 && (20)
\end{aligned}$$

where the coefficients of II are

$$\begin{aligned}
e &= \mathbf{N}\cdot\mathbf{x}_{uu} && (21)\\
f &= \mathbf{N}\cdot\mathbf{x}_{uv} && (22)\\
g &= \mathbf{N}\cdot\mathbf{x}_{vv} && (23)
\end{aligned}$$

In the above we have used the fact that

$$\begin{aligned}
0 = \frac{d}{du}(\mathbf{N}\cdot\mathbf{x}_u) &= \mathbf{N}_u\cdot\mathbf{x}_u + \mathbf{N}\cdot\mathbf{x}_{uu} \\
0 = \frac{d}{du}(\mathbf{N}\cdot\mathbf{x}_v) &= \mathbf{N}_u\cdot\mathbf{x}_v + \mathbf{N}\cdot\mathbf{x}_{vu} \\
0 = \frac{d}{dv}(\mathbf{N}\cdot\mathbf{x}_u) &= \mathbf{N}_v\cdot\mathbf{x}_u + \mathbf{N}\cdot\mathbf{x}_{uv} \\
0 = \frac{d}{dv}(\mathbf{N}\cdot\mathbf{x}_v) &= \mathbf{N}_v\cdot\mathbf{x}_v + \mathbf{N}\cdot\mathbf{x}_{vv}
\end{aligned}$$

And, in case you were wondering, yes:

$$\mathbf{x}_{uv} = \mathbf{x}_{vu}$$

4 The Normal Curvature

As we have already mentioned, the normal curvature is the length of the curvature vector, but the normal curvature can be expressed in terms of the first and second fundamental forms. (5) projected onto the normal:

$$\begin{aligned}
\kappa_n &= \mathbf{k}\cdot\mathbf{N} = \frac{d\mathbf{t}}{ds}\cdot\mathbf{N} && (24)\\
&= \frac{d\mathbf{t}}{dt}\left|\frac{d\mathbf{x}}{dt}\right|^{-1}\cdot\mathbf{N} && (25)\\
&= -\mathbf{t}\cdot\frac{d\mathbf{N}}{dt}\left|\frac{d\mathbf{x}}{dt}\right|^{-1} && (26)\\
&= -\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{N}}{dt}\left|\frac{d\mathbf{x}}{dt}\right|^{-2} && (27)\\
&= -\frac{\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{N}}{dt}}{\frac{d\mathbf{x}}{dt}\frac{d\mathbf{x}}{dt}} && (28)\\
&= -\frac{\mathbf{w}\cdot d\mathbf{N}(\mathbf{w})}{\mathbf{w}\cdot\mathbf{w}} && (29)\\
&= -\frac{II(\mathbf{w})}{I(\mathbf{w})} && (30)
\end{aligned}$$

In the above, we use the fact that the normal and the tangent are always orthogonal, hence

$$0 = \frac{d}{dt}(\mathbf{t} \cdot \mathbf{N}) = \frac{d\mathbf{t}}{dt} \cdot \mathbf{N} + \mathbf{t} \cdot \frac{d\mathbf{N}}{dt}$$

Also remember that

$$\mathbf{w} = \frac{d\mathbf{x}}{dt} = \mathbf{x}_u u' + \mathbf{x}_v v'$$

and that

$$\frac{d\mathbf{N}}{dt} = \mathbf{N}_u u' + \mathbf{N}_v v'$$

What this really means is that the normal curvature at a given point, for a given direction is the ratio of the first and second fundamental forms.

5 The Principal Curvatures

Any tangent vector can be expressed in terms of the basis $\mathbf{x}_u$ and $\mathbf{x}_v$. Let $\mathbf{w} = (\mathbf{x}_u, \mathbf{x}_v)(u', v')^T$ be a tangent vector expressed in that basis. The map $d\mathbf{N}(\mathbf{w})$ is, as we have previously mentioned, a map which takes a tangent vector and maps it to the change of the normal in the direction of the tangent. It is an important fact that $d\mathbf{N}(\mathbf{w})$ is always perpendicular to $\mathbf{N}$, because this means that $d\mathbf{N}(\mathbf{w})$ can be expressed in the $\mathbf{x}_u, \mathbf{x}_v$ basis:

$$d\mathbf{N}((u', v')^T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{31}$$

The exciting question, now, is what the values of a, b, c, and d really are. Do Carmo has made the relevant computations, so I shall skip this step. Look at [1] pages 153–156. The important result is that a, b, c, and d can be expressed in terms of the coefficients from the First and Second Fundamental

Forms:

$$\begin{aligned} a &= \frac{fF - eG}{EG - F^2} && (32) \\ b &= \frac{gF - fG}{EG - F^2} && (33) \\ c &= \frac{eF - fE}{EG - F^2} && (34) \\ d &= \frac{fF - gE}{EG - F^2} && (35) \end{aligned}$$

At this point, it is very important to observe that $d\mathbf{N}$ is a self adjoint linear map, and this means ([1] p. 216) that it has two eigenvalues which correspond to the principal curvatures. That is, if $\mathbf{w}_1 = (\mathbf{x}_u, \mathbf{x}_v)(u_1', v_1')^T$ is a tangent vector in the direction of greatest principal curvature, and $\mathbf{w}_2 = (\mathbf{x}_u, \mathbf{x}_v)(u_2', v_2')^T$ is a vector in the direction of smallest principal curvature, then

$$\begin{aligned} d\mathbf{N}(\mathbf{w}_1) &= -k_1(u_1', v_1')^T && (36) \\ d\mathbf{N}(\mathbf{w}_2) &= -k_2(u_2', v_2')^T && (37) \end{aligned}$$

where the two eigenvalues are the principal curvatures. They can be found in the usual way by

$$0 = ||d\mathbf{N} - (-k)I|| = \left|\left| \begin{array}{cc} a+k & b \\ c & d+k \end{array} \right|\right| \quad (38)$$

which yields the quadratic equation

$$(a+k)(d+k) - cb = k^2 + (a+d)k + ad - cb = 0 \quad (39)$$

whose solution is

$$k = \frac{-(a+d) + - \sqrt{(a+d)^2 - 4(ad - cb)}}{2} \quad (40)$$

6 Example: The Ellipsoid

The parametric map which maps a point in 2D onto the surface of an ellipseoid $E : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ can be defined as

$$E(\theta, \phi) = \begin{pmatrix} E_x(\theta, \phi) \\ E_y(\theta, \phi) \\ E_z(\theta, \phi) \end{pmatrix} = \begin{pmatrix} a\cos(\theta)\sin(\phi) \\ b\sin(\theta)\sin(\phi) \\ c\cos(\phi) \end{pmatrix} \quad (41)$$

First, we compute the coefficients of the first and second fundamental forms, but to do so we must find the first and second order derivatives of the parametric expression:

$$\mathbf{E}_\theta = \begin{pmatrix} -a\sin(\theta)\sin(\phi) \\ b\cos(\theta)\sin(\phi) \\ 0 \end{pmatrix} \tag{42}$$

$$\mathbf{E}_\phi = \begin{pmatrix} a\cos(\theta)\cos(\phi) \\ b\sin(\theta)\cos(\phi) \\ -c\sin(\phi) \end{pmatrix} \tag{43}$$

$$\mathbf{E}_{\theta\theta} = \begin{pmatrix} -a\cos(\theta)\sin(\phi) \\ -b\sin(\theta)\sin(\phi) \\ 0 \end{pmatrix} \tag{44}$$

$$\mathbf{E}_{\phi\phi} = \begin{pmatrix} -a\cos(\theta)\sin(\phi) \\ -b\sin(\theta)\sin(\phi) \\ -c\cos(\phi) \end{pmatrix} \tag{45}$$

$$\mathbf{E}_{\theta\phi} = \begin{pmatrix} -a\sin(\theta)\cos(\phi) \\ b\cos(\theta)\cos(\phi) \\ 0 \end{pmatrix} \tag{46}$$

It is not convenient to carry symbolic computations much further, because the number of terms gets quite large for the second fundamental form.

Therefore, to compute the principal curvatures, plug numerical values into the formulae above, find the normal and calculate the coefficients for I and II. Calculate a,b,c, and d, and plug them into (40).

7 Alternative Method

In all the preceding sections, we have assumed that the ellipsoid was represented parametrically. Of course, we might also have an implicitly represented ellipsoid:

$$f(\mathbf{p}) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 - 1 = 0 \tag{47}$$

where $\mathbf{p} = (x, y, z)$ is a point in space. It is also possible to find the curvature starting from this equation. Once more we need the tangent

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} \tag{48}$$

to an arc length parameterized curve $\mathbf{x}(s)$ which we will assume is a normal section. This means that its curvature is the normal curvature in the direction of the tangent. Assuming that either the normal points the same way as the curvature vector or that the curvature is negative, it follows that

$$\frac{d\mathbf{t}}{ds} = \kappa_n \mathbf{N} = \kappa_n \frac{\nabla f}{||\nabla f||} \tag{49}$$

where κ_n is the normal curvature, $\mathbf{N}$ is the normal and $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})^T$ is the gradient of f. For simplicity, we will write $\mathbf{g}$ instead of ∇f in the following. Incidentally, we observe that $\kappa_n \, ds \, \mathbf{N}$ is the change of the tangent as we take a ds length step along the curve.

We know that the tangent is always orthogonal to the gradient, hence

$$0 = \frac{d(\mathbf{g} \cdot \mathbf{t})}{ds} = \frac{d\mathbf{g}}{ds} \cdot \mathbf{t} + \mathbf{g} \cdot \frac{d\mathbf{t}}{ds} \tag{50}$$

Also, we know that according to the chain rule

$$\frac{d\mathbf{g}}{ds} = \frac{\partial \mathbf{g}}{\partial x}\frac{dx}{ds} + \frac{\partial \mathbf{g}}{\partial y}\frac{dy}{ds} + \frac{\partial \mathbf{g}}{\partial z}\frac{dz}{ds} = H\mathbf{t} \tag{51}$$

where H is the Hessian, i.e. the matrix of the second order partial derivatives

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} \tag{52}$$

Using (49), (50) and (51) we may now obtain an expression for the curvature using just the tangent, the Hessian and the gradient

$$\begin{aligned} \kappa_n \frac{\mathbf{g}}{||\mathbf{g}||} &= \frac{d\mathbf{t}}{ds} && (53) \\ \Leftrightarrow \kappa_n ||\mathbf{g}|| &= \mathbf{g} \cdot \frac{d\mathbf{t}}{ds} && (54) \\ \Leftrightarrow \kappa_n &= \mathbf{g} \cdot \frac{d\mathbf{t}}{ds}\frac{1}{||\mathbf{g}||} && (55) \\ \Leftrightarrow \kappa_n &= -\mathbf{t} \cdot \frac{d\mathbf{g}}{ds}\frac{1}{||\mathbf{g}||} && (56) \\ \Leftrightarrow \kappa_n &= -\frac{\mathbf{t}^T H \mathbf{t}}{||\mathbf{g}||} && (57) \end{aligned}$$

7.1 Applying the Alternative Method

For an ellipsoid we easily derive the Hessian

$$H = \begin{bmatrix} 2/a^2 & 0 & 0 \\ 0 & 2/b^2 & 0 \\ 0 & 0 & 2/c^2 \end{bmatrix} \tag{58}$$

and the gradient is just

$$\mathbf{g} = \nabla f = 2 \begin{bmatrix} x/a^2 \\ y/b^2 \\ z/c^2 \end{bmatrix} \tag{59}$$

and a unit length tangent in arbitrary direction can easily be constructed from the gradient, so (57) suffices to calculate the normal curvature at a given point on an ellipsoid in any direction.

Using a Lagrange multiplier technique, it is possible to find the direction of maximum curvature and the maximum curvature at a given point using (57) and the already mentioned constraints that the tangent be unit length and orthogonal to the surface. This text should be updated to include that method.

References

[1] Manfredo P. Do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice Hall, 1976.

[2] David Hilbert and S Cohn-Vossen. *Geometry and the Imagination.* AMS Chelsea Puplishing, 1990.

[3] Jan J. Koenderink. *Solid Shape.* MIT Press, 1990.

[4] Eric Weisstein. http://mathworld.wolfram.com/. WWW.