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IT University Technical Report Series

TR-2005-66

ISSN 1600–6100

July 2005

ISSN 1600–6100

ISBN 87-7949-096-4

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Inge Li Gørtz*

Abstract

In this paper we give approximation algorithms and inapproximability results for various asymmetric k-center with minimum coverage problems. In the k-center with minimum coverage problem, each center is required to serve a minimum number of clients. These problems have been studied by Lim *et al.* [Theor. Comput. Sci. 2005] in the symmetric setting.

In the *q-all-coverage k-center problem* each center must serve at least q vertices (including itself). In the *q-coverage k-center problem* each center must serve at least q non-center nodes. We provide $O(\log^* n)$-approximation algorithms for the *asymmetric* q-all-coverage and q-coverage problems in both the unweighted and weighted case. This is optimal within a constant factor. Lim *et al.* also study the *q-coverage k-supplier problem* and the priority version of all the mentioned problems in the symmetric setting. We show that the asymmetric *q-coverage k-supplier problem* and the priority versions of asymmetric q-coverage k-center and asymmetric q-all-coverage k-center are inapproximable.

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1 Introduction

Imagine you have a delivery service. You want to place your k delivery hubs at locations that minimize the maximum distance between customers and their nearest hubs. This is the *k-center problem*—a type of clustering problem that is similar to the facility location [18] and k-median [3] problems. The motivation for the *asymmetric* k-center problem, in our example, is that traffic patterns or one-way streets might cause the travel time from one point to another to differ depending on the direction of travel. In the *k-center with minimum coverage problem* each center is required to serve a minimum number of clients. The motivation is to try to balance the workload between the centers, such that you are not wasting resources on e.g. delivery hubs that are almost never used because they are located in isolated places. This problem has also been addressed by Charikar *et al.* [5]. They studied the k-center problem with outliers, where a small subset of costumers may be denied service. Unfortunately, this problem cannot be approximated in the asymmetric case unless P $=$ NP [10]. Other examples where the k-center problem with minimum coverage is useful is in planning location of hospitals. Requiring the hospitals to serve at least a certain number of neighborhoods/patients allows for economies of scale and specialization. A small hospital cannot have a specialist in every area, whereas a larger hospital can hire more specialized people and also possibly be more effective. In this paper we study asymmetric version of k-center with minimum coverage problems.

Symmetry is a vital concept in graph approximation algorithms. Recently, the k-center problem was shown to be $\Omega(\log^* n)$ hard to approximate [7, 8, 11], even though the symmetric version has a factor 2 approximation. Facility location and k-median both have constant factor algorithms in the symmetric case, but are provably $\Omega(\log n)$ hard to approximate without symmetry [1]. The traveling salesman problem is a little better, in that no super-constant hardness is known, but without symmetry no algorithm better than $\frac{4}{3}\log n)$ [15] has been found either.

Definition 1.1 (k-Center)**.** Given $G = (V, E)$, a complete graph with nonnegative (but possibly infinite) edge costs, and a positive integer k, find a set S of k vertices, called *centers*, with minimum covering radius. The covering radius of a set S is the minimum distance R such that every vertex in V is within distance R of some vertex in S.

Kariv and Hakimi [16] showed that the k-center problem is NP-hard. Without the triangle inequality the problem is NP-hard to approximate within any factor (there is a straightforward reduction from the dominating set problem). We henceforth assume that the edge costs satisfy the triangle inequality. Hsu and Nemhauser [14], using the same reduction, showed that the metric k-center problem cannot be approximated within a factor of $(2 - \epsilon)$ unless P $=$ NP. In 1985 Hochbaum and Shmoys [12] provided a (best possible) factor 2 algorithm for the metric k-center problem. In 1996 Panigrahy and Vishwanathan [19, 21] gave the first approximation algorithm for the asymmetric problem, with factor $O(\log^* n)$. Archer [2] proposed two $O(\log^* k)$ algorithms based on many of the ideas of Panigrahy and Vishwanathan. The complementary $\Omega(\log^* n)$ hardness result [7, 8, 11] shows that these approximation algorithms are asymptotically optimal.

1.1 k-Center with Minimum Coverage

A number of variants of the k-center problem have been explored in the context of symmetric graphs [4–6, 13, 17, 20] and in the asymmetric setting [4, 10].

In this paper we give approximation algorithms and inapproximability results for various asymmetric k-center with minimum coverage problems. These problems have been studied by Lim *et al.* [17] in the symmetric setting. In k-center with minimum coverage, each center is required to serve a minimum of clients. This problem is motivated by requirements to balance the workload of centers. Lim *et al.* studied the following problems:

- The *q-all-coverage k-center problem*, where each center must cover at least q vertices (including itself).
- The *q-coverage k-center problem*, where each center must cover at least q non-center nodes.
- The *q-coverage k-supplier problem*. Here V is divided into two disjoint subsets S and C. The object is to find a subset U of S, $|U| \leq k$, that minimizes R such that U covers C within radius R and each center in U covers at least q demands in C.

Furthermore, Lim *et al.* studied both the weighted and the priority versions of these problems. In the weighted k-center problem instead of a restriction on the number of centers we can use, each vertex has a weight and we have a budget k that limits the total weight of centers. In the priority k-center problem each vertex has a priority and the distance we try to minimize is the prioritized distance: Given vertex v and center s the distance from s to v is $d(s, v) \cdot p_v$, where p_v is the priority of v.

For the q-all-coverage k-center problem Lim *et al.* gave an 2-approximation algorithm, and an 3-approximation algorithm for the weighted and priority versions of the problem. For the q-coverage k-center problem they gave an 2-approximation algorithm, and an 4-approximation algorithm for the weighted and priority versions of the problem. For the q-coverage k-supplier problem they gave an 3-approximation algorithm for both the basic, the weighted, and the priority version.

Our Results We give $O(\log^* n)$-approximation algorithms for the *asymmetric* q-all-coverage and q-coverage problems in both the unweighted and weighted case. Of course, the algorithm for the weighted case also works for the unweighted case (set all weights = 1), but the algorithm for the unweighted case is simpler and the hidden constant in $O(\log^* n)$ is smaller using this algorithm.

In [10] it is showed that the asymmetric priority k-center and asymmetric k-supplier problems cannot be approximated within any factor unless P $=$ NP. Since the q-all-coverage k-center problem and the q-cover k-center problem are generalizations of the k-center problem (set $q = 1$ and $q = 0$, respectively), the priority version of these problems cannot be approximated within any factor in the asymmetric case unless P $=$ NP. Since the q-coverage k-supplier problem is a generalization of the k-supplier problem ($q = 0$), it cannot be approximated within any factor in the asymmetric version unless P $=$ NP.

2 Definitions

To avoid any uncertainty, we note that $\log$ stands for $\log_2$ by default, while $\ln$ stands for $\log_e$.

Definition 2.1. For every integer $i > 1$, $\log^i x = \log(\log^{i-1} x)$, and $\log^1 x = \log x$. We let $\log^* x$ represent the smallest integer i such that $\log^i x \leq 2$.

The input to the asymmetric k-center problem is a distance function d on every *ordered* pair of vertices—distances are allowed to be infinite—and a bound k on the number of centers. Note that we assume that the edges are *directed*.

Definition 2.2. Vertex c *covers* vertex v *within* r, or c r*-covers* v, if $d_{cv} \leq r$. We extend this definition to a sets so that a set C r-covers a set A if for every $a \in A$ there is some $c \in C$ such that c covers a within r. Often we abbreviate "1-covers" to "covers".

Many of the algorithms for k-center and its variants do not, in fact, operate on graphs with edge costs. Rather, they consider bottleneck graphs [13], in which only those edges with distance lower than some threshold are included, and they appear in the bottleneck graph with unit cost. Since the optimal value of the covering radius must be one of the $n(n-1)$ distance values, many algorithms essentially run through a sequence of bottleneck graphs of every possible threshold radius in ascending order. This can be thought of as *guessing* the optimal radius R_{OPT}. The approach works because the algorithm either returns a solution, within the specified factor of the current threshold radius, or it fails, in which case R_{OPT} must be greater than the current radius.

Definition 2.3 (Bottleneck Graph G_r)**.** For $r > 0$, define the bottleneck graph G_r of the graph $G = (V, E)$ to be the graph $G_r = (V, E_r)$, where $E_r = \{(i, j) : d_{ij} \leq r\}$ and all edges have unit cost.

Most of the following definitions apply to *bottleneck* graphs.

Definition 2.4 (Power of Graphs)**.** The t^{th} power of a graph $G = (V, E)$ is the graph $G^t = (V, E^{(t)})$, $t > 1$, where $E^{(t)}$ is the set of ordered pairs of distinct vertices that have a path of at most t edges between them in G.

Definition 2.5. For $i \in \mathbb{N}$ define

$$\Gamma_i^+(v) = \{u \in V \mid (v, u) \in E^i\} \cup \{v\}, \qquad \Gamma_i^-(v) = \{u \in V \mid (u, v) \in E^i\} \cup \{v\} \ ,$$

i.e., in the bottleneck graph there is a path of length at most i from v to u, respectively u to v.

Notice that in a symmetric graph $\Gamma_i^+(v) = \Gamma_i^-(v)$. We extend this notation to sets so that $\Gamma_i^+(S) = \{u \in V \mid u \in \Gamma_i^+(v) \text{ for some } v \in S\}$, with $\Gamma_i^-(S)$ defined similarly. We use $\Gamma^+(v)$ and $\Gamma^-(v)$ instead of $\Gamma_1^+(v)$ and $\Gamma_1^-(v)$.

Definition 2.6. For $i \in \mathbb{N}$ define

$$\Upsilon_q(v) = \{u \mid u \in \Gamma^-(v) \text{ and } \deg(u) \geq q\},$$

i.e., u covers v and has degree at least q.

Note that $\Upsilon_0(v) = \Gamma^-(v)$.

Definition 2.7 (Center Capturing Vertex (CCV))**.** A vertex v is a *center capturing vertex* (CCV_q) if $\Upsilon_{q-1}(v) \in \Gamma^+(v)$, .i.e., v covers every vertex of degree $q - 1$ that covers v.

We use CCV instead of CCV_2. To get some intuition about the notion of CCV assume we have an instance of the q-all-coverage k-center problem. In the graph $G_{R_{\mathsf{OPT}}}$ the optimum center that covers v must lie in $\Upsilon_q(v)$; for a CCV_q v, it lies in $\Gamma^+(v)$, hence the name. In symmetric graphs all vertices are CCVs and this property leads to the 2-approximation for the standard k-center problem.

The following two problems, related to k-center, are both NP-complete [9].

Definition 2.8 (Dominating Set)**.** Given a graph $G = (V, E)$, and a weight function $w : V \to \mathbb{Q}^+$ on the vertices, find a minimum weight subset $D \subseteq V$ such that every vertex $v \in V$ is covered by D, i.e., $\Gamma^+(D) = V$.

Definition 2.9 (Set Cover)**.** Given a universe $\mathcal{U}$ consisting of n elements, a collection $\mathcal{S} = \{S_1, \ldots, S_k\}$ of subsets of $\mathcal{U}$, and a weight function $w : \mathcal{S} \to \mathbb{Q}^+$, find a minimum weight sub-collection of $\mathcal{S}$ that includes all elements of $\mathcal{U}$.

3 Asymmetric k-Center Review

In this section we review the $O(\log^* n)$-approximation algorithm for the standard asymmetric k-center problem by Panigrahy and Vishwanathan [19]. It forms a basis for our approximation algorithms for the asymmetric k-center with minimum coverage problems. The algorithm by Panigrahy and Vishwanathan has two phases, the *halve* phase, sometimes called the *reduce* phase, and the *augment* phase. As described above, the algorithm guesses R_{OPT}, and works in the bottleneck graph $G_{R_{\mathsf{OPT}}}$. In the halve phase we find a CCV v, include it in the set of centers, mark every vertex in $\Gamma_2^+(v)$ as covered, and repeat until no CCVs remain unmarked. The CCV property ensures that, as each CCV is found and vertices are marked, the unmarked portion of the graph can be covered with one fewer center. Hence if k'' CCVs are obtained, the unmarked portion of the graph can be covered with $k' = k - k''$ centers. The authors then prove that this unmarked portion, CCV-free, can be covered with only $k'/2$ centers if we use radius 5 instead of 1. That is to say, $k'/2$ centers suffice in the graph $G^5_{R_{\mathsf{OPT}}}$.

The k-center problem in the bottleneck graph is identical to the dominating set problem. This is a special case of set cover in which the sets are the Γ^+ terms. In the augment phase, the algorithm recursively uses the greedy set cover procedure. Since the optimal cover uses at most $k'/2$ centers, the first cover has size at most $\frac{k'}{2} \log \frac{2n}{k'}$.

The centers in this first cover are themselves covered, using the greedy set cover procedure, then the centers in the second cover, and so forth. After $O(\log^* n)$ iterations the algorithm finds a set of at most k' vertices that, together with the CCVs, $O(\log^* n)$-covers the unmarked portion, since the optimal solution has $k'/2$ centers. Combining these with the k'' CCVs, we have k centers covering the whole graph within $O(\log^* n)$.

We now know that this approximation algorithms is asymptotically optimal [7, 8, 11].

4 Approximation of q-All-Coverage k-Center

In this section we give a $O(\log^* n)$-approximation algorithm for the asymmetric q-all-coverage k-center problem.

Definition 4.1 (q-All-Coverage k-Center)**.** Given $G = (V, E)$, a complete graph with nonnegative (but possibly infinite) edge costs, and a positive integer k, find a set S of k vertices, called *centers*, with minimum covering radius R, such that each center covers at least q vertices within radius R.

Our algorithm is based on Panigrahy and Vishwanathan's technique for the asymmetric k-center problem [19]. Just as their algorithm, our algorithm guesses R_{OPT}, and works in the bottleneck graph $G_{R_{\mathsf{OPT}}}$.

First we note that if we are in the right bottleneck graph any node either has out-degree at least $q-1$ or is covered by a node with out-degree at least $q-1$.

In the halve phase we find a CCV$_q$ v, include it in the set of centers, mark every vertex in $\Gamma_2^+(v)$ as covered, and repeat until no CCV$_q$s remain unmarked. The CCV$_q$ property ensures that, as each CCV$_q$ is found and vertices are marked, the unmarked portion of the graph can be covered with one fewer center. Hence if k'' CCV$_q$s are obtained, the unmarked portion of the graph can be covered with $k' = k - k''$ centers.

We will prove that this unmarked portion, CCV$_q$-free, can be covered with only $k'/2$ centers if we use radius 5 instead of 1. That is to say, $k'/2$ centers suffice in the graph $G^5_{R_{\mathsf{OPT}}}$.

Panigrahy and Vishwanathan [19] show the following lemma.

Lemma 4.2 (Panigrahy and Vishwanathan [19])**.** *Let $G = (V, E)$ be a digraph with unit edge costs. Then there is a subset $S \subseteq V$, $|S| \leq |V|/2$, such that every vertex with positive indegree is reachable in at most 2 steps from some vertex in S.*

Henceforth call the vertices not yet covered/marked *active*. Using Lemma 4.2 we can show that after removing the CCVs from the graph, we can cover the active set with half the weight of an optimum cover if we are allowed to use distance 5 instead of 1.

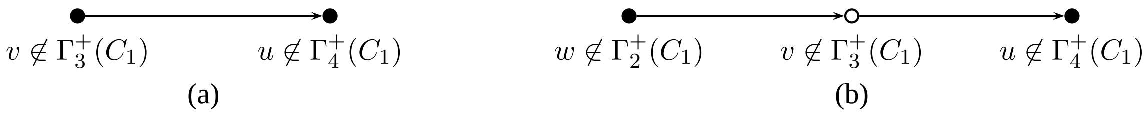

Figure 1: In (a) v is a center. Since $v \notin \Gamma_3^+(C_1)$, v is not in C_1, and thus $v \in U$. In (b) v is not a center. Since $v \notin \Gamma_3^+(C_1)$, v is in A, and thus v is covered by a center $w \in U$.

Lemma 4.3. *Consider a subset $A \subseteq V$ with the following properties: i) A has a cover consisting of vertices of size k, and each vertex in the cover covers at least q vertices. ii) A contains no CCV$_q$s. Assume there exists a set C_1 such that C_1 3-covers exactly $V \setminus A$, and every vertex in C_1 3-covers at least q vertices. Then there exists a set of vertices S of size $k/2$ that, together with C_1, 5-covers A, and every vertex in S covers at least q vertices.*

Proof. Let U be a subset of the optimal centers that covers A. We call $u \in U$ a *near* center if it can be reached in 4 steps from C_1, and a *far* center otherwise. Since C_1 5-covers all of the nodes covered by near centers, it suffices to choose S to 4-cover the far centers, so that S will 5-cover all the nodes they cover. We also need to ensure that any vertex in S 5-covers at least q vertices.

Define an auxiliary graph H on the (optimal) centers U as follows. There is an edge from x to y in H if and only if x 2-covers y in G (and $x \neq y$). The idea is to show that any far center has positive indegree in H. As a result, Lemma 4.2 shows there exists a set $S \in U$ with $|S| \leq k/2$ such that S 2-covers the far centers in H, and thus 4-covers them in G. Since $S \subseteq U$ and U is the set of optimal centers, all vertices in S covers at least q vertices.

Let u be any far center: note that $u \in A$. Since A contains no CCV$_q$s, there exists $v \in \Upsilon_q(u)$ that is not covered by u. Since u is a far center $u \notin \Gamma_4^+(C_1)$, and thus $v \notin \Gamma_3^+(C_1)$. Therefore, we have $v \in A$, since everything not 3-covered by C_1 is in A (see also Figure 1). If $v \in U$ then u is covered by another center in U, and thus has positive indegree in H. If v is not a center, there exists a vertex $w \in U$ that covers v and therefore 2-covers u, since v is covered in the optimal solution. Since $v \notin \Gamma^+(u)$, $w \neq u$. Hence u has positive indegree in H. □

In the augment phase we use the greedy set cover algorithm, which has approximation guarantee $1 + \ln(n/k)$, where n is the number of elements and k is the optimum number of sets. Only nodes that have degree at least $q - 1$ in the bottleneck graph G_i before the removal of CCVs are possible centers. It is easy to check wether it is possible to cover the graph with only these nodes. If not then we are not in the right bottleneck graph.

We now show that the tradeoff between the covering radius and the optimal cover size leads to an $O(\log^* n)$ approximation.

Lemma 4.4. *Given $A \subseteq V$, such that A has a cover of size k, where all centers in the cover covers at least q vertices, and a set $C_1 \subseteq V$ that covers $V \setminus A$, where all centers in C_1 covers at least q vertices. We can then find in polynomial time a set of centers of size at most $2k$ that, together with C_1, covers A (and hence V) within a radius of $O(\log^* n)$, such that all centers cover at least q vertices.*

Proof. We will apply the greedy set cover algorithm recursively. The initial set of centers S_0 is constructed as follows. For any vertex v for which $\Gamma^+(v) \cap A$ is non-empty, and which has out-degree at least $q - 1$ construct a set containing $\Gamma^+(v)$, identified by v.

The greedy algorithm set cover algorithm has approximation guarantee $O(\log(n/k))$, which is less than $\log_{1.5}(n/k)$ when $n \geq 2k$. Applying this algorithm thus results in a set S_1 of centers (the identifiers of the sets found by the algorithm) that covers A and has size at most $k \cdot \log_{1.5}(n/k))$, assuming $n \geq 2k$.

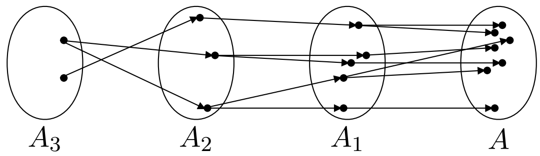

Figure 2: Example of recursive application of the greedy set cover algorithm. In each step we get fewer centers. The centers in A_3 3-covers everything in A.

The set C_1 covers $S_1 \setminus A$, so we need only consider $A_1 = S_1 \cap A$. We apply the greedy set cover algorithm again to obtain a set S_2 of size at most

$$k \cdot (\log_{1.5}(|A_1|/k)) = k \cdot (\log_{1.5}(k \log_{1.5}(n/k)/k)) = k \cdot (\log_{1.5}(\log_{1.5}(n/k))) ,$$

that covers A_1. We continue this procedure and note that at the ith iteration we have

$$|S_i| \leq k \cdot \log_{1.5}(|S_{i-1}|/k) .$$

By induction, after $O(\log^* n)$ iterations the size of our solution set, S_i, is at most $2k$. □

We can now combine Lemma 4.3 and Lemma 4.4 to get an approximation algorithm.

Theorem 4.5. *The q-all-coverage k-center problem can be approximated within a factor of $O(\log^* n)$ in polynomial time.*

Proof. Guess the optimum radius, R_{OPT}, and work in the bottleneck graph $G_{R_{\mathsf{OPT}}}$. Initially, the active set A is V. Repeat the following as many times as possible: Pick a CCV$_q$ v in A, add v to our solution set of centers, and remove the set $\Gamma_2^+(u)$ from A. Since v is covered by an optimum center in $\Gamma^-(v)$, and this optimum center lies in $\Gamma^+(v)$, $\Gamma_2^+(v)$ includes everything covered by it.

Let C_1 be the centers chosen in this first phase. We know the remainder of the graph, A, has a cover of total size $k' = k - |C_1|$.

Lemma 4.3 shows that we can cover the remaining uncovered vertices with at most $k'/2$ centers if we use covering radius 5. Let the active set A be $V \setminus \Gamma_5^+(C_1)$, and recursively apply the greedy algorithm as described in the proof of Lemma 4.4 on the graph $G_{R_{\mathsf{OPT}}}^5$. As a result, we have a set of size $2(k'/2) = k'$ that covers A within radius $O(\log^* n)$. □

5 Approximation of q-Coverage k-Center

Definition 5.1 (q-Coverage k-Center)**.** Given $G = (V, E)$, a complete graph with nonnegative (but possibly infinite) edge costs, and a positive integer k, find a set S of k vertices, called *centers*, with minimum covering radius R, such that each center R-covers at least q vertices in $V \setminus S$.

We use the algorithm from the previous section to find a set S of centers for the $(q+1)$-all-coverage k-center problem. First we note that the centers found in the halve phase all cover at least q non-centers, since when we pick a CCV$_{q+1}$ as v a center we mark $\Gamma_2^+(v)$ as covered and thus none of these at least q vertices will later be picked as centers. The potentially problematic centers are the centers found in the augment phase. These centers all cover q vertices, but they might not cover q non-centers.

Lemma 5.2. *Let S be a set of centers covering all vertices, such that each center in S covers at least q vertices. Then there is a set $S' \subseteq S$ of centers 2 covering all vertices, such that each center in S' 2-covers at least q vertices from $V \setminus S$. Moreover, S' can be found in polynomial time.*

Proof. Let P be the set of problematic centers, i.e., centers that do not cover q non-centers. To construct the set S' repeat the following as long as P is non-empty: Pick a center v from P. Remove all vertices $\Gamma^+(v) \cap S$ except v from S (and P), and remove all vertices in $\Gamma^-(v) \cap P$ from P. When P is empty set $S' = S' \cup S$.

Let v be a center in S'. We need to show that v 2-covers at least q non-center vertices. If v was never in P then clearly v covers at least q non-center vertices, as $S' \subseteq S$. Assume v was initially in P. Then either v was picked or some center in $\Gamma^+(v)$ was picked. If v was picked, then since v covers at least q vertices and all vertices covered by v now are non-centers, v covers at least q non-centers. If some center $u \in \Gamma^+(v)$ was picked then as u covers at least q non-centers v 2-covers at least q non-centers.

We must now show that S' 2-covers all vertices. Assume $v \in S$ was picked. Since all vertices in $\Gamma^-(v)$ are removed from P, v remains a center and thus $v \in S'$. Assume $v \in S$ was not picked by the procedure. If $v \notin S'$ then it must be the case that some vertex $u \in \Gamma^-(v)$ was picked. As just argued $u \in S'$. All vertices in $\Gamma^+(v)$ are 2-covered by u. Therefore, S' 2-covers all vertices covered by S. □

Using Lemma 5.2 together with Theorem 4.5 we get an $O(\log^*)$-approximation algorithm for the q-coverage k-center problem.

Theorem 5.3. *The q-coverage k-center problem can be approximated within factor $O(\log^* n)$ in polynomial time.*

Proof. Apply the algorithm from the previous section to find a set S of centers for the $(q+1)$-all-coverage k-center problem. Let α be the actual approximation ratio obtained by the $(q+1)$-all-coverage k-center algorithm on this instance.

Now apply the procedure from Lemma 5.2 on S in the graph $G^{\alpha}_{R_{\mathsf{OPT}}}$. This gives us a set of centers that 2α-covers all the vertices, and all the centers 2α-covers at least q non-center vertices. Since $\alpha = O(\log^* n)$ this gives an $O(\log^* n)$-approximation. □

6 Weighted Versions

In [10] an $O(\log^* n)$-approximation algorithm for the asymmetric weighted set cover problem is given. The algorithm works on bottleneck graphs and has a halve phase and an augment phase as the algorithm for the standard k-center problem. In the halve phase, the algorithm recursively finds a CCV, v, picks the lightest vertex u in $\Gamma^-(v)$ (which might be v itself) as a center, and mark everything in $\Gamma^+_3(u)$ as covered. It is shown that when there are no more CCVs left the unmarked vertices can be 49-covered by a set of weight at most a quarter of the optimum. In the augment phase, a greedy procedure for weighted sets and elements is applied recursively $O(\log^* n)$ times.

We can approximate the weighted version of the q-all-coverage k-center problem and the q-coverage k-center problem with a factor of $O(\log^* n)$ by adapting our algorithm for the weighted k-center problem to the approaches in the previous sections.

6.1 Weighted q-all-coverage k-center

The halve phase proceeds as follows: Find a CCV$_q$, pick the lightest vertex u in $\Upsilon_q(v)$ as a center, and mark $\Gamma^+_3(u)$ as covered. We will show that we can cover the remaining graph with weight no more than a quarter of the optimum if we use distance 49 instead of 1. We need the following lemma from [10].

Lemma 6.1 ([10]). *Let $G = (V, E)$ be a digraph with weighted vertices, but unit edge costs. Then there is a subset $S \subseteq V$, $w(S) \leq w(V)/2$, such that every vertex with positive indegree is reachable in at most 3 steps from some vertex in S.*

We can now show a lemma analog to Lemma 4.3. The proof is similar to that of Lemma 4.3.

Lemma 6.2. *Consider a subset $A \subseteq V$ with the following properties: i) A has a cover consisting of vertices of total weight W, and each vertex in the cover covers at least q vertices. ii) A contains no CCV$_q$s. Assume there exists a set C_1 such that C_1 3-covers exactly $V \setminus A$, and every vertex in C_1 3-covers at least q vertices. Then there exists a set of vertices S of weight $W/2$ that, together with C_1, 7-covers A, and every vertex in S covers at least q vertices.*

We will use the following greedy heuristic for the dominating set problem in weighted graphs to complete the algorithm: All vertices with outdegree at least $q-1$ are potential members of the dominating set (i.e. centers). Pick the most *efficient* vertex, i.e., the vertex that maximizes $w(A \cap \Gamma^+(v))/w(v)$. In [10] it is shown that this algorithm has an approximation guarantee of $2 + \ln(w(A)/w^*$, where w^* is the weight of an optimum solution. This is less than $\log_{1.5}(w(A)/w^*)$ when $w(A) \geq 4w^*$. We can now show the following lemma.

Lemma 6.3. *Given $A \subseteq V$, such that A has a cover of weight W, where all centers in the cover covers at least q vertices, and a set $C_1 \subseteq V$ that covers $V \setminus A$, where all centers in C_1 covers at least q vertices. We can then find in polynomial time a set of centers of total weight at most $2W$ that, together with C_1, covers A (and hence V) within a radius of $O(\log^* n)$, such that all centers cover at least q vertices.*

Proof. We will apply the greedy set cover algorithm recursively. The initial set of centers S_0 is constructed as follows. For any vertex v with $w(v) \leq W$ for which $\Gamma^+(v) \cap A$ is non-empty, and which has out-degree at least $q-1$ construct a set containing $\Gamma^+(v)$. The total weight of these centers is at most nW. Applying the greedy dominating set algorithm thus results in a set S_1 that covers A and has weight at most

$$w(S_1) \leq W \log_{1.5}(\frac{nW}{W}) = W \log_{1.5} n ,$$

assuming $n \geq 4$. The set C_1 covers $S_1 \setminus A$, so we need only consider $A_1 = S_1 \cap A$. We continue this procedure and note that at the ith iteration we have $|S_i| \leq k \cdot \log_{1.5}(|S_{i-1}|/k)$. By induction, after $O(\log^* n)$ iterations the size of our solution set, S_i, is at most $4W$. □

Combining Lemma 6.2 and Lemma 6.3 we get,

Theorem 6.4. *We can approximate the asymmetric q-all-coverage weighted k-center problem within factor $O(\log^* n)$ in polynomial time.*

Proof. Guess the optimum radius, R_{OPT}, and work in the bottleneck graph $G_{R_{\mathsf{OPT}}}$. Initially, the active set A is V. Repeat the following as many times as possible: Pick a CCV$_q$ v in A, add the lightest vertex u in $\Upsilon^-(v)$ to our solution set of centers, and remove the set $\Gamma_3^+(u)$ from A. Since v is covered by an optimum center in $\Upsilon^-(v)$, u is no heavier than this optimum center. Moreover, since the optimum center lies in $\Gamma^+(v)$, $\Gamma_3^+(u)$ includes everything covered by it.

Let C_1 be the centers chosen in this first phase. We know the remainder of the graph, A, has a cover of total weight $W' = W - w(C_1)$, because of our choices based on CCV and weight.

Lemma 6.2 shows that we can cover the remaining uncovered vertices with weight no more than $W'/2$ if we use covering radius 7. Applying the lemma again, we can cover the remaining vertices with weight $W'/4$ centers if we allow radius 49. So let the active set A be $V \setminus \Gamma_{49}^+(C_1)$, and recursively apply the greedy algorithm as described in the proof of Lemma 6.3 on the graph $G_{R_{\mathsf{OPT}}}^{49}$. As a result, we have a set of size W' that covers A within radius $O(\log^* n)$. □

6.2 Weighted q-coverage k-center

Using Theorem 6.4 and Lemma 5.2 we get the following theorem.

Theorem 6.5. *We can approximate the asymmetric weighted q-coverage k-center problem within factor $O(\log^* n)$ in polynomial time.*

The proof is similar to the proof of Theorem 5.3.

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