TIME SERIES ANALYSIS

Solutions to problems in Chapter 8
Solution 8.1

Question 1.

\[ Y_t = \sum_{p=0}^{k} a_p X_{t-p} = \left( \sum_{p=0}^{k} a_p B^p \right) X_t \Rightarrow \]

\[ \mathcal{H}(\omega) = \sum_{p=0}^{k} a_p e^{-i\omega p} \]

The spectral density for \( \{Y_t\} \) is found as

\[ f_y(\omega) = G^2(\omega) f_x(\omega) = \mathcal{H}(\omega) \overline{\mathcal{H}(\omega)} f_x(\omega), \]

where

\[ \mathcal{H}(\omega) \overline{\mathcal{H}(\omega)} = \sum_{p=0}^{k} a_p e^{-i\omega p} \sum_{q=0}^{k} a_q e^{i\omega q} \]

\[ = \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q e^{-i\omega q} e^{i\omega p} \]

\[ = \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q e^{-i\omega(p-q)} \]

\[ = \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \cos(\omega(p-q)) + i \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \sin(\omega(p-q)) \]

\[ = \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \cos(\omega(p-q)) \]

I.e.

\[ f_y(\omega) = \sum_{p=0}^{k} \sum_{q=0}^{k} a_p a_q \cos(\omega(p-q)) f_x(\omega) \]

Question 2.
\[ Y_t = \left( \frac{1}{k+1} \sum_{p=0}^{k} B^p \right) X_t = \frac{1}{k+1} \left( \frac{1 - B^{k+1}}{1 - B} \right) X_t \]

I.e.

\[ H(\omega)\overline{H(\omega)} = \frac{1}{(k+1)^2} \frac{1 - e^{-i\omega(k+1)}}{1 - e^{-i\omega}} \times \frac{1 - e^{i\omega(k+1)}}{1 - e^{i\omega}} \]

\[ = \frac{1}{(k+1)^2} \frac{2 - e^{-\omega(k+1)} - e^{i\omega(k+1)}}{2 - e^{-i\omega} - e^{i\omega}} \]

\[ = \frac{1}{(k+1)^2} \frac{2(1 - \cos(\omega(k+1)))}{2(1 - \cos(\omega))} = \frac{1}{(k+1)^2} \frac{\sin^2(\omega(k+1)/2)}{\sin^2(\omega/2)} \]

The spectral density for \( \{Y_t\} \) becomes

\[ f_y(\omega) = \frac{1}{(k+1)^2} \frac{\sin^2(\omega(k+1)/2)}{\sin^2(\omega/2)} f_x(\omega) \]

\[ \text{Question 3.} \]

The spectral density for \( \{Z_t\} \) is

\[ f_z(\omega) = H(\omega)\overline{H(\omega)} f_y(\omega) = \frac{1}{(k+1)^2} \frac{\sin^2(\omega(k+1)/2)}{\sin^2(\omega/2)} f_y(\omega) \]

I.e.

\[ f_z(\omega) = \frac{1}{(k+1)^4} \frac{\sin^4(\omega(k+1)/2)}{\sin^4(\omega/2)} f_x(\omega) \]

\[ \text{Question 4.} \]

Given

\[ Z_t = 0.5Y_t + 0.5Y_{t-1} \]
\[ Y_t = 0.5X_t + 0.5X_{t-1} \]
The expression for \( \{Z_t\} \) described by values of \( \{X_t\} \) is

\[
Z_t = 0.25X_t + 0.5X_{t-1} + 0.25X_{t-2}
\]

The impulse response function is found by sending a pulse (i.e. a one) through the system.

\[
h_k = \begin{cases} 
0 & k < 0 \\
0.25 & k = 1 \\
0.5 & k = 2 \\
0.25 & k = 3 \\
0 & k > 2
\end{cases}
\]

**Question 5.**

The frequency response function is

\[
\mathcal{H}(\omega) = \sum_k h_k e^{i\omega k} = 0.25 + 0.5e^{-i\omega} + 0.25e^{-i2\omega}
\]

The amplitude (or gain) is determined as

\[
G^2(\omega) = \mathcal{H}(\omega)\overline{\mathcal{H}(\omega)}
\]

\[
= (0.25 + 0.5e^{-i\omega} + 0.25e^{-i2\omega})(0.25 + 0.5e^{i\omega} + 0.25e^{i2\omega})
\]

\[
= \frac{1}{16} + \frac{1}{8}e^{i\omega} + \frac{1}{8}e^{i2\omega} + \frac{1}{16}e^{-i\omega} + \frac{1}{16}e^{-i2\omega} + \frac{1}{8}e^{-i\omega} + \frac{1}{16}e^{-i2\omega} + \frac{1}{8}e^{i\omega} + \frac{1}{16}e^{i2\omega}
\]

\[
= \frac{3}{8} + \frac{1}{2}\cos(\omega) + \frac{1}{8}\cos(2\omega)
\]

\[
= \frac{3}{8} + \frac{1}{2}\cos(\omega) + \frac{1}{8}(2\cos^2(\omega) - 1)
\]

\[
= \left(\frac{1}{2}\right)^2 (1 + \cos(\omega))^2 \Rightarrow
\]

\[
G(\omega) = 0.5(1 + \cos(\omega))
\]
The phase is

\[ \Phi(\omega) = \text{arg}(\mathcal{H}(\omega)) = \arctan \left( \frac{-0.25 \sin(\omega) - 0.25 \sin^2(\omega)}{0.25 + 0.25 \cos(\omega) + 0.25 \cos^2(\omega)} \right) \]

\[ = \arctan \left( \frac{-0.25 \sin(\omega) - 0.25 \sin^2(\omega)}{0.25 + 0.25 \cos(\omega) + 0.25 (2 \cos^2(\omega) - 1)} \right) \]

\[ = \arctan \left( \frac{-\sin(1 - \cos(\omega))}{\cos(\omega)(1 - \cos(\omega))} \right) \]

\[ = \arctan \left( \frac{\sin(-\omega)}{\cos(-\omega)} \right) = -\omega \]

The amplitude or phase could also be calculated more easily by applying that

\[ \mathcal{H}(\omega) = (0.25e^{-i\omega} + 0.5 + 0.25e^{i\omega})e^{-i\omega} = \mathcal{H}_1(\omega)\mathcal{H}_2(\omega) \Rightarrow \]

\[ G(\omega) = G_1(\omega)G_2(\omega) = 0.25(1.\cos(\omega)) \cdot 1 = 0.5(1.\cos(\omega)) \]

\[ \Phi(\omega) = \Phi_1(\omega)\Phi_2(\omega) = 0 + (-\omega) = -\omega \]

I.e. the composed filter is seen as a symmetric filter \((h_{-1} = 0.25, h_0 = 0.5, h_1 = 0.25)\) coupled to a time delay (with \(\mathcal{H}(\omega) = e^{-i\omega}\)). The amplitude and phase is plotted in Figure 1. From the amplitude plot it is clear that the composed filter is a low-pass filter.

![Figure 1: The amplitude (left) and phase (right) of the composed filter.](image)

*Question 6.*
If the composed filter is changed to a filter with the following impulse response function
\[
h_k = \begin{cases} 
0 & k < -1 \\
0.25 & k = -1 \\
0.5 & k = 0 \\
0.25 & k = 1 \\
0 & k > 1 
\end{cases}
\]
the amplitude will be maintained but a phase shift is avoided ($\Phi(\omega) = 0$).
Solution 8.2

Question 1. The following intervention model is considered

\[ Y_t - 200 = \frac{100B}{1 - 0.9B} I_t + \frac{1 + 0.3B}{1 - 0.9B} \epsilon_t \]

It is assumed that the effect of previous campaigns is negligible.

We introduce

\[ X_t = Y_t - 200 \]

A prediction of for instance \( X_{8|7} \) require information about \( \epsilon_7 \). This can be found by setting e.g. \( \epsilon_4 = 0 \) and from here determining the one-step prediction and prediction error. First the intervention model is rewritten to

\[ X_t - 0.9X_{t-1} = \epsilon_t + 0.3\epsilon_{t-1} \Rightarrow X_{t+1} - 0.9X_t = \epsilon_{t+1} + 0.3\epsilon_t \]

I.e.

\[ \hat{X}_{t+1|t} = 0.9X_t + 0.3\epsilon_t \]

And the one-step predictions are

\[ \hat{X}_{5|4} = 0.9X_4 + 0.3 \cdot 0 = 0.9 \cdot 8 = 7.2 \Rightarrow \epsilon_5 = X_5 - \hat{X}_{5|4} = 15 - 7.2 = 7.8 \]

\[ \hat{X}_{6|5} = 0.9 \cdot 15 + 0.3 \cdot 7.8 = 15.8 \Rightarrow \epsilon_6 = X_6 - \hat{X}_{6|5} = -4 - 15.8 = -19.8 \]

\[ \hat{X}_{7|6} = 0.9 \cdot (-4) - 0.3 \cdot 19.8 = -9.54 \Rightarrow \epsilon_7 = X_7 - \hat{X}_{7|6} = 7 + 9.54 = 16.5 \]

Thus, the expected sales in week 8,9,10 and 11 are

\[ \hat{X}_{8|7} = 0.9 \cdot 7 + 0.3 \cdot 16.5 = 11.3 \Rightarrow \hat{Y}_{8|7} = 211.3 \]

\[ \hat{X}_{9|7} = 0.9 \hat{X}_{8|7} = 0.9 \cdot 11.3 = 10.2 \Rightarrow \hat{Y}_{9|7} = 210.2 \]

\[ \hat{X}_{10|7} = 0.9 \hat{X}_{9|7} = 0.9 \cdot 10.2 = 9.2 \Rightarrow \hat{Y}_{10|7} = 209.2 \]

\[ \hat{X}_{11|7} = 0.9 \hat{X}_{10|7} = 0.9 \cdot 9.2 = 8.3 \Rightarrow \hat{Y}_{11|7} = 208.3 \]
Question 2.

A sales campaign is undertaken in week 8, i.e. \( I_8 = 1 \). The expected sales in week 8 is as estimated in Question 1. For week 9-11 we get the following estimates

\[
X_9 - 0.9X_8 = 100 \cdot 1 + \epsilon_9 + 0.3\epsilon_8 \\
\hat{X}_{9|7} = 0.9\hat{X}_{8|7} + 100 = 110.2 \Rightarrow \hat{Y}_{9|7} = 310.2
\]

\[
X_{10} - 0.9X_9 = \epsilon_{10} + 0.3\epsilon_9 \\
\hat{X}_{10|7} = 0.9\hat{X}_{9|7} = 99.2 \Rightarrow \hat{Y}_{10|7} = 299.2
\]

\[
X_{11} - 0.9X_{10} = \epsilon_{11} + 0.3\epsilon_{10} \\
\hat{X}_{11|7} = 0.9\hat{X}_{10|7} = 89.3 \Rightarrow \hat{Y}_{10|7} = 289.3
\]

In order to determine a 95% confidence interval for the future sales we wish to rewrite the process such that we obtain the \( \Psi \)-weights.

\[
X_t = \frac{100B}{1-0.9B}I_t + N_t
\]

As \( V[I_t] = 0 \) only \( N_t \) contribute to the prediction error, and the \( \Psi \)-weights are found by rewriting \( N_t \) into MA-form.

\[
N_t = (1 + 0.3B)(1 + 0.9B + 0.9^2B^2 + 0.9^3B^3 + \ldots)\epsilon_t
\]

\[
= (1 + 1.2B + 1.08B^2 + 0.97B^2 + \ldots)\epsilon_t
\]

\[
= (1 + \Psi_1B + \Psi_2B^2 + \Psi_3B^3 + \ldots)\epsilon_t
\]

An approximate 95% confidence interval is obtained as

\[
\hat{X}_{t+\ell|t} \pm 2 \cdot \sqrt{(1 + \Psi_1^2 + \ldots + \Psi_{t-1}^2)} \sigma_{\epsilon_t}
\]

I.e.

- week 8: 211.3 ± 40.0 units/week
- week 9: 310.2 ± 62.5 units/week
- week 10: 299.2 ± 76.0 units/week
- week 11: 289.3 ± 85.3 units/week