A Spectral Element Method for Nonlinear and Dispersive Water Waves

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Motivation

The use of flexible mesh discretisation methods are important for simulation of nonlinear wave-structure interactions in offshore and marine settings such as harbour and coastal areas. For real applications, development of efficient models for wave propagation based on unstructured discretisation methods is of key interest. We present a high-order general-purpose three-dimensional numerical model solving fully nonlinear and dispersive potential flow equations with a free surface.

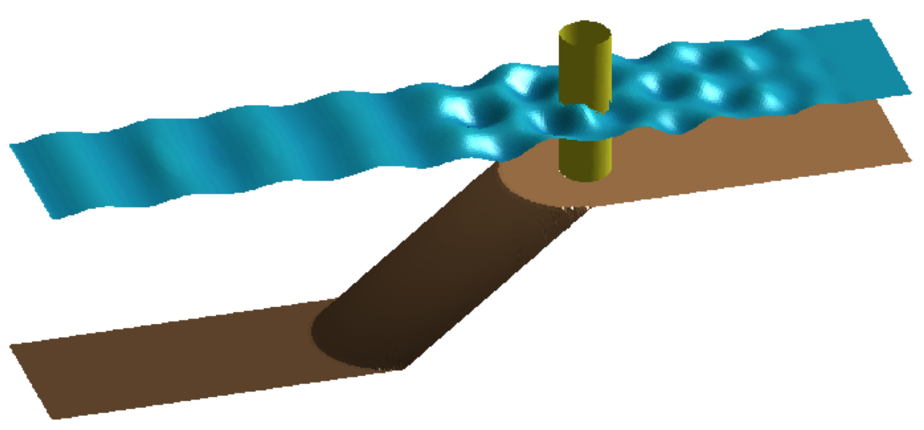

Figure 1 : *Snapshot of scaled free surface showing diffraction and refraction patterns in the free surface.*

Governing equations

Let both $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ and $\Omega' \subset \mathbb{R}^{d-1}$ be bounded, connected domains with piecewise smooth boundaries Γ and Γ', respectively. Let $T : t \geq 0$ be the time domain. Introduce the free surface boundary $\Gamma^{FS} \subset \Gamma$ and the bottom boundary $\Gamma^b \subset \Gamma$. The mathematical problem is to find a scalar velocity potential function $\phi(x, z, t) : \Omega \times T \to \mathbb{R}$ and to determine the evolution of the free surface elevation $\eta(x, t) : \Omega' \times T \to \mathbb{R}$.

The Eulerian description of the unsteady kinematic and a dynamic free surface boundary conditions is expressed in the Zakharov form. In $\Omega' \times T$, find $\eta, \tilde{\phi}$

$$\partial_t \eta = -\nabla\eta \cdot \nabla\tilde{\phi} + \tilde{w}(1 + \nabla\eta \cdot \nabla\eta)$$

$$\partial_t \tilde{\phi} = -g\eta - \frac{1}{2}\left(\nabla\tilde{\phi} \cdot \nabla\tilde{\phi} - \tilde{w}^2(1 + \nabla\eta \cdot \nabla\eta)\right)$$

Here $\nabla = (\partial_x, \partial_y)$ and the '$\sim$' symbol denote functionals defined in the free surface plane. The vertical velocity $\tilde{w} \equiv \partial_z \phi|_{z=\eta}$ is at a given time determined from the solution of a Laplace problem

$$\begin{aligned} \phi = \tilde{\phi}, \quad & z = \eta \quad \text{on} \quad \Gamma^{FS} \\ \nabla^2\phi + \partial_{zz}\phi = 0, \quad & -h(x) < z < \eta \quad \text{in} \quad \Omega \\ \partial_z\phi + \nabla h \cdot \nabla\phi = 0, \quad & z = -h(x) \quad \text{on} \quad \Gamma^b \end{aligned}$$

where $h(x) : \Omega' \mapsto \mathbb{R}$ describes the still water depth.

A basis for efficient simulations is the classical σ-transformation of the vertical coordinate

$$\sigma \equiv (z + h(x))d(x, t)^{-1}, \quad 0 \leq \sigma \leq 1,$$

where $d(x, t) = \eta(x, t) + h(x)$ is the height of the water column.

Following Cai Et al. (1998), we express the σ-transformed system in a form where variable depth is accounted. Let $\Omega^c \subset \mathbb{R}^d$ $(d = 2, 3)$ be the time-independent computational domain $\Omega^c = \{(x, y, \sigma) | (x, y) \in \Omega', 0 \leq \sigma \leq 1\}$. The Jacobian of the map $\chi : \Omega \to \Omega^c$ is then

$$J(x, z, t) = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial \sigma} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial \sigma} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial \sigma} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{h_x}{d} - \frac{\sigma d_x}{d} & \frac{h_y}{d} - \frac{\sigma d_y}{d} & \frac{1}{d} \end{bmatrix}$$

enabling the σ-transformed system to be expressed in the differential form

$$\nabla^c \cdot (K\nabla^c\Phi) = 0 \quad \text{in} \quad \Omega^c$$

where $\nabla^c = (\nabla, \partial_\sigma)$ is introduced and the symmetric coefficient matrix is

$$K(x, t) = \frac{1}{\det J} J J^T$$

The velocity field can be determined from Φ using the relation $(\mathbf{u}, w) = (\nabla + \nabla\sigma\partial_\sigma, \partial_z\sigma\partial_\sigma)\Phi$.

Numerical Discretization

We form a partition of the domain $\Omega_h \subseteq \Omega$ to obtain a tessellation $\mathcal{T}_h$ of Ω_h consisting of N_{el} non-overlapping shape-regular elements $\mathcal{T}_k$ such that $\cup_{k=1}^{N_{el}} \mathcal{T}_k = \mathcal{T}_h$ with k denoting the k'th element. For approximation of functions we introduce the finite element approximation space of continuous, piece-wise polynomial functions $V = \{v_h \in C^0(\Omega_h); \forall k \in \{1, ..., K\}, v_{h|\mathcal{T}_k} \in \mathbb{P}^q\}$ where $\mathbb{P}^q$ is the space of polynomials of degree at most q.

Introduce the approximations

$$f_h = \sum_{i=1}^{N^{FS}} f_i(t) N_i(x)$$

where $\{N_i\}_{i=1}^{N_{FS}} \in V$ is the set of global finite element basis functions with cardinal property $N_i(\mathrm{x}_j) = \delta_{ij}$ at mesh nodes. Choose $v(x) \in \{N_i\}_i^{N_{FS}}$. The discretization in two spatial horizontal dimensions becomes

$$M' \frac{d}{dt}\eta_h = -\left(A_x^{\tilde{\phi}_x} + A_y^{\tilde{\phi}_y}\right)\eta_h + M'\tilde{w}_h + \left(A_x^{\tilde{w}_h(\eta_h)_x} + A_y^{\tilde{w}_h(\eta_h)_y}\right)\eta_h$$

$$M' \frac{d}{dt}\tilde{\phi}_h = -M'g\eta_h - \frac{1}{2}\left[\left(A_x^{(\tilde{\phi}_h)_x} + A_y^{(\tilde{\phi}_h)_y}\right)\tilde{\phi}_h + M^{\tilde{w}_h}\tilde{w}_h - \left(A_x^{\tilde{w}_h^2(\eta_h)_x} + A_y^{\tilde{w}_h^2(\eta_h)_y}\right)\right]\eta_h$$

where the following global matrices have been introduced

$$M'_{ij} = \iint_{\Omega'} N_i N_j dx,$$

$$M^b_{ij} = \iint_{\Omega'} b N_i N_j dx,$$

$$(A^b_q)_{ij} = \iint_{\Omega'} b(x) N_i \frac{\partial}{\partial q} N_j dx$$

The gradient of the globally continuous basis functions will be discontinuous across element interfaces in the classical sense. To guarantee global continuity of derivatives a gradient recovery procedure can be used.

We represent the global approximation of components of the horizontal first derivatives in C^0

$$\mathrm{u} = \nabla\phi = \sum_{i=1}^{N} \mathrm{u}_i N_i(x)$$

By a global Galerkin projection of the form

$$\iiint_\Omega \mathrm{u} v(x) dx = \iiint_\Omega \nabla\phi v(x) dx$$

two linear systems of equations are generated

$$Mu = D_x\phi, \quad Mv = D_y\phi$$

for the velocity vectors.

Consider the discretisation of the governing equations for the σ-transformed Laplace problem. We seek to construct a linear system

$$L\Phi_h = \mathrm{b}, \quad L \in \mathbb{R}^{N \times N}, \quad \Phi_h, \mathrm{b} \in \mathbb{R}^N$$

where N is the total degrees of freedom in the discretisation.

The weak formulation of the symmetric Laplace problem can be expressed as: find $\Phi \in V$ such that

$$-\iiint_\Omega (K\nabla^c\Phi) \cdot \nabla^c v dx = 0, \quad \forall v \in V$$

having assumed impermeable wall boundaries. Thus, the discrete system operator is defined from

$$L_{ij} = -\iiint_\Omega (K\nabla^c N_j) \cdot \nabla^c N_i dx$$

The elemental integrals are approximated as

$$\iiint_{\Omega^k} (K\nabla^c N_j) \cdot \nabla^c N_i dx = \iiint_{\mathcal{T}_r} |J^k| (K\nabla^c N_j) \cdot \nabla^c N_i dr$$

where J^k is the Jacobian of the affine mapping $\chi^k : \mathcal{T}_k \to \mathcal{T}_r$ where $\mathcal{T}_r$ is a computational reference element.

Numerical Results

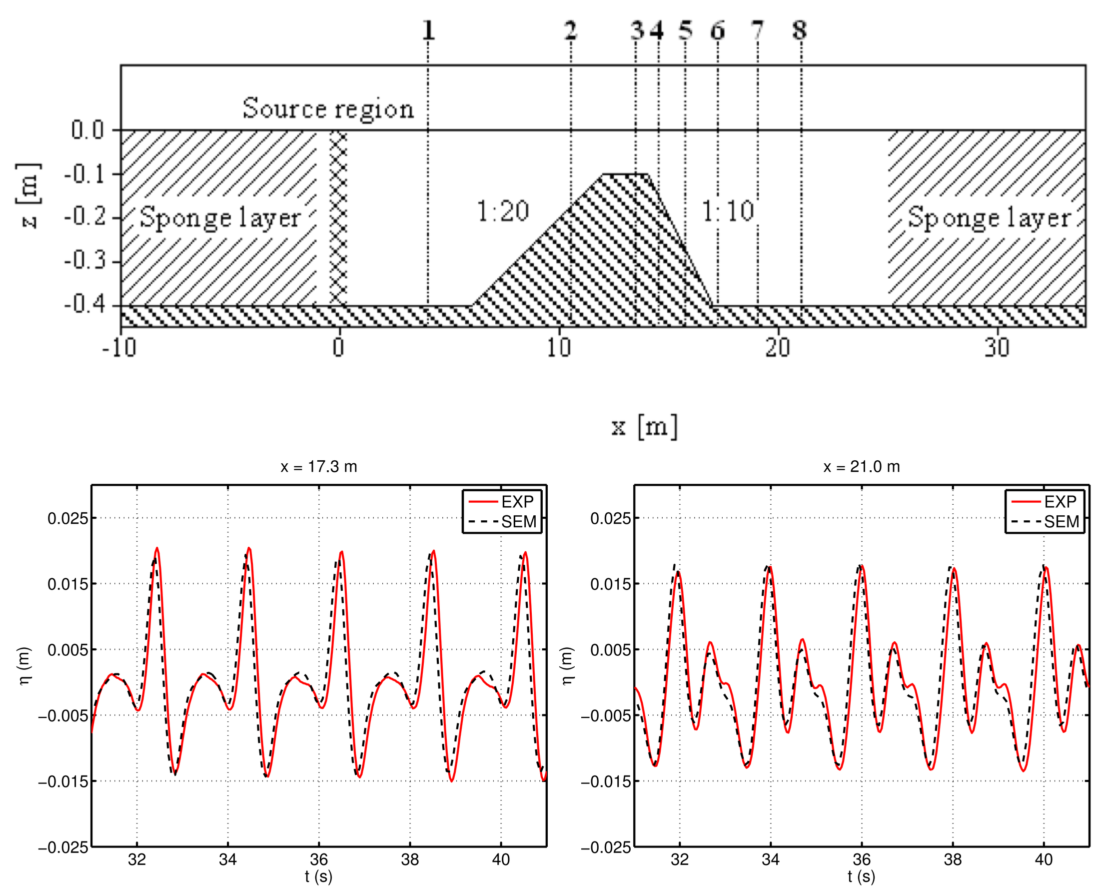

Figure 2 : *Computed and measured time series of free surface elevations at two gauges after the bar in submerged bar test benchmark.*

Contributions

- Spectral Element Method for solving Nonlinear and Dispersive Water Waves efficiently and accurately.
- General discretization framework based on a Galerkin Method in space.
- Numerical analysis and validation of the model using benchmark for dispersive and nonlinear waves in 2D.

Outlook

- Improve numerical stability for marginally resolved/highly nonlinear waves.
- Advanced industry size applications in areas with marine structures.

References

- Engsig-Karup, A. P., Glimberg, L. S., Nielsen, A. S. and Lindberg, O. 2013. Fast hydrodynamics on heterogenous many-core hardware. Part of: Raphäel Couturier (Ed). Designing Scientific Applications on GPUs, 2013, CRC Press / Taylor & Francis Group.
- Cai, X., Langtangen, H. P., Nielsen, B. F., and Tveito, A. 1998 A finite element method for fully nonlinear water waves. *J.* Comput. Phys., 143, pp. 544–568.

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