DTU Compute
Department of Applied Mathematics and Computer Science

Parallel Algebraic Multiscale Linear Solver for Reservoir Models

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Motivation

Accurate and efficient simulation of multiphase flow in large-scale heterogenous natural formations is crucial for a wide range of application, including hydrocarbon production optimisation, risk managementof carbon capture and storage, water resources utilisations and geothermal power extraction. Therefore, developing fast linear solvers is an ongoing challenge and is necessary due to the complexity of the models and the increasing fidelity of the subsurface description.

In order to solve efficiently large-scale elliptic problems and to take into account the details of the physics of different scales in complex geological formations (see Fig.1), the Algebraic Multiscale (AMS) solver [Wang et al., 2012] has been developed. AMS is a multilevel algorithm that constructs the solution (fine-scale) by mean of a set of local problems coupled by a conservative global (coarse-scale) problem, employing domain decomposition with a localization assumption.

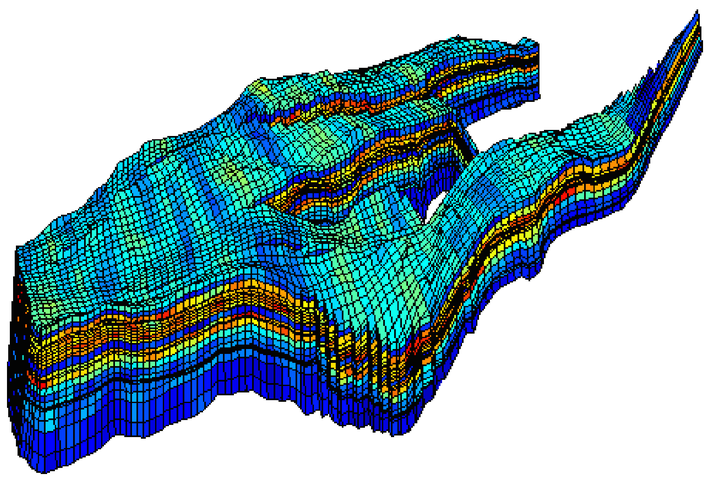

Figure 1: *Real-field reservoir model.*

AMS Method

In reservoir simulation, a considerable portion of the simulation time is spent in solving a variant of the following elliptic partial differential equation, which represent the mass balance equation of an incompressible flow through a porous medium

$$-\nabla \cdot (\lambda_t \cdot \nabla p) = q, \qquad \text{on } \Omega \tag{1}$$

where p is the pressure, λ_t is the total mobility defined as the ratio between the porous medium permeability and to the fluid viscosity, and q is the source or sink term.

If we discretise the problem with a standard two-point flux approximation scheme on a fine grid, see Fig.2

$$F_{ij} = T_{ij}(p_j - p_i)$$

where F_{ij} is the flux between fine-cell i and fine-cell j, and T_{ij} is the transmissibility, we obtain the following algebraic system of equations

$$A^f p^f = q^f$$

where p^f refers to the discretised pressure unknowns on a given fine grid computational domain Ω^f.

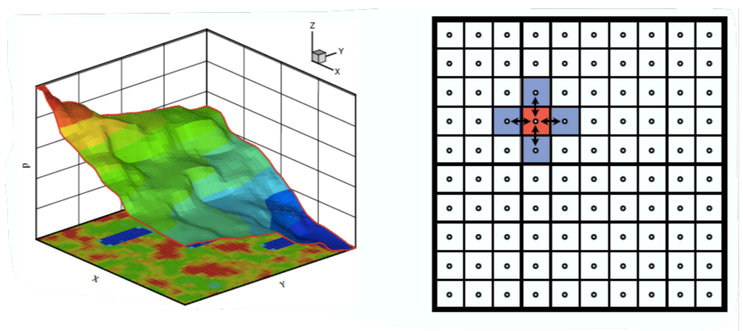

Figure 2: *Two-point flux approximation scheme.*

We construct the multiscale grids, see Fig.3, by first dividing the fine grid domain, Ω^f, into subdomains, Ω^c_k, called *primal* coarse grid blocks. Then we define the *dual* coarse grid, Ω^d_m, which is constructed by joining the centres of the primal coarse grid blocks.

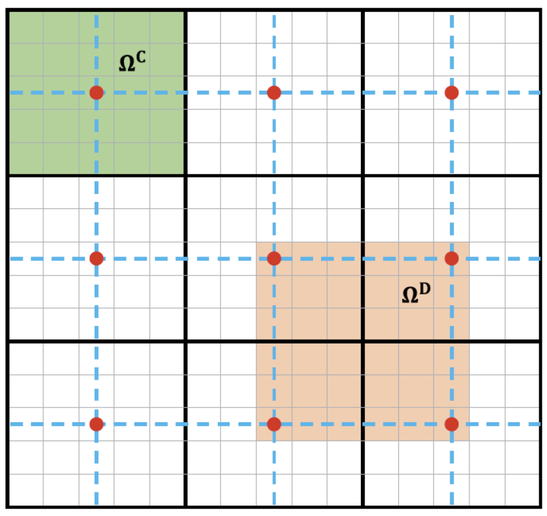

Figure 3: *Primal coarse, dual coarse and fine cells.*

Each dual coarse cell Ω^d_m defines a local subdomain on which Equation 1 is solved locally using reduced boundary conditions in order to obtain basis functions. Algebraically, the basis functions are obtained by solving

$$\begin{aligned} \nabla \cdot (\lambda_t \nabla \phi^i_j) &= 0, \quad &\in \Omega^d_j \\ \nabla_{||} \cdot (\lambda_t \nabla \phi^i_j)_{||} &= 0, \quad &\in \partial\Omega^d_j \\ \nabla \cdot (\lambda_t \nabla \phi^i_j) &= \delta_{ik}, \quad &\forall x_k \in \{1, ..., N_c\} \end{aligned}$$

where ϕ^i_j is the basis function associated with the coarse node i in the dual coarse cell Ω^d_j, and $||$ denotes the projection of the vector or operator along the tangential direction of $\partial\Omega^d_j$.

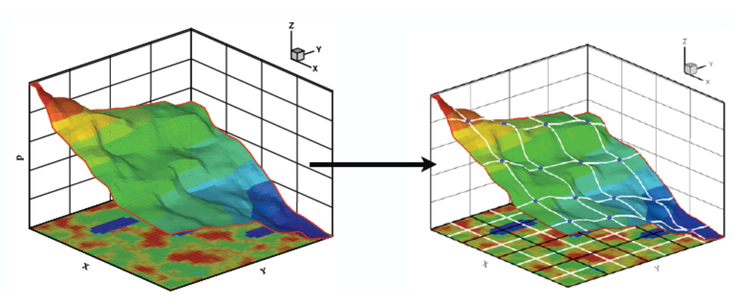

Figure 4: *From fine-scale to coarse-scale.*

The multiscale solution, $\tilde{p}^f$, will be then calculated by juxtaposition of the basis function with the coarse pressure solution, p^c, as coefficients, see Fig.4, namely

$$p \approx \tilde{p}^f = \sum_{j=1}^{N_d} \sum_{i=1}^{N_c} \phi^i_j p^c_i = \mathcal{P} p^c$$

where $\mathcal{P}$ is the $N_f \times N_c$ prolongation operator

$$\mathcal{P} = [\phi^1, \phi^2, \ldots, \phi^{N_c}].$$

In order to construct the coarse-scale operator, A^c, we need to map the fine-space into the coarse-space. To this aim, we define the $Nc \times N_f$ restriction operator

$$\mathcal{R} = \mathcal{P}^T.$$

So the $N_c \times N_c$ coarse-scale operator is

$$A^c = \mathcal{R} A^f \mathcal{P}$$

which is much smaller and less expensive to solve than the original fine-scale system. So the multiscale solution is

$$\tilde{p}^f = \mathcal{P}(A^c)^{-1} q^c = \mathcal{P}(\mathcal{R} A^f \mathcal{P})^{-1} \mathcal{R} q^f$$

from which we can derive the global-stage multiscale preconditioner

$$M_g^{-1} = \mathcal{P}(\mathcal{R} A^f \mathcal{P})^{-1} \mathcal{R}$$

If used alone, M_g^{-1} is rank deficient by $N_f - N_c$ and does not yield a convergent scheme [Manea et al., 2015]. Due to its inability to eliminate high-frequency error modes, so we must use it together with a *local* preconditioner, M_l^{-1}. The overall preconditioning scheme, called two-stage algebraic multiscale solver (TAMS) [Wang et al., 2012] can be written as

$$M^{-1}_{TAMS} = M_g^{-1} + M_l^{-1} - M_l^{-1} A^f M_g^{-1}$$

Fig.5 shows an application of AMS in our sequantial 2-D reservoir simulator. In particular we can notice the reduced number of pressure dof.

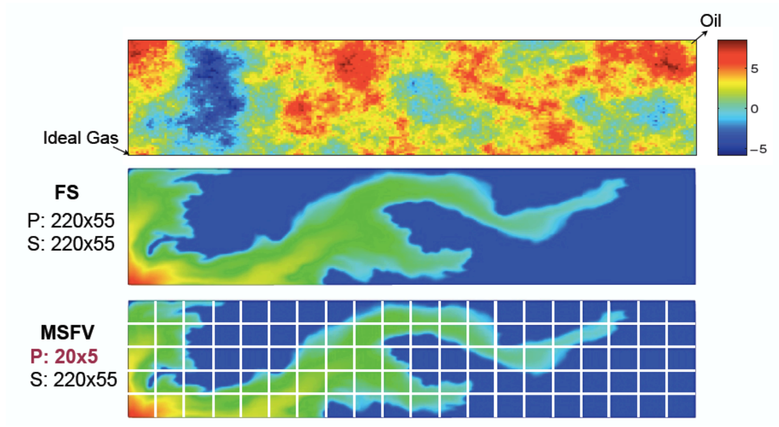

Figure 5: *SPE 10 top layer, fine-scale vs. Multiscale solutions.*

AMS and PETSC

We carried out the parallel implementation within PETSC, a framework of data structures and routines for scalable and parallel solution of applications modelled by PDE, which supports MPI, CUDA, OpenCL, and hybrid MPI-GPU parallelism. In this preliminary work we developed a one-dimensional parallel linear solver for a heterogeneous case without forcing term.

For the fine-scale solution we implemented the following scheme

$$\frac{\lambda_t^{i-1/2}}{\Delta x^2}(p_i - p_{i-1}) + \frac{\lambda_t^{i+1/2}}{\Delta x^2}(p_i - p_{i+1}) = q_i$$

where λ_t is randomly generated

$$\lambda_t = 10^{-5} \cdot e^{\xi}, \quad \xi \sim \mathcal{N}(0, 1)$$

Fig.6 shows a realisation of λ_t and its harmonic average at the fine-cell interface, the basis function, the solution, and a scalability test on 16 cores, performed on a dual socket configuration of Intel Xeon Processor X5550, which is sufficiently close to be $O(N)$.

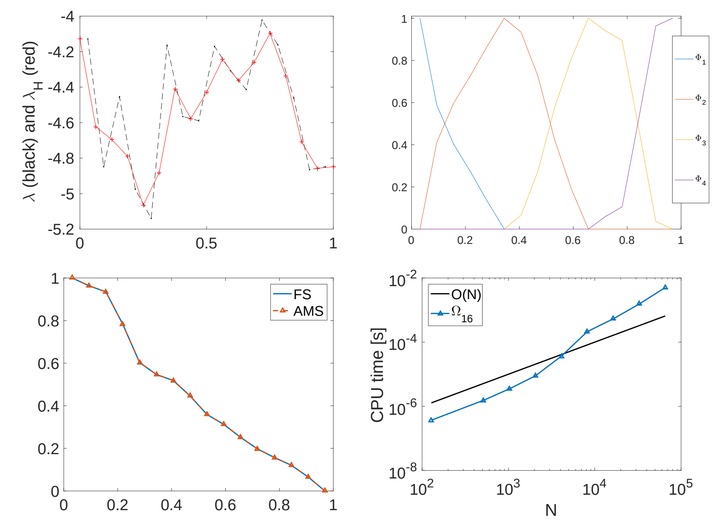

Figure 6: *1-D AMS, λ_t, basis functions, solution, and scalability on 16 cores.*

Conclusions and Perspectives

- Promising results of AMS scalability on large-scale problems.
- Fit larger problems on same hardware and extend to multi-dimensional simulations.
- Next step: profile the code and show the speed-up.

References

Y. Wang, H. Hajibeygi, and H.A. Tchelepi, *Algebraic Multiscale Solver for Flow in Heterogeneous Porous Media*, Jour. Comput. Phys. , 2014.

A.M. Manea, J. Sewall, H.A. Tchelepi, *Parallel Multiscale Linear Solver for Highly Detailed Reservoir Models*, SPE, 2015.

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