Kriging Example

The main result in kriging is concerned with estimation of the value

$$Z(\boldsymbol{r}_0)$$

(also referred to as Z_0) based on the observed values

$$\{Z_1, \cdots, Z_N\} = \boldsymbol{Z}^T$$

We will consider *linear estimators* only

$$\hat{Z}_0 = \sum_{i=1}^{N} w_i Z_i = \boldsymbol{w}^T \boldsymbol{Z}$$

We demand that the estimator is *unbiased* and get

$$\boldsymbol{w}^T \boldsymbol{1} = \sum w_i = 1$$

The kriging estimate is obtained by choosing $\boldsymbol{w}$ so that the estimation variance

$$\sigma_E^2 = E\{[Z_0 - \hat{Z}_0]^2\}$$

is minimised.

Let

$$\begin{aligned}
D\{\boldsymbol{Z}\} &= \begin{bmatrix} V\{Z_1\} & \cdots & C\{Z_1, Z_N\} \\ \vdots & & \vdots \\ C\{Z_N, Z_1\} & \cdots & V\{Z_N\} \end{bmatrix} \\
&= \begin{bmatrix} C_{11} & \cdots & C_{1N} \\ \vdots & & \vdots \\ C_{N1} & \cdots & C_{NN} \end{bmatrix} \\
&= \boldsymbol{C} \\
C\{\boldsymbol{Z}, Z_0(\boldsymbol{r})\} &= \begin{bmatrix} C\{Z_1, Z_0(\boldsymbol{r})\} \\ \vdots \\ C\{Z_N, Z_0(\boldsymbol{r})\} \end{bmatrix} = \begin{bmatrix} C_{10} \\ \vdots \\ C_{N0} \end{bmatrix} \\
&= \boldsymbol{D}
\end{aligned}$$

Using this, the mean squared error expression can be written

$$\begin{aligned}
\sigma_E^2 &= E\{(Z_0(\boldsymbol{r}) - \hat{Z}_0(\boldsymbol{r}))^2\} \\
&= V\{Z_0(\boldsymbol{r}) - \hat{Z}_0(\boldsymbol{r})\} \\
&= C_{00} + \boldsymbol{w}^T \boldsymbol{C} \boldsymbol{w} - 2\boldsymbol{w}^T \boldsymbol{D}
\end{aligned}$$

The problem is now to minimise

$$C_{00} + \boldsymbol{w}^T \boldsymbol{C} \boldsymbol{w} - 2\boldsymbol{w}^T \boldsymbol{D}$$

with the constraint that

$$\boldsymbol{w}^T \mathbf{1} = 1$$

We introduce a Lagrange multiplier -2λ and minimise

$$F = C_{00} + \boldsymbol{w}^T \boldsymbol{C} \boldsymbol{w} - 2\boldsymbol{w}^T \boldsymbol{D} + 2\lambda(\boldsymbol{w}^T \mathbf{1} - 1)$$

without constraints.

By differentiation we get

$$\begin{aligned} \frac{\partial F}{\partial \boldsymbol{w}} &= 2\boldsymbol{C}\boldsymbol{w} - 2\boldsymbol{D} + 2\lambda \mathbf{1} = \mathbf{0} \\ \frac{\partial F}{\partial \lambda} &= 2\boldsymbol{w}^T \mathbf{1} - 2 = 0 \end{aligned}$$

or

$$\begin{bmatrix} \boldsymbol{C} & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{D} \\ 1 \end{bmatrix},$$

which is the so-called *ordinary kriging system*.

By direct solution we get

$$\begin{aligned} \boldsymbol{C}\boldsymbol{w} + \lambda \mathbf{1} = \boldsymbol{D} \quad & i.e. \quad \boldsymbol{w} + \lambda \boldsymbol{C}^{-1}\mathbf{1} = \boldsymbol{C}^{-1}\boldsymbol{D} \\ & i.e. \quad \mathbf{1}^T \boldsymbol{w} + \lambda \mathbf{1}^T \boldsymbol{C}^{-1} \mathbf{1} = \mathbf{1}^T \boldsymbol{C}^{-1} \boldsymbol{D} \end{aligned}$$

Since $\mathbf{1}^T \boldsymbol{w} = 1$ we have

$$\boxed{\lambda = \frac{\mathbf{1}^T \boldsymbol{C}^{-1} \boldsymbol{D} - 1}{\mathbf{1}^T \boldsymbol{C}^{-1} \mathbf{1}} = \frac{\boldsymbol{D}^T \boldsymbol{C}^{-1} \mathbf{1} - 1}{\mathbf{1}^T \boldsymbol{C}^{-1} \mathbf{1}}}$$

If we insert this above we get

$\boxed{\boldsymbol{w} = \boldsymbol{C}^{-1}[\boldsymbol{D} - \lambda \mathbf{1}]}$

The weights can all be determined from the spatial lay-out and the semivariogram.

The matrix inversion includes the observations we want to krige from only and not the point we want to krige to. This point is included via there covariances $\boldsymbol{D}$ linearly in λ and in $\boldsymbol{w}$.

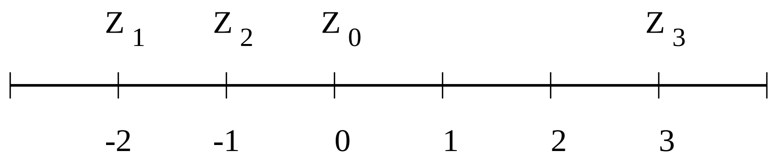
The minimal variance is the so-called *ordinary kriging variance*, which is

$$\begin{aligned} V\{Z_0 - \boldsymbol{w}^T \boldsymbol{Z}\} &= C_{00} + \boldsymbol{w}^T \boldsymbol{C} \boldsymbol{w} - 2\boldsymbol{w}^T \boldsymbol{D} \\ &= C_{00} + \boldsymbol{w}^T [\boldsymbol{D} - \lambda \mathbf{1}] - 2\boldsymbol{w}^T \boldsymbol{D} = \end{aligned}$$

$\boxed{\sigma^2_{OK} = C_{00} - \boldsymbol{w}^T \boldsymbol{D} - \lambda}$

Some results on kriging weights

We consider the following situation



$$\hat{Z}_0 = \boldsymbol{w}^T \boldsymbol{Z} = w_1 Z_1 + w_2 Z_2 + w_3 Z_3$$

We want to krige a value for Z_0 based on the observations $\{Z_1, Z_2, Z_3\}$. We assume that we have the semivariogram

$$\hat{\gamma}(h) \quad = \quad \begin{cases} 0 & h = 0 \\ C_0 + C_1 \left(\frac{3}{2}\frac{h}{R} - \frac{1}{2}(\frac{h}{R})^3 \right) & 0 < h < R \\ C_0 + C_1 & R \leq h. \end{cases}$$

i.e. a spherical model with nugget effect.

We assume that $R = 6$, and for $C_0 = 0$ and $C_1 = 1$ we get

h	0	1	2	3	4	5	6
$\gamma(h)$	0	.2477	.4815	.6875	.8519	.9606	1.0000

This gives the correlation function

h	$C(h)$	
0	C_0 +	C_1
1		C_1 0.7523
2		C_1 0.5185
3		C_1 0.3125
4		C_1 0.1481
5		C_1 0.0394
6		0

Therefore we have

$$D\left\{\begin{bmatrix} Z_0 \\ Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}\right\} =$$

$$\begin{bmatrix} C_0+ & C_1 & & & & & & \\ & C_1 0.5185 & C_0+ & C_1 & & & & \\ & C_1 0.7523 & & C_1 0.7523 & C_0+ & C_1 & & \\ & C_1 0.3125 & & C_1 0.0394 & & C_1 0.1481 & C_0 + C_1 & \end{bmatrix}$$

From this we have

$$\begin{aligned} C\left\{Z_0, \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}\right\} &= \boldsymbol{D} = \begin{bmatrix} 0.5185 \\ 0.7523 \\ 0.3125 \end{bmatrix} C_1 \\ D\left\{\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}\right\} &= \boldsymbol{C} = C_0 \boldsymbol{I} + C_1 \begin{bmatrix} 1 & & \\ .7523 & 1 & \\ .0394 & .1481 & 1 \end{bmatrix} \end{aligned}$$

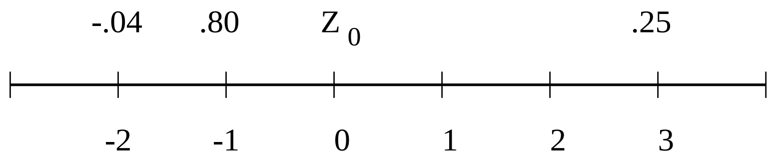
The solution to the kriging system is

$$\begin{aligned} \lambda &= [\boldsymbol{D}^T \boldsymbol{C}^{-1} \mathbf{1} - 1]/[\mathbf{1}^T \boldsymbol{C}^{-1} \mathbf{1}] \\ \boldsymbol{w} &= \boldsymbol{C}^{-1}[\boldsymbol{D} - \lambda \mathbf{1}] \\ \sigma_{OK}^2 &= C_0 + C_1 - \boldsymbol{w}^T \boldsymbol{D} - \lambda \end{aligned}$$

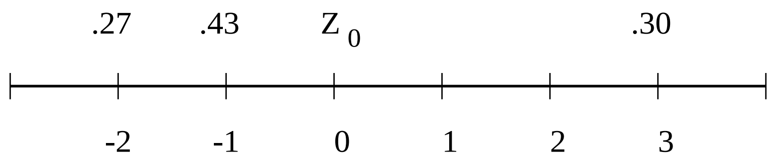
These values have been determined for different values of C_0 and C_1 and are shown in the table below.

It is seen that we for a pure nugget effect, i.e. $C_1 = 0$ have the weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$, i.e. just a smoothing. The kriging variance $V\{Z_0 - \boldsymbol{w}^T \boldsymbol{Z}\} = \sigma^2 + \sigma^2/3 = \frac{4}{3}\sigma^2$ is also seen.

For neglectable nugget effect we notice that the weights converge towards



If we have equal amounts of nugget effect og "dependent variance" we get the weights



We also notice that the weights do not depend upon the scaling of the variances. The dependence is only on the ratio between the variances C_0 and C_1.

C_0	C_1	w_1	w_2	w_3	λ	$\hat{\sigma}^2_{OK}$
0	1	–.0407	.7955	.2452	–.0489	.3949
0	2	–.0407	.7955	.2452	–.0978	.7898
0	3	–.0407	.7955	.2452	–.1467	1.1847
1	1	.2702	.4316	.2982	–.3584	1.8004
1	2	.2219	.4932	.2849	–.3933	2.2431
1	3	.1861	.5365	.2774	–.4325	2.6721
2	1	.3004	.3894	.3103	–.6878	3.1422
2	2	.2702	.4316	.2982	–.7168	3.6008
2	3	.2442	.4654	.2904	–.7502	4.0478
3	1	.3113	.3727	.3161	–1.0193	4.4788
3	2	.2899	.4046	.3055	–1.0452	4.9449
3	3	.2702	.4316	.2982	–1.0752	5.4013
1	0	.3333	.3333	.3333	–.3333	1.3333
0.1	1	.0746	.6658	.2596	–.0746	.5539
0.01	1	–.0250	.7780	.2470	–.0512	.4117
1	100	–.0250	.7780	.2470	–5.1238	41.1720
1	10000	–.0406	.7954	.2452	–489.3	3950.9
1	0.01	.3327	.3347	.3327	–.3335	1.3382
1	0.1	.3268	.3461	.3271	–.3349	1.3816