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ENCLOSING SOLUTIONS OF INTEGRAL EQUATIONS

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Abstract

A method to enclose the solution of an integral equation of the *Fredholm type* is presented. The enclosure is based on an interval mean value extension of the integral operator T for which $y^\star = T(y^\star)$ and $y^\star$ is a solution. If T is a contraction near $y^\star$ then the method is proven to converge for appropriate initial conditions. If the region specified by the user does not contain any solution, then the method may be able to disprove the existence of solutions in this region. The performance of the algorithm is demonstrated by numerical examples.

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Enclosing Solutions of Integral Equations

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1 Introduction

We consider integral operators of the type

$$Ty(x) = f(x) + \int_a^b g(x,t,y(x),y(t))dt \ , \ \ x \in [a,b] \tag{1}$$

where $[a,b]$ is a real interval and f,g are given smooth functions.

We wish to solve the integral equation

$$Ty = y, \tag{2}$$

and to provide guaranteed bounds for a solution. The set of functions considered in this paper is the set of all functions y for which the integral of (1) exists. This set is denoted by $\mathcal{F}([a,b])$. A *function interval* Y is an interval of functions in $\mathcal{F}([a,b])$,

$$Y = [\underline{y},\overline{y}] = \{y \in \mathcal{F}([a,b]) \mid \forall t \in [a,b] \ : \ \underline{y(t)} \leq y(t) \leq \overline{y(t)}\}$$

where $\underline{y},\overline{y} \in \mathcal{F}([a,b])$. In $\mathcal{F}([a,b])$ we use a function norm $||\cdot||$ defined as

$$||y|| = \sup_{a \leq t \leq b} |y(t)|$$

The width of a function interval Y is

$$\mathrm{w}(Y) = \sup_{a \leq t \leq b} \{\mathrm{w}(Y(t))\},$$

where $\mathrm{w}(Y(t))$ is the normal interval width. The distance between two function intervals is

$$\mathrm{d}(Y_1,Y_2) = \max\{\sup_{y_1 \in Y_1}(\inf_{y_2 \in Y_2}(||y_1 - y_2||)), \sup_{y_2 \in Y_2}(\inf_{y_1 \in Y_1}(||y_1 - y_2||))\}.$$

T is said to be a *contraction* over the function interval Y if there exists a real constant $q < 1$ such that

$$||Ty_1 - Ty_2|| \leq q||y_1 - y_2||, \quad \text{for } y_1,y_2 \in Y. \tag{3}$$

In order to construct a narrow function interval $Y^\star$ which is guaranteed to contain a solution $y^\star$ of (2) we use iterated sets: Suppose we were able to calculate

$$TY = \{Ty \mid y \in Y\}$$

for any function interval Y. Then the iteration

$$Y^{(k+1)} = Y^{(k)} \cap TY^{(k)} \quad k = 0, 1, \ldots \tag{4}$$

defines a nested sequence of function intervals with the following properties (where $y^\star$ is a solution of (2) see [2]) :

$$\begin{aligned} y^\star \in Y^{(k)} \quad &\Rightarrow \quad y^\star \in Y^{(k+1)} &(5)\\ Y^{(k+1)} = \emptyset \quad &\Rightarrow \quad \text{no solution exists in } Y^{(k)} &(6)\\ T \text{ contraction in } Y^{(0)} \quad &\Rightarrow \quad \mathrm{w}(Y^{(k+1)}) \le q^k \mathrm{w}(Y^{(0)}) &(7)\end{aligned}$$

However, it is usually not possible to calculate TY, unless severe assumptions are made. Rall [8] solves the problem using a monotonicity assumption on T. In the present paper we avoid this type of assumption by using interval analysis. Moore [6] considers a subdivision

$$X_k = [x_k, x_{k+1}] \quad , k = 0, \ldots, n-1 \tag{8}$$

of $[a,b]$. If $x \in X_i$ then TY can be included as follows

$$TY(x) \in F(X_i) + \sum_{j=0}^{n-1} G(X_i, X_j, \tilde{Y}_i, \tilde{Y}_j)\mathrm{w}(X_j) \tag{9}$$

where F and G are interval extensions of f and g, and $\tilde{Y}_k = [\underline{y_k}, \overline{y_k}]$ is a real interval which contains

$$Y(X_k) = \{y(t) \mid y \in Y \wedge t \in X_k\}, \; k = 0, .., n-1 \tag{10}$$

However, the extension (9) of T does not necessarily maintain a contraction property of T, even if we let the maximum width of the subdivision (8) go to zero. This is demonstrated in [2].

Instead of (9) we base our practical version of (4) on the mean value form of T [3],

$$T_m Y = Ty_m + (T'Y)(Y - y_m) \tag{11}$$

where $y_m \in Y$. More detailed $T_m Y$ is the following [3, 10]

$$\begin{aligned} T_m Y(x) \quad = \quad & Ty_m(x) + (Y(x) - y_m(x)) \int_a^b G_3'(x, t, Y(x), Y(t))dt \\ & + \int_a^b G_4'(x, t, Y(x), Y(t))(Y(t) - y_m(t))dt \end{aligned} \tag{12}$$

where G_3' and G_4' are interval extensions of the partial derivatives of g with respect to its third and fourth variable. The interval integrals are defined in [4]. Since the distance between T_mY and TY is of the order $\mathrm{O}(\mathrm{w}(Y)^2)$, under realistic conditions on G_3' and G_4', T_mY will conserve a contraction property of T in the neighbourhood of a solution $y^\star$ of (2), [3].
As we demonstrate in the next section this means that we can find Riemann sums in the style of (9) which include T_mY and provide a contraction.

2 Practical mean value forms

In general the expression $T_mY(x)$ of (12) can not be calculated directly. Therefore we have to use outer approximations to the integrals, and it turns out that Riemann sums suffice for our purpose. In the following we assume that y_m is the minimum function of Y, i.e.

$$y_m(x) = \underline{y}(x) = \inf\{Y\}(x). \tag{13}$$

Consider the subdivision (8), and let $Y_k = Y(X_k)$. Using the theory of interval integrals [4] we can now include the integrals in (11) as follows,

$$\begin{aligned} T_mY(x) &= Ty_m(x) + (Y(x) - y_m(x)) \sum_{j=0}^{n-1} \int_{X_j} G_3'(x,t,Y(x),Y(t))dt \\ &\quad + \sum_{j=0}^{n-1} \int_{X_j} G_4'(x,t,Y(x),Y(t))(Y(t) - y_m(t))dt \\ &\subseteq Ty_m(x) + (Y(x) - y_m(x)) \sum_{j=0}^{n-1} G_3'(x,X_j,Y(x),Y_j)\mathrm{w}(X_j) \\ &\quad + \sum_{j=0}^{n-1} \int_{X_j} G_4'(x,X_j,Y(x),Y_j)(Y(t) - y_m(t))dt \\ &= Ty_m(x) + (Y(x) - y_m(x)) \sum_{j=0}^{n-1} G_3'(x,X_j,Y(x),Y_j)\mathrm{w}(X_j) \\ &\quad + \sum_{j=0}^{n-1} G_4'(x,X_j,Y(x),Y_j) \int_{X_j} (Y(t) - y_m(t))dt \end{aligned} \tag{14}$$

where the last equality is due to the fact that $Y - y_m$ is a non-negative interval function [10].

Assume for the time being that $Ty_m(x)$ can be calculated exactly. Then the following interval extension of the last expression in (14) is computable in practice:

$$\begin{aligned} IT_mY(x) &= Ty_m(x) + (Y(x) - y_m(x)) \sum_{j=0}^{n-1} G_3'(X_i,X_j,Y_i,Y_j)\mathrm{w}(X_j) \\ &\quad + \sum_{j=0}^{n-1} G_4'(X_i,X_j,Y_i,Y_j) \int_{X_j} (Y(t) - y_m(t))dt \end{aligned} \tag{15}$$

where $x \in X_i$.

Theorem 1 *Let $\Delta = \max_{0 \le j \le n-1} w(X_j)$. Assume G_3' and G_4' are linear interval extensions of g_3' and g_4'. Assume further that all Y under consideration contain a (fixed) function $y^\star$ which satisfies a Lipschitz condition. Then*

$$d(IT_mY, T_mY) = O(w(Y)\max(\Delta, w(Y))) \tag{16}$$

Proof

We must show that each of the two integrals in (12) are close to the corresponding sums in (15). Since these two arguments are quite similar we only treat the last one here.

Let $x \in X_i$ and let

$$J_j(x,Y) = \int_{X_j} G_4'(X_i, X_j, Y_i, Y_j)(Y(t) - y_m(t))dt \tag{17}$$

$$I_j(x,Y) = \int_{X_j} G_4'(x, t, Y(x), Y(t))(Y(t) - y_m(t))dt \tag{18}$$

Since G_4' is a linear interval extension we have for $t \in X_j$.

$$\mathrm{d}(G_4'(X_i, X_j, Y_i, Y_j), G_4'(x, t, Y(x), Y(t))) = \mathrm{O}(\max(\mathrm{w}(X_i), \mathrm{w}(X_j), \mathrm{w}(Y_i), \mathrm{w}(Y_j))) \tag{19}$$

Furthermore, since Y contains a Lipschitz function $y^\star$

$$\begin{aligned} \mathrm{w}(Y_i) &= \sup_{t \in X_i}(y^\star(t)) - \inf_{t \in X_i}(y^\star(t)) + \mathrm{O}(\mathrm{w}(Y)) \\ &= \mathrm{O}(\mathrm{w}(X_i)) + \mathrm{O}(\mathrm{w}(Y)) \end{aligned}$$

Hence, the right hand side of (19) is $\mathrm{O}(\max(\Delta, \mathrm{w}(Y)))$. Therefore

$$\begin{aligned} J_j(x,Y) &= \int_{X_j} (G_4'(x,t,Y(x),Y(t)) + \mathrm{O}(\max(\Delta, \mathrm{w}(Y))))(Y(t) - y_m(t))dt \\ &= I_j(x,Y) + \int_{X_j} \mathrm{O}(\max(\Delta, \mathrm{w}(Y)))(Y(t) - y_m(t))dt \end{aligned}$$

i.e.

$$\mathrm{d}(J_j(x,Y), I_j(x,Y)) = \mathrm{O}(\max(\Delta, \mathrm{w}(Y)))\mathrm{w}(Y)\mathrm{w}(X_j).$$

This implies that the distance between the last term in (15) and the last term in (12) is of the order

$$\sum_{j=0}^{n-1} \mathrm{O}(\max(\Delta, \mathrm{w}(Y)))\mathrm{w}(Y)\mathrm{w}(X_j) = \mathrm{O}(\mathrm{w}(Y)\max(\Delta, \mathrm{w}(Y)))$$

Now the theorem follows from a similar argument applied to the other integrals or sums in (12) and (15).

□

Theorem 2 *Assume T is a contraction over Z, that a solution $y^\star$ of (2) is included in the function interval $Y^{(0)} \subseteq Z$, and that G'_3 and G'_4 are linear interval extensions. Let*

$$Y^{(k+1)} = IT_m Y^{(k)} \cap Y^{(k)} \quad ,k = 0,1,2,... \tag{20}$$

Then there exists a constant $c < 1$ such that

$$w(Y^{(k+1)}) \leq cw(Y^{(k)}) \quad ,k = 0,1,2,... \tag{21}$$

provided that $\Delta = \max(w(X_i))$ and $w(Y_0) > 0$ are sufficiently small.

<u>Proof</u>
From (3) and Theorem 4 of [3] we have

$$\begin{aligned} \mathrm{w}(TY^{(k)}) &\leq q\mathrm{w}(Y^{(k)}) &(22)\\ \mathrm{d}(TY^{(k)}, T_m Y^{(k)}) &= \mathrm{O}(\mathrm{w}(Y^{(k)})^2) &(23) \end{aligned}$$

Therefore (16) implies

$$\mathrm{w}(IT_m Y^{(k)}) \leq (q + \mathrm{O}(\max(\mathrm{w}(Y^{(k)}), \Delta)))\mathrm{w}(Y^{(k)}) \tag{24}$$

and the result follows by letting Δ and $\mathrm{w}(Y^{(0)})$ be so small that the factor to $\mathrm{w}(Y^{(k)})$ in (24) is less than 1 for $k = 0$.

□

Theorem 2 shows that a contraction property of T is not wasted by extending T to its mean value form and enclosing the interval integrals by Riemann sums. We still have $Y^{(k)} \to [y^\star, y^\star]$.

Therefore, the only source of error will be the "real" term in (15): $Ty_m(x)$ (and rounding errors when it comes to actual calculations). However, it turns out that this term can be calculated as accurately as we want (Section 3).
Assume we enclose the real term Ty_m in the function interval ITy_m. Then we obtain the following extension of IT_mY:

$$\hat{I}T\,Y(x) = ITy_m(x) + J^{(3)}(x,Y) + J^{(4)}(Y) \tag{25}$$

where $J^{(3)}$ and $J^{(4)}$ are the last terms of (15). The width of this extension can be bounded as follows:

Theorem 3 *Let the assumption of Theorem 2 hold, and let*

$$Y^{(k+1)} = Y^{(k)} \cap \hat{I}T\,Y^{(k)} \quad ,k = 0,1,\dots \tag{26}$$

and assume $w(ITy_m) \leq \delta$ *for* $k = 0,1,\dots$. *If* $y^\star \in Y^{(0)}$ *and* $w(Y^{(0)})$, Δ *are as small as required in Theorem 2, then* $y^\star \in Y^{(k)}$ $,k = 0,1,2,\dots$, *and*

$$w(Y^{(k+1)}) \leq cw(Y^{(k)}) + \delta \tag{27}$$

where $c < 1$ *is defined in Theorem 2. Furthermore* $Y^{(k)} \to Y^\star$ *with* $y^\star \in Y^\star$ *and*

$$w(Y^\star) \leq \frac{\delta}{1-c} \tag{28}$$

Proof
(27) is an immediate consequence of Theorem 2, and (28) follows from Theorem 6 of [1].

□

Notice that since $TY \subseteq \hat{I}T\,Y$ the iteration (26) satisfies implication (6).

3 Calculation of Ty_m

In formula (26) we express the final interval iteration for enclosing a solution to (2). In this section we give three different formulas for calculating the enclosure ITy_m of Ty_m. They are based on zero, first and second order Taylor approximations to f and g, and the subdivision (8) of $[a,b]$. Δ is $\max(\mathrm{w}(X_i))$.

3.1 Zero order approximation

This is the Riemann sum enclosure which has been used for the other terms as well: If $x \in X_i$ then $Ty_m \subseteq I_0T\ y_m$ where

$$I_0T\ y_m(x) = F(X_i) + \sum_{j=0}^{n-1} G(X_i, X_j, y_m(X_i), y_m(X_j))\mathrm{w}(X_j) \tag{29}$$

This means that $TY(x)$ can be enclosed by

$$\begin{aligned}\hat{I}_0T\ Y(x) \quad = \quad & F(X_i) + \sum_{j=0}^{n-1} G(X_i, X_j, y_m(X_i), y_m(X_j))\mathrm{w}(X_j) \\ & + (Y(x) - y_m(x)) \sum_{j=0}^{n-1} G_3'(X_i, X_j, Y_i, Y_j)\mathrm{w}(X_j) \\ & + \sum_{j=0}^{n-1} G_4'(X_i, X_j, Y_i, Y_j) \int_{X_j} (Y(t) - y_m(t))dt \end{aligned} \tag{30}$$

where $x \in X_i$ If F and G are linear interval extensions then

$$\begin{aligned}\mathrm{w}(I_0Ty_m(x)) \quad \leq \quad & \mathrm{w}(F(X_i)) + \sum_{j=0}^{n-1} \mathrm{w}(G(X_i, X_j, y_m(X_i), y_m(X_j)))\mathrm{w}(X_j) \\ = \quad & \mathrm{O}(\max(\Delta, \max_i(\mathrm{w}(y_m(X_i))))) \end{aligned} \tag{31}$$

If we assume that all y_m under consideration satisfy a Lipschitz condition with an upper bound on the Lipschitz constant (or if the condition on $y^\star$ in Theorem 1 holds), then $\mathrm{w}(y_m(X_i)) = \mathrm{O}(\mathrm{w}(X_i))$ and hence

$$\mathrm{w}(I_0Ty_m) = \mathrm{O}(\Delta). \tag{32}$$

Under the conditions of Theorem 3 we then obtain convergence to $Y^\star$ with

$$\mathrm{w}(Y^\star) = \mathrm{O}(\Delta) \tag{33}$$

3.2 First order approximation

When Ty_m is calculated y_m is considered as a real function (not a variable) and hence we can consider g as a function of x and t only,

$$h(x,t) = g(x,t,y_m(x),y_m(t)). \tag{34}$$

Note that

$$\begin{aligned} h'_1(x,t) &= g'_1(x,t,y_m(x),y_m(t)) + g'_3(x,t,y_m(x),y_m(t))y'_m(x) \\ h'_2(x,t) &= g'_2(x,t,y_m(x),y_m(t)) + g'_4(x,t,y_m(x),y_m(t))y'_m(t) \end{aligned}$$

Let H'_1 and H'_2 be interval extensions of h'_1 and h'_2. From the interval mean value theorem [3] we have for any $\hat{x}_i, x \in X_i$ and any $\hat{x}_j, t \in X_j$ that

$$h(x,t) \in h(\hat{x}_i,\hat{x}_j) + H'_1(X_i,X_j)(x-\hat{x}_i) + H'_2(X_i,X_j)(t-\hat{x}_j)$$

and

$$f(x) \in f(\hat{x}_i) + F'(X_i)(x-\hat{x}_i)$$

Therefore, if we choose $\hat{x}_k = x_k$ for $k = 0,\ldots,n-1$ then we obtain $Ty_m \subseteq I_1T\ y_m(x)$ with

$$\begin{aligned} I_1T\ y_m(x) &= f(x_i) + F'(X_i)(x-x_i) + \sum_{j=0}^{n-1}\Big\{(h(x_i,x_j) \\ &\quad + H'_1(X_i,X_j)(x-x_i))\mathrm{w}(X_j) + \int_{X_j} H'_2(X_i,X_j)(t-x_j)dt\Big\} \\ &= f(x_i) + F'(X_i)(x-x_i) + \sum_{j=0}^{n-1}\Big\{(h(x_i,x_j) \\ &\quad + H'_1(X_i,X_j)(x-x_i))\mathrm{w}(X_j) + H'_2(X_i,X_j)\cdot\frac{1}{2}\mathrm{w}(X_j)^2\Big\} \end{aligned} \tag{35}$$

where $x \in X_i$. Hence we can define our first order integral operator by

$$
\begin{aligned}
\hat{I}_1 T\, Y(x) = f(x_i) + F'(X_i)(x - x_i) + \sum_{j=0}^{n-1} \Big\{ (h(x_i, x_j) \\
+ H_1'(X_i, X_j)(x - x_i))\mathrm{w}(X_j) + H_2'(X_i, X_j) \cdot \frac{1}{2}\mathrm{w}(X_j)^2 \Big\} \\
+ (Y(x) - y_m(x)) \sum_{j=0}^{n-1} G_3(X_i, X_j, Y_i, Y_j)\mathrm{w}(X_j) \\
+ \sum_{j=0}^{n-1} G_4(X_i, X_j, Y_i, Y_j) \int_{X_j} (Y(t) - y_m(t))dt
\end{aligned} \tag{36}
$$

where $x \in X_i$. Notice that $\inf\{(\hat{I}_1 T\, Y)(x)\}$ is piecewise linear as a function of x.

If F', H_1' and H_2' are linear interval extensions then

$$
\begin{aligned}
\mathrm{w}(I_1 T y_m(x)) &\le \mathrm{w}(F'(X_i)) \cdot \Delta \\
&+ \sum_{j=0}^{n-1} \Big\{ \mathrm{w}(H_1'(X_i, X_j))\Delta^2 + \frac{1}{2}\mathrm{w}(H_2'(X_i, X_j))\Delta^2 \Big\} \\
&= \mathrm{O}(\Delta^2) + \sum_{j=0}^{n-1} \mathrm{O}(\Delta^2)\mathrm{w}((X_j) \\
&= \mathrm{O}(\Delta^2).
\end{aligned} \tag{37}
$$

(Notice that the Lipschitz condition assumed in connection with (32) is implicitly assumed here through the assumption on H_1' and H_2'.)

Under the conditions of Theorem 3 we then obtain convergence to $Y^\star$ with

$$
\mathrm{w}(Y^\star) = \mathrm{O}(\Delta^2) \tag{38}
$$

3.3 Second order approximation

This can be based on using Taylor's formula with one more term than in section 3.2. Skipping the details we obtain $Ty_m \subseteq I_2T\, y_m(x)$ for $x \in X_i$ where

$$\begin{aligned} I_2T\, y_m(x) = f(x_i) + f'(x_i)^{\mathrm{T}}(x-x_i) + \sum_{j=0}^{n-1} \int_{X_j} \Big\{ (h(x_i,x_j) \\ + h'_1(x_i,x_j)(x-x_i) + h'_2(x_i,x_j)(t-x_j) \Big\} dt \\ + \{\text{Interval remainder term}\} \end{aligned} \tag{39}$$

Introducing the notations F'', H''_{ij} for interval extensions of the second order partial derivatives f'', h''_{ij}, $i,j = 1,2$, we can write the interval remainder term as follows,

$$\begin{aligned} \frac{1}{2}F''(X_i)(x-x_i)^2 + \frac{1}{2}\sum_{j=0}^{n-1}\Big\{ H''_{11}(X_i,X_j)(x-x_i)^2 \mathrm{w}(X_j) \\ + H''_{12}(X_i,X_j)(x-x_i)\mathrm{w}(X_j)^2 + \frac{1}{3}H''_{22}(X_i,X_j)\mathrm{w}(X_j)^3 \Big\} \end{aligned} \tag{40}$$

where it is assumed that $h''_{12} = h''_{21}$.

Now $\inf\{(\hat{I}_2T\, Y)(x)\}$ is a piecewise quadratic function if Y has this property. If F'', H''_{ij} are linear interval extensions then

$$\begin{aligned} \mathrm{w}(I_2T\, Y(x)) \;&\le\; \frac{1}{2}\mathrm{w}(F''(X_i))\cdot\Delta^2 + \sum_{j=0}^{n-1}\Big\{ \mathrm{w}(H''_{11}(X_i,X_j))\Delta^2 \\ &\quad + \mathrm{w}(H''_{12}(X_i,X_j))\Delta^2 + \mathrm{w}(H''_{22}(X_i,X_j))\Delta^2 \Big\} \mathrm{w}(X_j) \\ &= \mathrm{O}(\Delta^3) + \sum_{j=0}^{n-1}\mathrm{O}(\Delta^3)\mathrm{w}(X_j) = \mathrm{O}(\Delta^3). \end{aligned} \tag{41}$$

Under the conditions of Theorem 3 we then obtain convergence to $Y^\star$ with

$$\mathrm{w}(Y^\star) = \mathrm{O}(\Delta^3) \tag{42}$$

4 Numerical examples

In this section we consider the application of the zero and the first order method described in section 3. The algorithms is based upon an iteration procedure:

Generate a subdivision $X_k = [x_k, x_{k+1}]$ $\quad, k = 0, \ldots, n-1$,
and a function interval $Y(x)$ according to the subdivision.
do
  $Y_{\text{old}} := Y$
  for $i := 0$ to $n-1$ do
    for $x \in X_i$
      $Y(x) := \hat{I}T\ Y(x)$
  If $Y \subseteq Y_{\text{old}}$ a solution exists in Y
while $Y \neq Y_{\text{old}}$

Note that the function interval Y is updated each time a new piece of the function has been calculated. This Gauss-Seidel like method speeds up the convergence. Another important simplification of the method is that instead of using (26) in the iteration we use $Y := \hat{I}T\ Y$.

In some interval-libraries the width of an interval is defined as an upper bound, in this article it is crucial that the width is calculated as an interval containing the true width.

4.1 The Saaty integral equation

The first integral operator we will discuss is the Saaty integral equation,

$$y(x) = \int_0^1 g(x,t,y(x),y(t))dt \tag{43}$$

with

$$g(x,t,y(x),y(t)) = \begin{cases} t(1-x)\{\alpha^2 \sin(y(t)) - \sin(t\pi)\} & ,t \leq x, \\ x(1-t)\{\alpha^2 \sin(y(t)) - \sin(t\pi)\} & ,t > x. \end{cases}$$

The partial derivatives with respect to the third and fourth variable are

$$\begin{aligned} g_3'(x,t,y(x),y(t)) &= 0 \\ g_4'(x,t,y(x),y(t)) &= \begin{cases} t(1-x)\alpha^2 \cos(y(t)) & ,t < x, \\ x(1-t)\alpha^2 \cos(y(t)) & ,t \geq x. \end{cases} \end{aligned}$$

The interval extensions G, G'_3 and G'_4 are easily derived

$$\begin{aligned}
G(X_i,X_j,Y_i,Y_j) &= \begin{cases} X_j(1-X_i)\{\alpha^2\sin(Y_j)-\sin(X_j\pi)\} & ,t \le x, \\ X_i(1-X_j)\{\alpha^2\sin(Y_j)-\sin(X_j\pi)\} & ,t > x. \end{cases} \\
G'_3(X_i,X_j,Y_i,Y_j) &= 0 \\
G'_4(X_i,X_j,Y_i,Y_j) &= \begin{cases} X_j(1-X_i)\alpha^2\cos(Y_j) & ,t \le x, \\ X_i(1-X_j)\alpha^2\cos(Y_j) & ,t > x. \end{cases}
\end{aligned}$$

But a problem occurs in (14) if we do not know which one of x,t is the smallest, and therefore do not know which of the cases in G and G'_4 to insert. This problem is reflected in (30) when $i = j$. Fortunately in this case it does not matter because the cases in G and G'_4 are equal when $X_i = X_j$. The zero order Saaty integral operator is now given by

$$\begin{aligned}
\hat{I}T\,Y(x) &= \sum_{j=0}^{n-1} S_{ij}(X_i,X_j)(\alpha^2\sin(y_m(X_j))-\sin(X_j\pi))\mathrm{w}(X_j) \\
&\quad + \sum_{j=0}^{n-1} S_{ij}(X_i,X_j)\alpha^2\cos(Y_j))\int_{X_j}(Y(t)-y_m(t))dt
\end{aligned}$$

where $x \in X_i$ and the function S_{ij} is defined by

$$S_{ij}(X_i,X_j) = \begin{cases} X_j(1-X_i)\ , & j <= i \\ X_i(1-X_j)\ , & j > i \end{cases}$$

The partial derivatives of g with respect to its first and second variable is

$$\begin{aligned}
g'_1(x,t,y(x),y(t)) &= \begin{cases} t(\sin(t\pi)-\alpha^2\sin(y(t))) & ,t < x, \\ (1-t)(\alpha^2\sin(y(t))-\sin(t\pi)) & ,t > x. \end{cases} \\
g'_2(x,t,y(x),y(t)) &= \begin{cases} (1-x)(\alpha^2\sin(y(t))-\sin(t\pi)-t\pi\cos(t\pi)),t < x, \\ x((t-1)\pi\cos(t\pi)-\alpha^2\sin(y(t))+\sin(t\pi)),t > x. \end{cases}
\end{aligned}$$

In the case $i = j$ we do not know which cases in the interval extensions of G'_1 and G'_2 to use in (36), and in this case we are forced to use the zero order approximation of h.

4.2 The Ambartsumian integral equation

The next integral equation considered in this article is the Ambartsumian integral equation [9],

$$y(x) = 1 + \frac{\lambda}{2}\int_0^1 \frac{x}{x+t}y(x)y(t)dt, \text{ where } \lambda \in [0,1] \tag{44}$$

	zero order		first order	
i	$n = 15$	$n = 50$	$n = 15$	$n = 50$
0	0.30000	0.30000	0.30000	0.30000
1	0.08425	0.04100	0.06387	0.03181
2	0.06126	0.01858	0.03667	0.00887
3	0.05969	0.01745	0.03444	0.00787
4	0.05960	0.01740	0.03425	0.00782
5	0.05959	0.01740	0.03423	0.00782
...	...	...	...	...
∞	0.05959 (16)	0.01740 (15)	0.03423 (17)	0.00782 (16)

Table 1: The width of Y as a function of the iteration number i and the number of discretization intervals n. The number in parenthesis is the total number of iterations before the algorithm stops when $Y = Y_{\text{old}}$.

We define

$$g(x,t,y(x),y(t)) = \frac{x}{x+t}y(x)y(t).$$

To use the zero order method described in section 3.1 we have to implement the interval extensions G, G'_3, G'_4 of g, g'_3 and g'_4.

$$g'_3(x,t,y(x),y(t)) = \frac{x}{x+t}y(t), \quad g'_4(x,t,y(x),y(t)) = \frac{x}{x+t}y(x)$$

For $x > 0$ or $t > 0$ we use the natural interval extensions, but if $x,t \in [x_0,x_1] = [0,x_1]$ we make use of the fact that $x \le x+t$ to derive the interval extensions

$$G(X_0,X_0,Y_0,Y_0) = [0,1]Y_0Y_0$$

$$G'_3(X_0,X_0,Y_0,Y_0) = [0,1]Y_0, \quad G'_4(X_0,X_0,Y_0,Y_0) = [0,1]Y_0$$

The zero order Ambarzumian integral operator can now computed by using (30). To implement the first order approximation described in section 3.2, interval extensions of g'_1 and g'_2 are needed,

$$g'_1(x,t,y(x),y(t)) = \frac{x}{(x+t)^2}y(x)y(t),$$

$$g'_2(x,t,y(x),y(t)) = -\frac{x}{(x+t)^2}y(x)y(t).$$

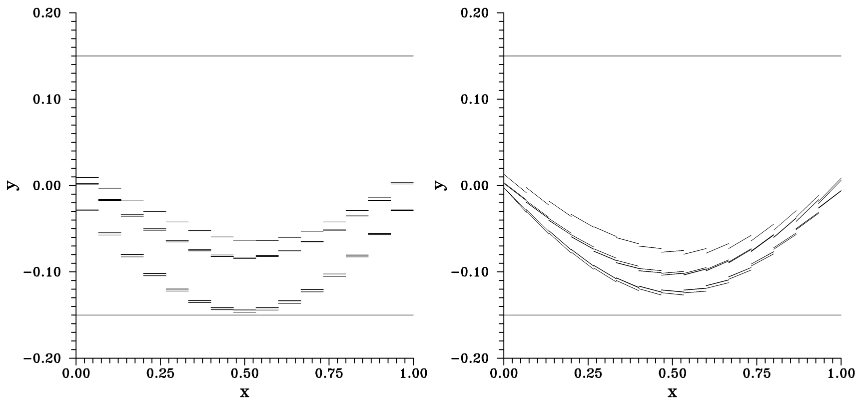


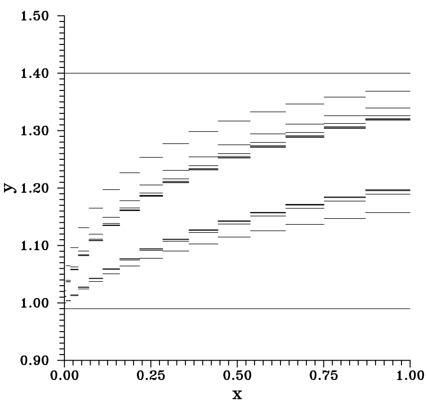
Figure 1: The iteration process solving the Saaty integral equation using $n = 15$ discretization intervals. The left figure shows the zero order calculation, and the right figure shows the first order calculation.

If $x > 0$ or $t > 0$ the natural interval extensions are used, but when $x,t \to 0$ the functions g_1' and g_2' are unbounded, and we cannot make interval extensions. This means that we do not have an interval extension of the mean value form for h. In this case (i=0), we have to use the zero order approximation of h in (36).

Using $\lambda = \frac{1}{2}$ some numerical calculations were performed using a non equidistant partition $x_i = \frac{i^2}{n^2}$, $i = 0,...,n$, and the constant interval $Y = [.99, 1.4]$ as a initial value for Y. Table 2 shows the width of Y as a function of the iteration number i and the number of discretization intervals n. In figure 2 the iteration process is displayed for the zero and the first order approximation, both using $n = 15$ discretization intervals. All solutions were self-validated, i.e. the final function interval was proven to contain a solution to (44).

	zero order		first order	
i	$n = 15$	$n = 50$	$n = 15$	$n = 50$
0	0.41000	0.41000	0.41000	0.41000
1	0.21141	0.14070	0.16132	0.12418
2	0.14997	0.06210	0.07382	0.03752
3	0.13061	0.04142	0.05370	0.01412
4	0.12437	0.03626	0.05333	0.00881
5	0.12232	0.03495	0.05323	0.00818
6	0.12164	0.03461	0.05321	0.00807
7	0.12141	0.03453	0.05320	0.00805
8	0.12133	0.03450	0.05320	0.00804
9	0.12130	0.03450	0.05319	0.00804
10	0.12129	0.03450	0.05319	0.00804
...	...	...	...	...
∞	0.12129 (34)	0.03450 (30)	0.05319 (33)	0.00804 (28)

Table 2: The width of Y as a function of the iteration number i and the number of discretization intervals n. The number in parenthesis is the total number of iterations before the algorithm stops when $Y = Y_{\text{old}}$.



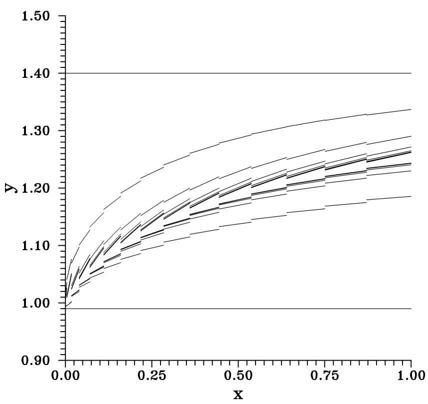


Figure 2: The iteration process solving the Ambartsumian integral equation using $n = 15$ discretization intervals. The left figure shows the zero order calculation, and the right figure shows the first order calculation.

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