Stochastic Simulation
Simulated annealing

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A general optimisation problem

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- $f^{\star}$

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- $f^{\star} = \min_x$

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This process is called **annealing**.

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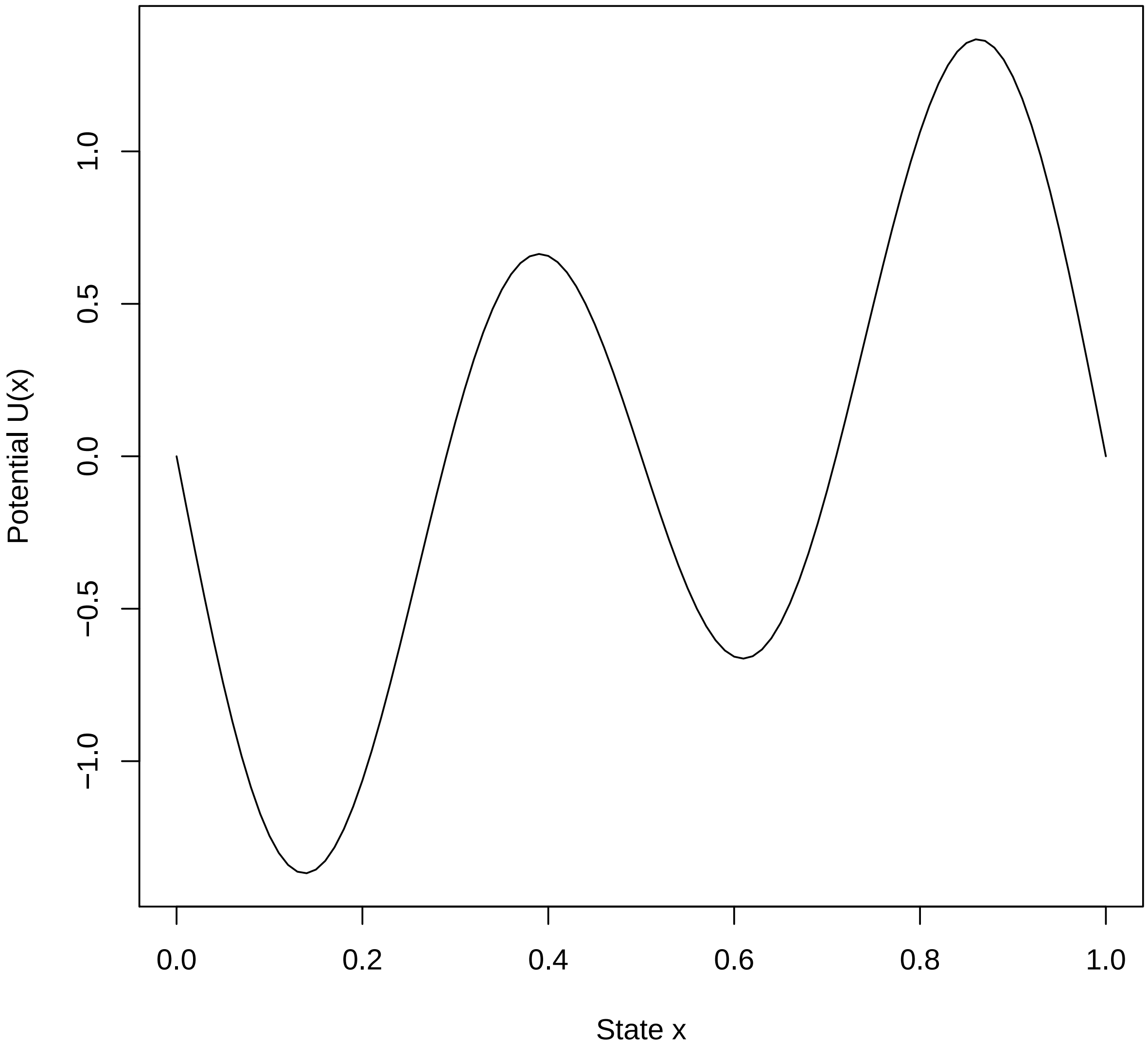
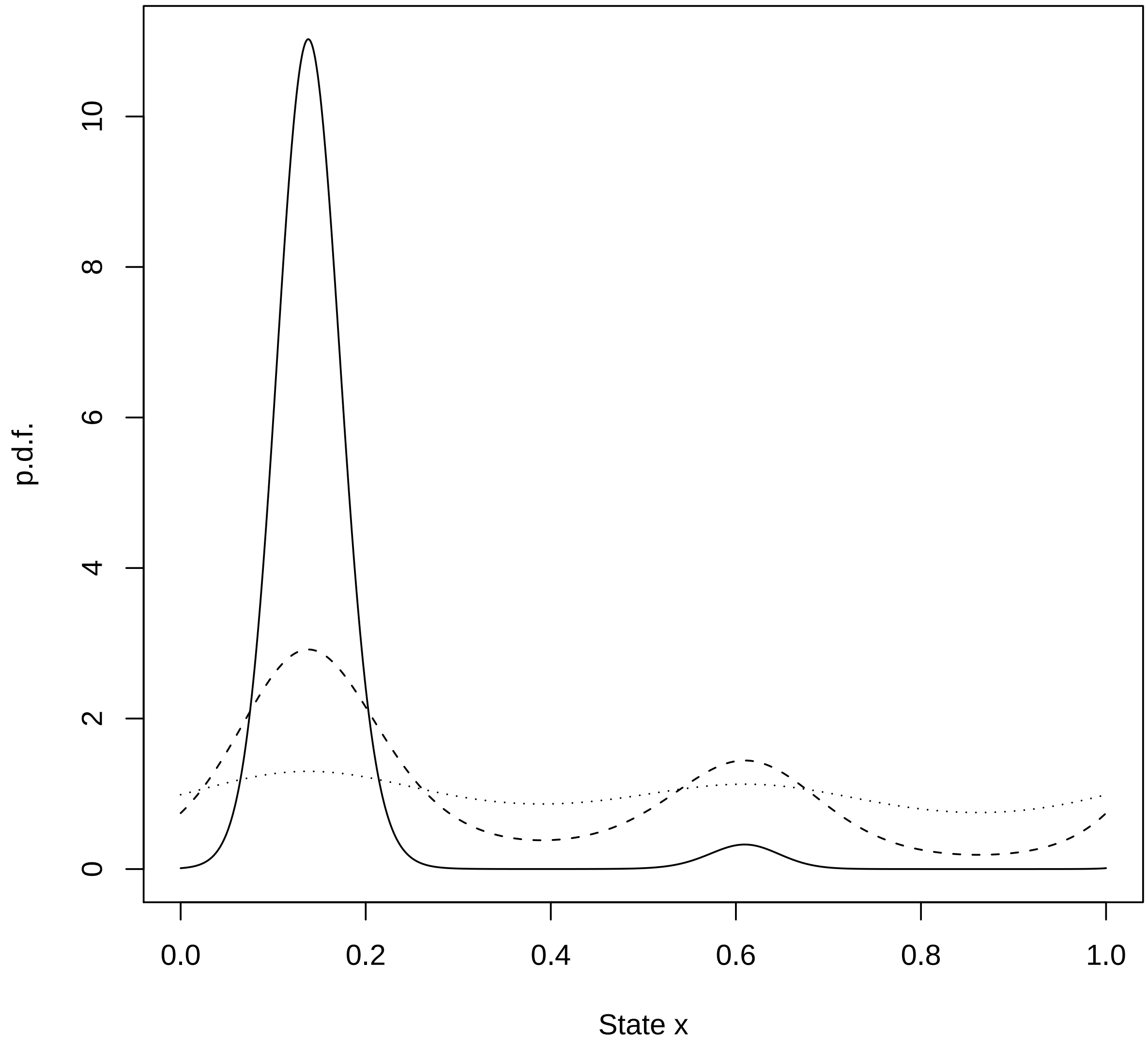
Example energy potential



Corresponding p.d.f., for $T = 0.2, 1, 5$



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Let the temperature be a decreasing function of time or iteration number - k.

At each time step, update the state according to the random walk Metropolis-Hastings algorithm for MCMC, where the target p.d.f. is $f(x, T_i)$.

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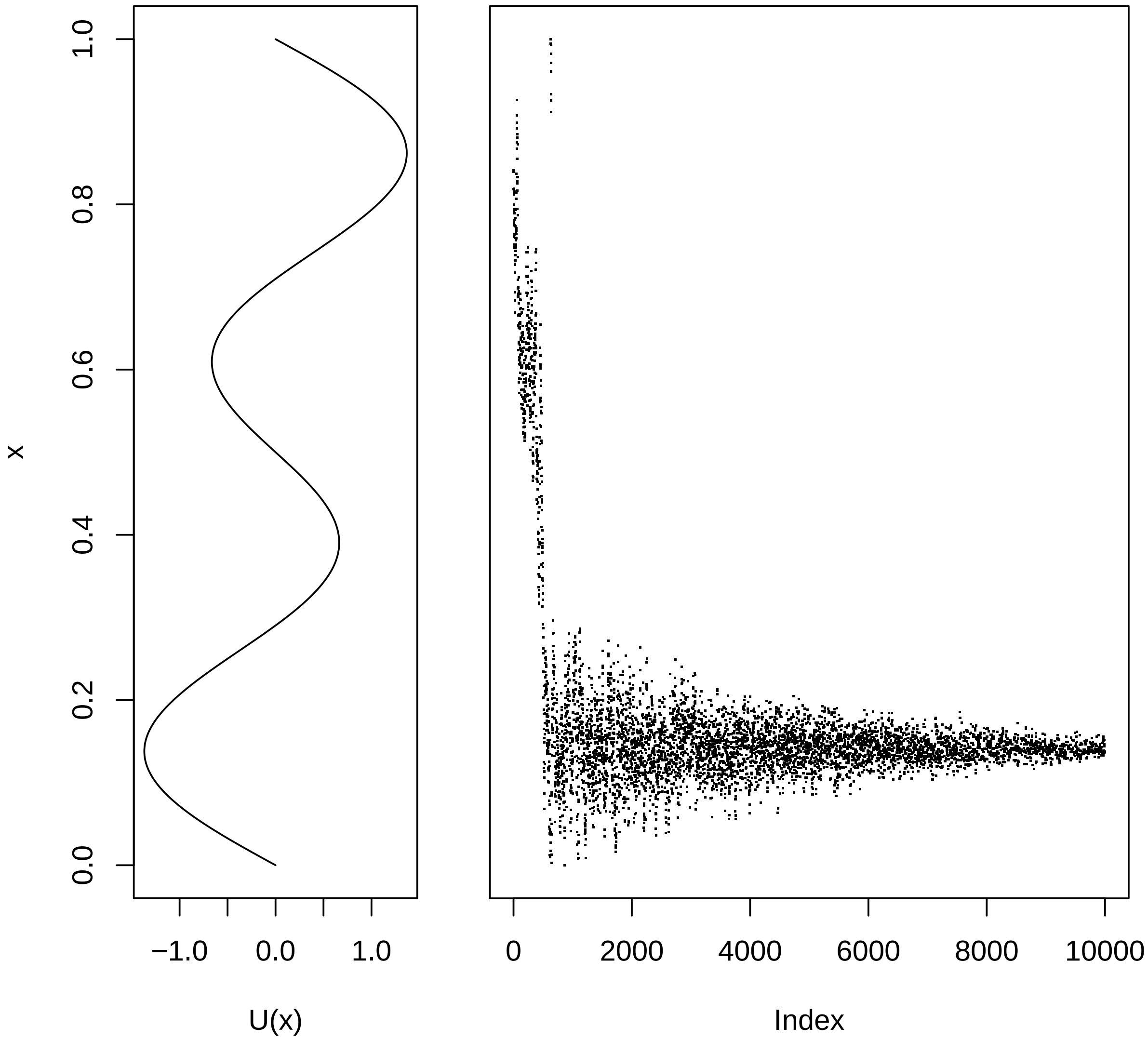
for a symmetric proposal distribution (to keep the probabilistic interpreation)



1.0
0.8
0.6
0.4
0.2
0.0
x
−1.0
0.0
1.0
U(x)
0
2000
4000
6000
8000
10000
Index

Different issues

Different issues

- Try with different schemes for lowering the temperature

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- Try with different schemes for lowering the temperature
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A basic problem in combinatorial optimisation

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$$\sum_{i=1}^{n-1} A(S_i, S_{i+1})$$

Cost matrix - an example

Town from	Town to 1	2	3	4	5	6
1	-	5	3	1	4	12
2	2	-	22	11	13	30
3	6	8	-	13	12	5
4	33	9	5	-	60	17
5	1	15	6	10	-	14
6	24	6	8	9	40	-

Cost matrix - an example

Town from	Town to 1	2	3	4	5	6
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- Initial solution: $\{1, 2, 3, 4, 5, 6, 1\}$

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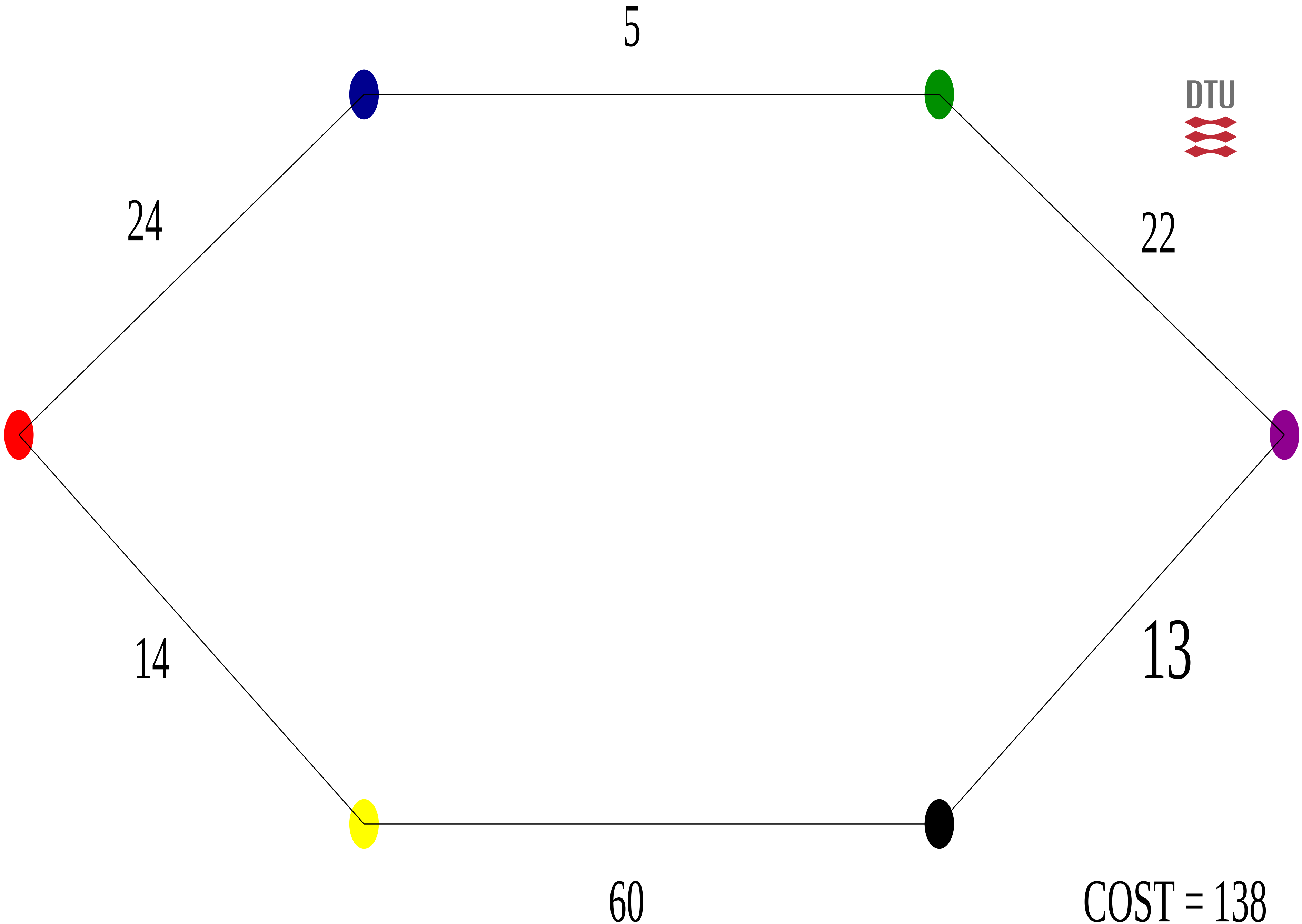
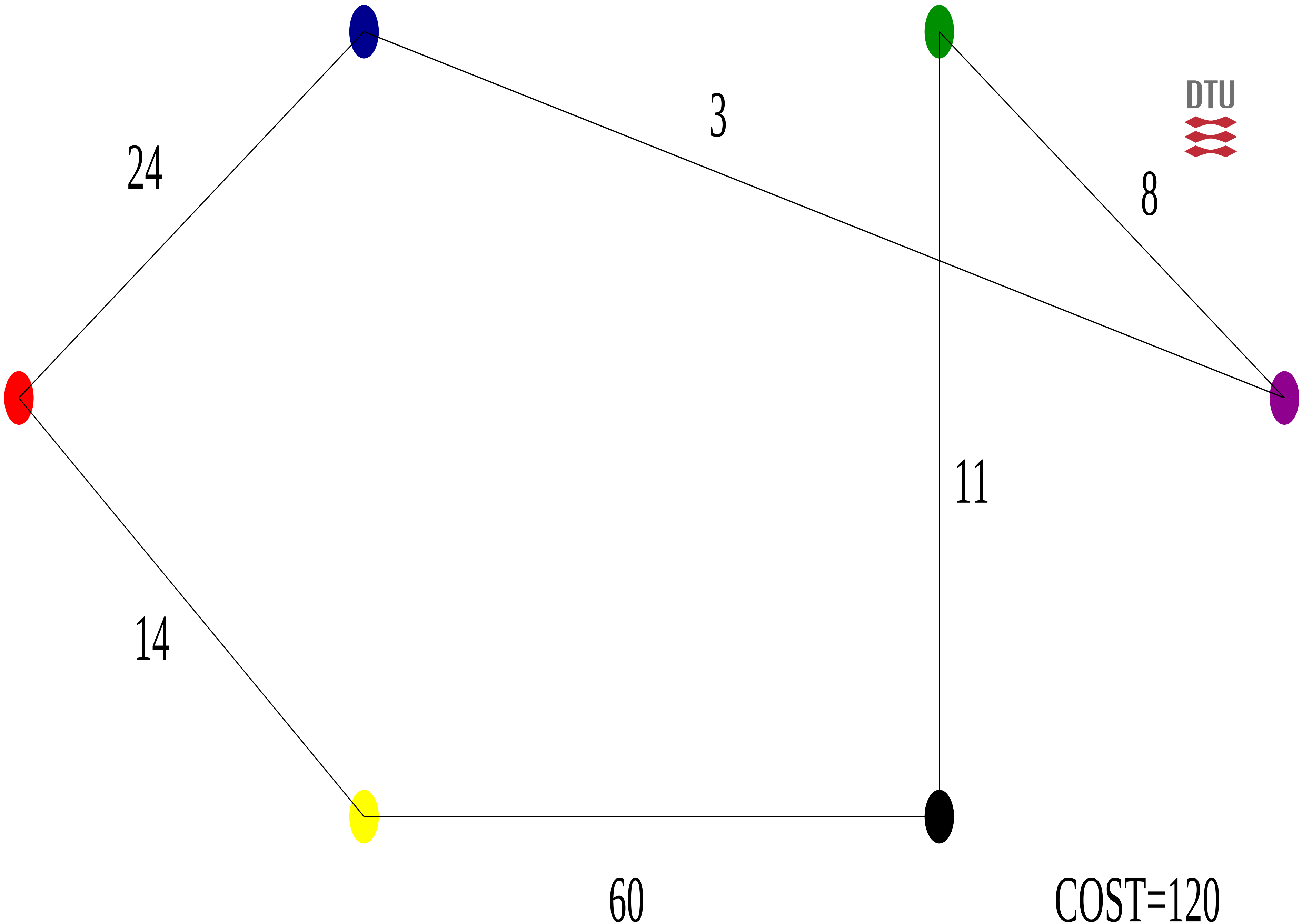
- Initial solution: $\{1, 2, 3, 4, 5, 6, 1\}$ initial cost: 5+22+13+60+14+24 $=$ 138


5
24
22
14
13
60
COST = 138


24
3
8
11
14
60
COST=120

Exercise 7

1. Implement simulated annealing for the travelling salesman. As proposal, permute two random stations on the route. As cooling scheme, you can use e.g. $T_k = 1/\sqrt{1+k}$. or $T_k = -\log(k+1)$, feel free to experiment with different choices. The route must end where it started. Initialise with a random permutation of stations.
   (a) Have input be positions in the plane of the n stations.
       Let the cost of going $i \mapsto j$ be the Euclidian distance between station i and j.
       Plot the resulting route in the plane.
       Debug with stations on a circle.
   (b) Then modify your progamme to work with costs directly and apply it to the cost matrix from the course homepage.