Stochastic Simulation
Markov Chains

Bo Friis Nielsen

Institute of Mathematical Modelling

Technical University of Denmark

2800 Kgs. Lyngby – Denmark

Email: bfni@dtu.dk

The queueing example

We simulated the system until "stochastic steady state".

We were then able to describe this steady state:

- What is the distribution of occupied servers
- What is the rejection probability

To obtain steady-state statistics, we used stochastic simulation

For Poisson arrival process and exponential service times the model was a "state machine", i.e. a Markov Chain.

Discrete time Markov chains on discrete state space

- We observe a sequence of X_ns taking values in some sample space $\mathcal{S} = \{1, 2, \ldots, N\}$, where $N = \infty$ is possible
- The next value in the sequence X_{n+1} is determined from some decision rule depending on the value of X_n only.
- For a discrete sample space we can express the decision rule as a matrix of transition probabilities $\boldsymbol{P} = \{P_{ij}\}$, $P_{ij} = \mathsf{P}(X_{n+1} = j | X_n = i)$
- We define the n-step transition probabilities $\boldsymbol{P}^{(n)} = \{P_{ij}\}$, $P_{ij}^{(n)} = \mathsf{P}(X_n = j | X_0 = i)$

Examples of Markov chain models

- (Discretised) cloud cover successive days in January
- Number of cars in stock at a car dealer at beginning at day
- Number of communication packets in buffer at beginning at transmission slot

The probability of X_n

- The behaviour of the process itself - X_n
- The behaviour conditional on $X_0 = i$ is $(P_{ij}(n))$
- Define $\mathsf{P}(X_n = j) = p_j(n)$ with $\mathsf{P}(X_0 = j) = p_j(0)$
- with $\boldsymbol{p}(n) = (p_1(n), p_2(n), \ldots, p_k(n))$ we find

$$\boldsymbol{p}(n) = \boldsymbol{p}(n-1)\boldsymbol{P} = \boldsymbol{p}(0)\boldsymbol{P}_n = \boldsymbol{p}(0)\boldsymbol{P}^n$$

- Under some technical assumptions we can find a stationary and limiting distribution $\boldsymbol{\pi}$. $\lim_{n\to\infty} P_{ij}(n) = \pi_j = \mathsf{P}(X_\infty = j)$.
- This distribution can be analytically found by solving

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P} \qquad \text{(equilibrium distribution)}$$

An example from Tuesday

- Consider the first Blocking system.
- At any given event we might have one or more customers being served and an arrival to come
- Now assume arrivals are Poisson and service times are exponential
- The exponential distribution is memoryless.

$X \sim \exp(\lambda) \quad \mathsf{P}(X > t + x | X > t) = \frac{\mathsf{P}(X>t+x, X>t)}{\mathsf{P}(X>t)} = \frac{\mathsf{P}(X>t+x)}{\mathsf{P}(X>t)}$
$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = \mathsf{P}(X > t)$

Now with $Y \sim \exp(\mu)$ we have $\mathsf{P}(Y > X) =$
$\int_0^\infty \mathsf{P}(Y > X | X = x) f_X(x) \mathrm{d}x = \int_0^\infty e^{-\mu x} \lambda e^{-\lambda x} \mathrm{d}x = \frac{\lambda}{\lambda + \mu}$

An example from Tuesday

$Z = \min(X, Y) \quad \mathsf{P}(Z > z) = \mathsf{P}(X > z, Y > z)$
$= \mathsf{P}(X > z)\mathsf{P}(Y > z) = e^{-\lambda z} e^{-\mu z} = e^{-(\lambda+\mu)z}$ i.e.
$Z \sim \exp(\lambda + \mu)$

Finally, we can show
$\mathsf{P}(Z = X | z = z) = \mathsf{P}(Z = X) = \mathsf{P}(X < Y) = \frac{\lambda}{\lambda+\mu}$

- So which state is next is independent of the time it takes to get there
- we can simulate the sequence of the states without the time if we like. we can simulate the time afterwards if we want it, as long as we know the sequence of states.

Estimating blocking probabilities - exponential case

$N_{arr} = 0$

$N_{block} = 0$

$STATE = 0$

$i = 0$

while $N_arr < n_{sim}$ do

if $U_i < \frac{lambda}{lambda + STATE * mu}$ then do

$N_{arr} = N_{arr} + 1$

if $STATE < n_{servers}$ then $STATE = STATE + 1$

else $N_{block} = N_{block} + 1$

end

else $state = state - 1$

end

$B = \frac{N_block}{n_{sim}}$

Global balance equations

The equilibrium (limiting) distribution

$$\boldsymbol{\pi} = \boldsymbol{\pi} \boldsymbol{P}$$

can be written elementwise as

$$\pi_j = \sum_{i=1}^{N} \pi_i P_{ij}$$

$$\pi_j \cdot 1 = \sum_{i=1}^{N} \pi_i P_{ij}$$

Global balance equations (continued)

$$\pi_j \cdot 1 = \sum_{i=1}^{N} \pi_i P_{ij}$$

$$\pi_j \sum_{i=1}^{N} P_{ji} = \sum_{i=1}^{N} \pi_i P_{ij}$$

$$\sum_{i=1}^{N} \pi_j P_{ji} = \sum_{i=1}^{N} \pi_i P_{ij}$$

true if

$$\pi_j P_{ji} = \pi_i P_{ij}, \qquad \forall (i,j)$$

local balance, reversible Markov chain

Small numerical example

$$\boldsymbol{P} = \begin{bmatrix} 1-p & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 1-q \end{bmatrix}$$

with $\boldsymbol{p}(0) = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right)$ we get

$$\boldsymbol{p}(1) = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right) \begin{bmatrix} 1-p & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 1-q \end{bmatrix} = \left(\frac{1-p}{3}, \frac{p}{3}, \frac{2q}{3}, \frac{2(1-q)}{3}\right)$$

and

$$\boldsymbol{p}(0) = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right),$$

$$\boldsymbol{P}^2 = \begin{bmatrix} (1-p)^2 + pq & (1-p)p & p^2 & 0 \\ q(1-p) & 2qp & 0 & p^2 \\ q^2 & 0 & 2qp & p(1-q) \\ 0 & q^2 & (1-q)q & (1-q)^2 + qp \end{bmatrix}$$

$$\boldsymbol{p}(2) = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right) \cdot$$

$$\begin{bmatrix} (1-p)^2 + pq & (1-p)p & p^2 & 0 \\ q(1-p) & 2qp & 0 & p^2 \\ q^2 & 0 & 2qp & p(1-q) \\ 0 & q^2 & (1-q)q & (1-q)^2 + qp \end{bmatrix}$$

$$= \left(\frac{(1-p)^2 + pq}{3}, \frac{(1-p)p}{3}, \frac{4qp}{3}, \frac{2p(1-q)}{3}\right)$$

Local balance for example

$$\boldsymbol{P} = \begin{bmatrix} 1-p & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 1-q \end{bmatrix}$$

$$\pi_i p = \pi_{i+1} q \Leftrightarrow \pi_{i+1} = \frac{p}{q}\pi_i$$

to give $\pi_i = \left(\frac{p}{q}\right)^{i-1} \pi_1$, $1 \leq i < 3$, with

$\pi_1 = \left(\sum_{i=1}^{4} \left(\frac{p}{q}\right)^i\right)^{-1}$

Markov chains - generalisations

- The theory can be extended to:
  - ◇ Continuous sample space (very relevant for MCMC) or
  - ◇ Continuous time: exercise 4 is an example of a Continuous time Markov chain - a Markov jump process