Stochastic Simulation
The Bootstrap method

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- A technique for estimating the variance (etc) of an estimator.

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- Non-parametric technique

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- This estimator has the variance $\frac{1}{n}\text{Var}(X)$. To estimate this, we (typically) just use the sample variance.

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- Assume we want to estimate the median, rather than the mean.
- (This makes much sense w.r.t. robustness)
- The natural estimator for the median is the sample median.
- But what is the variance of the estimator?

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We don't have the true distribution. But we have the **empirical** distribution!

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3.69

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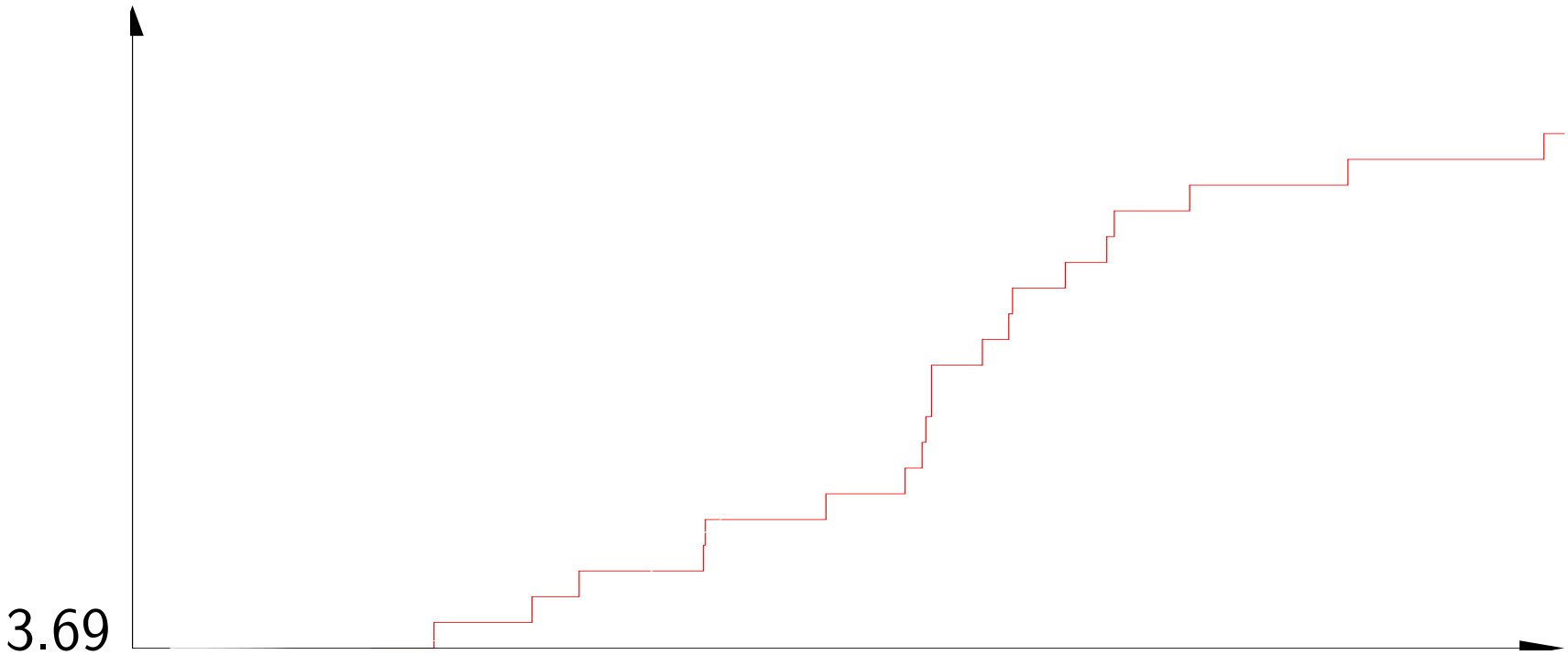

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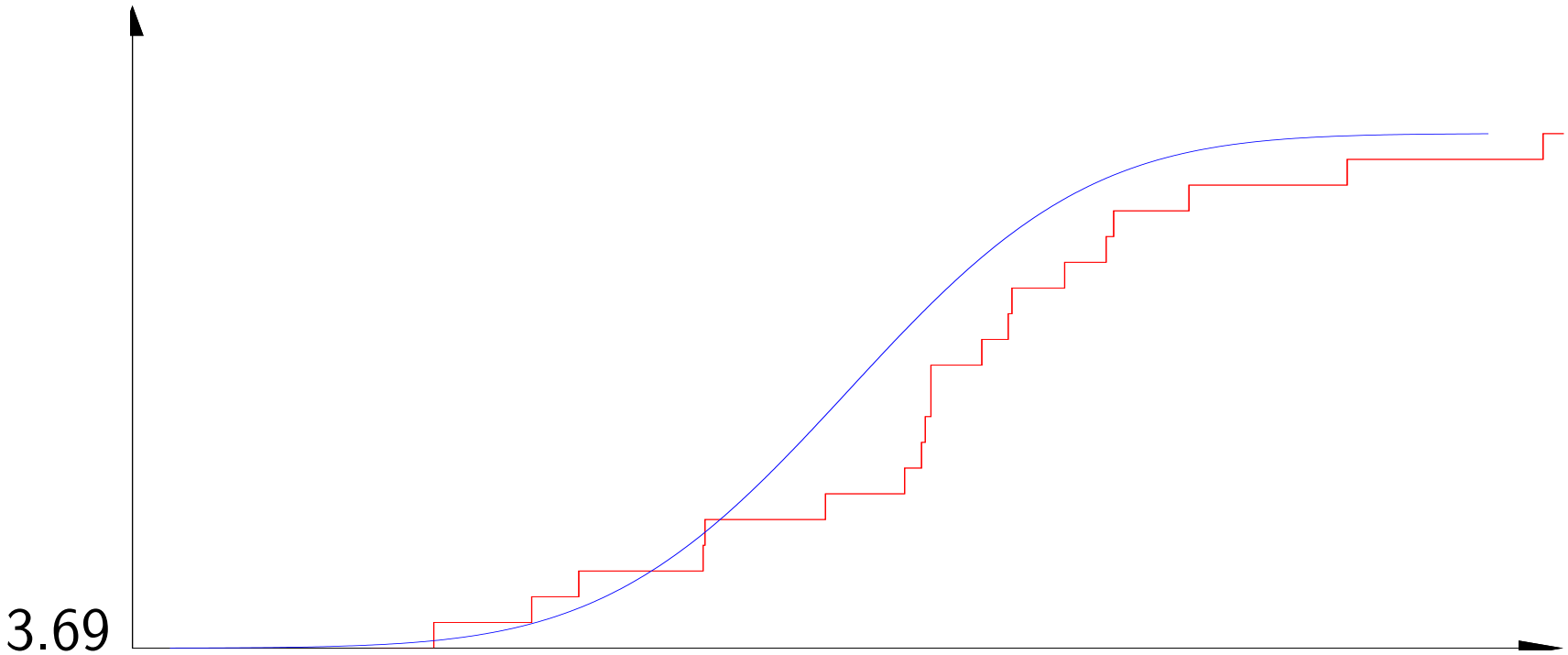

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- Generalizes e.g. to confidence intervals.

Exercise 8

1. Exercise 13 in Chapter 8 of Ross (P.152).
2. Exercise 15 in Chapter 8 of Ross (P.152).
3. Write a subroutine that takes as input a "data" vector of observed values, and which outputs the median as well as the bootstrap estimate of the variance of the median, based on $r = 100$ bootstrap replicates. Simulate $N = 200$ Pareto distributed random variates with $\beta = 1$ and $k = 1.05$.
   (a) Compute the mean and the median (of the sample)
   (b) Make the bootstrap estimate of the variance of the sample mean.
   (c) Make the bootstrap estimate of the variance of the sample median.
   (d) Compare the precision of the estimated median with the precision of the estimated mean.