Exercise 6: Markov Chain Monte Carlo

1. The number of busy lines in a trunk group (Erlang system) is given by a truncated Poisson distribution

$$P(i) = c \cdot \frac{A^i}{i!}, \quad i = 0, \ldots m$$

   Generate values from this distribution by applying the Metropolis-Hastings algorithm, verify with a χ^2-test. You can use the parameter values from exercise 4.

2. For two different call types the joint number of occupied lines is given by

$$P(i,j) = c \cdot \frac{A_1^i}{i!} \frac{A_2^j}{j!} \qquad 0 \leq i+j \leq m$$

   You can use $A_1, A_2 = 4$ and $m = 10$.

   (a) Use Metropolis-Hastings, directly to generate variates from this distribution.

(b) Use Metropolis-Hastings, coordinate wise to generate variates from this distribution.

(c) Use Gibbs sampling to sample from the distribution. This is (also) coordinate-wise but here we use the exact conditional distributions. You will need to find the conditional distributions analytically.

In all three cases test the distribution fit with a χ^2 test

The system can be extended to an arbitrary dimension, and we can add restrictions on the different call types.

3. We consider a Bayesian statistical problem. The observations are $X_i \sim \mathsf{N}\left(\Theta, \Psi\right)$, where the prior distribution of the pair $\left(\Xi, \Gamma\right) = \left(\log\left(\Theta\right), \log\left(\Psi\right)\right)$ is standard normal with correlation

$\rho = \frac{1}{2}$. The joint density $f(x, y)$ of (Θ, Ψ) is

$$f(x, y) = \frac{1}{2\pi xy\sqrt{1-\rho^2}} e^{-\frac{\log(x)^2 - 2\rho \log(x)\log(y) + \log(y)^2}{2(1-\rho^2)}}$$

which can be derived using a standard change of variable technique. The task of this exercise is now to sample from the posterior distribution of (Θ, Ψ) using Markov Chain Monte Carlo.

(a) Generate a pair (θ, ψ) from the prior distribution, i.e. the distribution for the pair (Θ, Ψ), by first generating a sample (ξ, γ) of (Ξ, Γ).

(b) Generate $X_i = 1, \ldots, n$ with the values of (θ, ψ) you obtained in item 3a. Use $n = 10$.

(c) Derive the posterior distribution of (Θ, Ψ) given the sample. **Hint 1** Apply Bayes theorem in the density version.

Hint 2 The sample mean and sample variance are independent. The sample mean follows a normal distribution, while a scaled version of the sample variance follows a χ^2 distribution. This can be used to simplify the expression.

(d) Generate MCMC samples from the posterior distribution of (Θ, Ψ) using the Metropolis Hastings method.

(e) Repeat item 3d with $n = 100$ and $n = 1000$, still using the values of (θ, ψ) from item 3a. Discuss the results.