Renewal processes
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Renewal processes

Phase type renewal process

For a Poisson process we have $Y_i \sim \exp(\lambda)$ or $Y_i \sim \mathrm{PH}((1), ||-\lambda||)$, where Y_i are (independent) interarrival times - distances between points.

Alternatively we can think of a process generated by a sequence $Y_i \sim \mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{S})$. In principle each Y_i has its own underlying Markov Jump process, however, they all have the "same" state space. So we can construct a concatenated Markov Jump process, by "gluing" together the individual processes over absorption points. We define a new Markov jump process $X(t)$

$$\begin{aligned} W_0 &= 0 \\ W_n &= \sum_{i=1}^{n} Y_i \\ X(t) &= \begin{cases} X_1(t) & \text{for} & t < Y_1 \\ X_2(t) & \text{for} & W_1 \leq t < W_2 \\ \vdots & \vdots & \vdots \\ X_n(t) & W_{n-1} \leq t < W_n \end{cases} \end{aligned}$$

$$\mathbb{P}\{X(t+h) = j | X(t) = i\} = S_{ij}h + s_i h \alpha_j + o(h)$$

We recognise the term $s_i h \alpha_j$ from the expression for the generator for a randum sum, and as a special case from the expression for a sum of two independent PH random variables.

We have a new Markov jump process with infinitessimal generator $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}$$

As we do not allow for more than one point of a time we must have $\boldsymbol{\alpha e} = 1$ ($\alpha_r = 0$ - impossibility of starting in an absorbing state) If $Y_i \sim \exp(\lambda)$ we have

$$\begin{aligned} N(t) &= \max_{n \in \mathbb{N} \geq 0} \{W_n \leq t\} \\ \mathbb{P}\{N(t) = n\} &= \frac{(\lambda t)^n}{n!} e^{-\lambda t} \end{aligned}$$

What if $Y_i \sim \text{PH}(\boldsymbol{\alpha}, \boldsymbol{S})$?

$$\boldsymbol{S}_n = \left\| \begin{array}{cccccc} \boldsymbol{S} & \boldsymbol{s\alpha} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S} & \boldsymbol{s\alpha} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{S} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots\ddots\ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{S} & \boldsymbol{s\alpha} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{S} \end{array} \right\|$$

what we could call a quasi birth process. We have $W_n \sim \text{PH}((\boldsymbol{\alpha}, \boldsymbol{0}, \dots, \boldsymbol{0}), \boldsymbol{S}_n)$, so $\mathbb{P}\{N(t) \geq t\} = \mathbb{P}\{W_n \leq t\}$, which we can calculate numerically.

For the Poisson process we have

$$\mathbb{E}(N(t)) = \lambda t = \lambda \int_0^t \mathrm{d}u = \int_0^t \lambda \mathrm{d}u$$

the integral over the intensity of having a point at all specific time points. We similarly first calculate the intensity (probability) of having a point at some specific time point t $\mathbb{P}\{\exists n : W_n \in [t; t + \mathrm{d}t[\}$ The probability of having a point in $[t; t + \mathrm{d}t[$ is the probability that $X(t)$ has a transition via the absorbing state (that $X(t)$ shifts from some $X_n(t)$ to some $X_{n+1}(t)$.

$$\begin{aligned} \mathbb{P}\{N(t + \mathrm{d}t) - N(t) = 1 | X(t) = i\} &= s_i \mathrm{d}t + o(\mathrm{d}t) \\ \mathbb{P}\{X(t) = i\} &= \boldsymbol{\alpha} e^{\boldsymbol{A}t} \boldsymbol{e}_i = \boldsymbol{\alpha} e^{(\boldsymbol{S} + \boldsymbol{s\alpha})t} \boldsymbol{e}_i \end{aligned}$$

where $\boldsymbol{e}_i$ is a column vector with 1 in the ith position and 0s elsewhere.

$$\mathbb{P}\{N(t + \mathrm{d}t) - N(t) = 1\} = \boldsymbol{\alpha} e^{(\boldsymbol{S} + \boldsymbol{s\alpha})t} \boldsymbol{s} \mathrm{d}t + o(\mathrm{d}t)$$

So

$$\mathbb{E}(N(t)) = \int_0^t \boldsymbol{\alpha} e^{\boldsymbol{A}u} \boldsymbol{s} \mathrm{d}u = \boldsymbol{\alpha} \int_0^t e^{\boldsymbol{A}u} \mathrm{d}u \boldsymbol{s}$$

Now $\boldsymbol{Ae} = \boldsymbol{0}$, as $\boldsymbol{A}$ has 0 as an eigenvalue, it is singular. First we note that $\boldsymbol{A}$ can be assumed to be irreducible without loss of generality, as otherwise there would be phases/states that are never visited, so the eigenvalue 0 has multiplicity 1 with left and right eigenvectors $\boldsymbol{\alpha}$ and $\boldsymbol{e}$. The matrix $\boldsymbol{e\pi} - \boldsymbol{A}$ has eigenvalue 1 associated with the pair $(\boldsymbol{\alpha}, \boldsymbol{e})$ all other eigenvectors and eigenvalues of of $\boldsymbol{A}$ is kept due to orthogonality of the eigenvectors, so this matrix is invertible. We can write

$$
\begin{aligned}
\mathbb{E}(N(t)) &= \boldsymbol{\alpha}\int_0^t e^{\boldsymbol{A}u}\mathrm{d}u\boldsymbol{s} = \boldsymbol{\alpha}\int_0^t (\boldsymbol{e\pi}-\boldsymbol{A})^{-1}(\boldsymbol{e\pi}-\boldsymbol{A})e^{\boldsymbol{A}u}\mathrm{d}u\boldsymbol{s} = \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\int_0^t(\boldsymbol{e\pi}-\boldsymbol{A})e^{\boldsymbol{A}u}\mathrm{d}u\boldsymbol{s}\\
e^{\boldsymbol{A}u} &= \sum_{k=0}^{\infty}\frac{(\boldsymbol{A}u)^k}{k!}\\
\boldsymbol{\pi}e^{\boldsymbol{A}u} &= = \sum_{k=0}^{\infty}\boldsymbol{\pi A}^k\frac{u^k}{k!} = \boldsymbol{\pi I} + \sum_{k=1}^{\infty}\boldsymbol{\pi A}^k\frac{u^k}{k!} = \boldsymbol{\pi} + \sum_{k=1}^{\infty}\boldsymbol{\pi A}^k\frac{u^k}{k!} = \boldsymbol{\pi}\\
\mathbb{E}(N(t)) &= \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\int_0^t(\boldsymbol{e\pi}-\boldsymbol{A})e^{\boldsymbol{A}u}\mathrm{d}u\boldsymbol{s}\\
&= \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\boldsymbol{e\pi s}t - \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\int_0^t \boldsymbol{A}e^{\boldsymbol{A}u}\mathrm{d}u\boldsymbol{s}\\
(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\boldsymbol{e} &= \boldsymbol{e}, \quad \boldsymbol{\alpha e} = 1\\
\mathbb{E}(N(t)) &= \boldsymbol{\pi s}t - \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}(e^{\boldsymbol{A}t}-\boldsymbol{I})\boldsymbol{s}\\
\boldsymbol{\pi s}\mathbb{E}(Y_i) &= 1, \quad (\boldsymbol{\pi s})^{-1} = \mathbb{E}(Y_i) = \mu\\
\mathbb{E}(N(t)) &= \frac{t}{\mu} + \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\boldsymbol{s} - \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}e^{\boldsymbol{A}t}\boldsymbol{s}\\
\boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}e^{\boldsymbol{A}t}\boldsymbol{s} &\overset{t\to\infty}{\to} \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\boldsymbol{e\pi s} = \boldsymbol{\pi s} = \mu^{-1}\\
\mathbb{E}(N(t)) - \frac{t}{\mu} &\overset{t\to\infty}{\to} \boldsymbol{\alpha}(\boldsymbol{e\pi}-\boldsymbol{A})^{-1}\boldsymbol{s} - \mu^{-1}
\end{aligned}
$$

What can be said in the general case where $\mathbb{P}\{Y_i \leq\}y = F(y)$, Y_i independent?

Not so much in fact!

$$
\begin{aligned}
\mathbb{P}\{N(t)\geq n\} &= \mathbb{P}\{W_n \leq t\}\\
\mathbb{E}(N(t)) &= \sum_{n=0}^{\infty} n\mathbb{P}\{N(t)=n\} = \sum_{n=1}^{\infty} n\mathbb{P}\{N(t)=n\} = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}1\right)\mathbb{P}\{N(t)=n\} = \sum_{n=1}^{\infty}\sum_{i=1}^{n}\mathbb{P}\{N(t)=n\}\\
&= \sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\mathbb{P}\{N(t)=n\} = \sum_{i=1}^{\infty}\mathbb{P}\{N(t)\geq i\} = \sum_{i=1}^{\infty}\mathbb{P}\{W_i\leq t\} = \sum_{i=1}^{\infty}F_n(t) = M(t)
\end{aligned}
$$

with $M(t)$ being the renewal function and $F_n(t)$ being the distribution function of the sum of n indepdent F distributed random variables. At time t the last (previous) point occured at time $W_{N(t)}$, the next point will occur at time $W_{N(t)}+1$

$$
W_{N(t)+1} = \sum_{i=1}^{N(t)+1} Y_i = Y_1 + \sum_{i=2}^{N(t)+1} Y_i
$$

the sum might be empty (if the next point is the first point, i.e. no points have yet occurred)

$$
\begin{aligned}
W_{N(t)+1} &= = Y_1 + \sum_{i=2}^{N(t)+1} Y_i = Y_i + \sum_{i=2}^{\infty} Y_i 1\{N(t)+1 \geq i\} = Y_i + \sum_{i=2}^{\infty} Y_i 1\{N(t) \geq i-1\} \\
&= Y_i + \sum_{i=2}^{\infty} Y_i 1 \left\{ \sum_{j=1}^{i-1} Y_j \leq t \right\} \\
\mathbb{E}(W_{N(t)+1}) &= \mathbb{E}\left[Y_i + \sum_{i=2}^{\infty} Y_i 1 \left\{ \sum_{j=1}^{i-1} Y_j \leq t \right\} \right] = \mathbb{E}(Y_i) + \mathbb{E}\left[\sum_{i=2}^{\infty} Y_i 1 \left\{ \sum_{j=1}^{i-1} Y_j \leq t \right\} \right] \\
&= \mathbb{E}(Y_i) + \sum_{i=2}^{\infty} \mathbb{E}\left[Y_i 1 \left\{ \sum_{j=1}^{i-1} Y_j \leq t \right\} \right] = \mathbb{E}(Y_i) + \sum_{i=2}^{\infty} \mathbb{E}(Y_i)\mathbb{E}\left[1 \left\{ \sum_{j=1}^{i-1} Y_j \leq t \right\} \right] \\
&= \mathbb{E}(Y_i) + \mathbb{E}(Y_i) \sum_{i=2} \mathbb{P}\left\{ \sum_{j=1}^{i-1} Y_j \leq t \right\} = \mathbb{E}(Y_i) + \mathbb{E}(Y_i) \sum_{k=1} \mathbb{P}\left\{ \sum_{j=1}^{k} Y_j \leq t \right\} \\
W_{N(t)+1} &= \mathbb{E}(Y_i)(1 + M(t)) = \mu(1 + M(t))
\end{aligned}
$$

For PH we have immediately

$$
\begin{aligned}
M(t) - \frac{t}{\mu} - a &\overset{t\to\infty}{\to} 0 \\
\frac{M(t)}{t} &\overset{t\to\infty}{\to} \mu^{-1}
\end{aligned}
$$

It is surprisingly hard (like two pages) to prove this in the general case, see e.g. Bladt& Nielsen if you need to see a proof. Actually it is easier to prove

$$
\begin{aligned}
\frac{N(t)}{t} &\overset{t\to\infty}{\to} \mu^{-1} \quad \text{in probability} \\
\frac{N(t)}{t} &= \frac{N(t)}{W_{N(t)} + (t - W_{N(t)})} \frac{N(t)}{W_{N(t)}} \frac{W_{N(t)}}{W_{N(t)} + (t - W_{N(t)})} = \frac{N(t)}{W_{N(t)}} \frac{1}{1 + \frac{t - W_{N(t)}}{W_{N(t)}}} \\
&= \left(\frac{W_{N(t)}}{N(t)} \right)^{-1} \frac{1}{1 + \frac{t - W_{N(t)}}{W_{N(t)}}} \overset{t\to\infty}{\to} \mu^{-1} \\
M(t) &= \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o\left(\frac{1}{t}\right) \\
\mathbb{P}\left(\frac{M(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \leq x \right) &\overset{t\to\infty}{\to} \Phi(x)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_t &= W_{N(t)+1} - t, \quad \text{(residual/excess life time)} \\
\delta_t &= t - W_{N(t)}, \quad \text{(age)}, = t \text{ if } N(t) = 0 \\
\beta_t &= W_{N(t)+1} - W_{N(t)}, \quad \text{(total life time/spread}
\end{aligned}
$$

Residual life time for PH $\gamma_t \sim \text{PH}\left(\boldsymbol{\alpha} e^{\boldsymbol{A}t}, \boldsymbol{S}\right)$ asymptotic distribution of γ_t - distribution of γ_∞ - $\gamma_\infty \sim \text{PH}(\boldsymbol{\pi}, \boldsymbol{S})$ (with $\boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{0}$) What can be said in the general case: Bus example $\mathbb{P}(\beta_\infty \in [x, x + \mathrm{d}x[) \cong x f(x) \mathrm{d}x$

$$
\begin{aligned}
f_1(x) &= \frac{x f(x)}{\mathbb{E}(X_i)} = \frac{x f(x)}{\mu}, \quad \text{first order moment distribution} \\
\mathbb{P}\{\beta_\infty \le x\} &= \frac{\int_0^x u f(u) \mathrm{d}u}{\mu} \\
f_j(x) &= \frac{x^j f(x)}{\mathbb{E}(X_i^j)}, \quad j\text{th order moment distribution} \\
\mathbb{P}(\gamma_\infty \le x) &= \frac{\int_0^x (1 - F(t)) \mathrm{d}t}{\mathbb{E}(X_i)} = \frac{\int_0^x (1 - F(t)) \mathrm{d}t}{\mu}
\end{aligned}
$$

Some examples

$$
\begin{aligned}
F(x) &= 1 - e^{-\lambda x} \\
\mathbb{P}(\gamma_\infty \le x) &= \int_0^x \frac{e^{-\lambda t}}{\frac{1}{\frac{1}{\lambda}}} \mathrm{d}t = 1 - e^{-\lambda x} \\
\gamma_\infty &\sim \exp(\lambda) \\
\mathbb{E}(\gamma_\infty) &= \frac{1}{\lambda} = \mathbb{E}(X_i) \\
f(x) &= \lambda (\lambda x) e^{-\lambda x} \\
F(x) &= 1 - e^{-\lambda x} - (\lambda x) e^{-\lambda x} \\
\mathbb{P}(\gamma_\infty \le x) &= \frac{\lambda}{2} \int_0^x \left(e^{-\lambda t} - (\lambda t) e^{-\lambda t}\right) \mathrm{d}t = \frac{1}{2} \int_0^x \lambda e^{-\lambda t} \mathrm{d}t + \frac{1}{2} \int_0^x \lambda (\lambda t) e^{-\lambda t} \mathrm{d}t \\
&= 1 - e^{-\lambda x} - \frac{1}{2} (\lambda x) e^{-\lambda x} \\
\mathbb{E}(\gamma_\infty) &= \frac{1}{2}\frac{1}{\lambda} + \frac{1}{2}\frac{1}{\lambda} = \frac{3}{2\lambda} < \frac{2}{\lambda} = \mathbb{E}(X_i)
\end{aligned}
$$

Additionally we have $X_i \sim \text{PH}\left((1,0), \left\| \begin{array}{cc} -\lambda & \lambda \\ 0 & -\lambda \end{array} \right\|\right)$ so $\boldsymbol{A} = \left\| \begin{array}{cc} -\lambda & \lambda \\ \lambda & -\lambda \end{array} \right\|$ and $\boldsymbol{\pi} = \left(\frac{1}{2}, \frac{1}{2}\right)$ so

$$\gamma_\infty \sim \mathrm{PH}\left(\left(\tfrac{1}{2},\tfrac{1}{2}\right), \left|\left|\begin{array}{cc} -\lambda & \lambda \\ 0 & -\lambda \end{array}\right|\right|\right)$$

$$\begin{aligned}
f(x) &= \alpha_1\lambda_1 e^{-\lambda_1 x} + \alpha_2\lambda_2 e^{-\lambda_2 x} \\
X_i &\sim \mathrm{PH}\left((\alpha_1,\alpha_2), \left|\left|\begin{array}{cc} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{array}\right|\right|\right) \\
\boldsymbol{A} &= \left|\left|\begin{array}{cc} -\lambda_1\alpha_2 & \lambda_1\alpha_2 \\ \lambda_2\alpha_1 & -\lambda_2\alpha_1 \end{array}\right|\right| \\
\boldsymbol{\pi} &= \left(\frac{\frac{\alpha_1}{\lambda_1}}{\frac{\alpha_1}{\lambda_1}+\frac{\alpha_2}{\lambda_2}}, \frac{\frac{\alpha_2}{\lambda_2}}{\frac{\alpha_1}{\lambda_1}+\frac{\alpha_2}{\lambda_2}}\right) \\
\gamma_\infty &\sim \mathrm{PH}\left(\left(\frac{\frac{\alpha_1}{\lambda_1}}{\frac{\alpha_1}{\lambda_1}+\frac{\alpha_2}{\lambda_2}}, \frac{\frac{\alpha_2}{\lambda_2}}{\frac{\alpha_1}{\lambda_1}+\frac{\alpha_2}{\lambda_2}}\right), \left|\left|\begin{array}{cc} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{array}\right|\right|\right) \\
\mathbb{E}(\gamma_\infty) &= \frac{\frac{\alpha_1}{\lambda_1}}{\frac{\alpha_1}{\lambda_1}+\frac{\alpha_2}{\lambda_2}}\frac{1}{\lambda_1} + \frac{\frac{\alpha_2}{\lambda_2}}{\frac{\alpha_1}{\lambda_1}+\frac{\alpha_2}{\lambda_2}}\frac{1}{\lambda_2} = \frac{\alpha_1}{\alpha_1+\alpha_2\frac{\lambda_1}{\lambda_2}}\frac{1}{\lambda_1} + \frac{\alpha_2}{\alpha_1\frac{\lambda_2}{\lambda_1}+\alpha_2}\frac{1}{\lambda_2} > \frac{\alpha_1}{\lambda_1}+\frac{\alpha_2}{\lambda_2} = \mathbb{E}(X_i)
\end{aligned}$$

Modified renewal process. For the phase type renewal process the initial distribution among the states could be given by some other probability distributin e.g. $\boldsymbol{\beta}$, more generally the first interval Y_i could have another distribution than the rest. Such a process is called a modified or delayed renewal process.

In the special case where Y_1 has the same distribution as γ_∞ the process is called a stationary (equilibrium) renewal process. For the PH renewal process this corresponds to initiating the Markov jump process $X(t)$ with $\boldsymbol{\pi}$, $\mathbb{P}\{X(0)=i\}=\pi_i$.

For a stationary renewal process we have $M(t)=\frac{t}{\mu}$ and $\mathbb{P}\{\gamma_t \le x\} = \frac{\int_0^x(1-F(u))\mathrm{d}u}{\mu}$ independent of t.

Joint distribution of δ_∞ and γ_∞

$\{\gamma_t \ge x \wedge \delta_t \ge y\} = \{\gamma_{t-y} \ge x+y\}$ so

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{P}\{\gamma_t \ge x, \delta_t \ge y\} &= \lim_{t\to\infty}\mathbb{P}\{\gamma_{t-y} \ge x+y\} = \frac{\int_{x+y}^\infty(1-F(u))\mathrm{d}u}{\mu} \\
f_{\gamma_\infty,\delta_\infty} &= \frac{f(x+y)}{\mu}, \quad \text{if } F \text{ has a density } f
\end{aligned}$$