Geometric Brownian Motion and Ornstein Uhlenbeck
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BFN/bfn

Geometric Brownian motion

$$
\begin{aligned}
X(t) &= X(0) + \mu t + \sigma B(t) \\
Z(t) &= \exp^{X(t)} = z \exp^{\mu t + \sigma B(t)} \\
\mathbb{E}(Z(t)) &= z e^{\mu t} \mathbb{E}\left(e^{\sigma B(t)}\right) \\
\mathbb{E}\left(e^{\sigma B(t)}\right) &= \int_{-\infty}^{\infty} e^{\sigma x} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{x^2}{2t}} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\sigma t x}{2t}} \mathrm{d}x = e^{\frac{1}{2}\sigma^2 t} \int_{-\infty}^{\infty} e^{-\frac{(x-\sigma t)^2}{2t}} \mathrm{d}x = e^{\frac{1}{2}\sigma^2 t} \\
\mathbb{E}(Z(t)) &= z e^{(\mu + \frac{1}{2}\sigma^2)t} \quad \left(z e^{\alpha t}\right)
\end{aligned}
$$

Introduce $\alpha = \mu + \frac{1}{2}\sigma^2$, (so $\mu = \alpha - \frac{1}{2}\sigma^2$)

$$
\begin{aligned}
X(t) &= X(0) + \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B(t) \\
Z(t) &= z e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B(t)} \\
\mathbb{V}\mathrm{ar}(Z(t)) &= z^2 e^{2\alpha} \left(e^{\sigma^2 t} - 1\right)
\end{aligned}
$$

Like N and LN, B and GM, i.e. Theorem 8.3

$$
\begin{aligned}
u(x) &= \mathbb{P}\{X(\tau) = b | X(0) = x\} = \mathbb{E}(\mathbf{1}_{X(\tau)=b} | X(0) = x) \\
u(x) &= \frac{e^{-\frac{2\mu}{\sigma^2}x} - e^{-\frac{2\mu}{\sigma^2}a}}{e^{-\frac{2\mu}{\sigma^2}b} - e^{-\frac{2\mu}{\sigma^2}a}} \\
T &= \min_{t \geq 0} \{Z(t) \in \{Z(0)A, Z(0)B\}\} \\
Z(t) &= Z(0)A \\
\log(Z(t)) &= \log(Z(0)) + \log(A) \\
\mathbb{P}\left\{\frac{Z(T)}{Z(0)} = B\right\} &= \frac{1 - A^{-\frac{2\alpha}{\sigma^2}}}{B^{-\frac{2\alpha}{\sigma^2}} - A^{-\frac{2\alpha}{\sigma^2}}}
\end{aligned}
$$

Black Scholes model

Stock price given by

$$
S(t) = S(0) e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B(t)}
$$

An option gives the right to buy a stock at or before (fixed) time τ at some fixed price a. The Black Scholes model gives the fair price $F(z,\tau)$ of the option under som idealistic assumptions. Here z is the current value of the stock and τ is the remaining time to expiration.

An important parameter is r the interest rate of a risk free financial instrument (think US treassuries - bonds)').

$$
\begin{aligned}
Z(\tau) &= S(0)e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}B(1)} \\
S(0) &= z \\
F(z,0) &= (z-a)^+ = \max(z-a,0) \\
F(Z,\tau) &= e^{-r\tau}\mathbb{E}\left[(Z(\tau)-a)^+|Z(0)=z\right], \quad \text{Black Scholes formula} \\
Z(\tau) &\geq a \Leftrightarrow ze^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}B(1)} \geq a \\
\log(z) + (r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}B(1) &\geq \log(a) \\
B(1) &\geq \frac{\log\left(\frac{a}{z}\right)-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = v_0 \\
\mathbb{E}\left[(Z(\tau)-a)^+|Z(0)=z\right] &= \int_{v_0}^{\infty}\left(ze^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}x}-a\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,\mathrm{d}x \\
&= ze^{(r-\frac{1}{2}\sigma^2)\tau}\int_{v_0}^{\infty}e^{\sigma\sqrt{\tau}x}\frac{1}{\sqrt{\tau}}e^{-\frac{x^2}{2}}\,\mathrm{d}x - a(1-\Phi(v_0)) \\
&= ze^{(r-\frac{1}{2}\sigma^2)\tau}e^{\frac{\sigma^2\tau}{2}}\int_{v_0}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma\sqrt{\tau})^2}{2}}\,\mathrm{d}x - a(1-\Phi(v_0)) \\
&= ze^{r\tau}\left(1-\Phi\left(v_0-\sigma\sqrt{\tau}\right)\right) - a(1-\Phi(v_0)) \\
&= ze^{r\tau}\left(\Phi(\sigma\sqrt{\tau}-v_0) - a\Phi(-v_0)\right) \\
&= ze^{r\tau}\Phi\left(\frac{\log\left(\frac{z}{a}\right)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - a\Phi\left(\frac{\log\left(\frac{z}{a}\right)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
F(z,\tau) &= z\Phi\left(\frac{\log\left(\frac{z}{a}\right)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - ae^{-r\tau}\Phi\left(\frac{\log\left(\frac{z}{a}\right)+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)
\end{aligned}
$$

Ornstein Uhlenbeck process

Pathwise/distributional definition

$$
\begin{aligned}
V(t) &= ve^{-\beta t} + \frac{e^{-\beta t}\sigma}{\sqrt{2\beta}} B\left(e^{2\beta t} - 1\right) \\
\mathbb{E}(V(t)) &= ve^{-\beta t} \\
\mathbb{V}\mathrm{ar}(V(t)) &= \frac{e^{-2\beta t}\sigma^2}{2\beta}\left(e^{2\beta t} - 1\right) = \sigma^2 \frac{1 - e^{-2\beta t}}{2\beta} \\
\mathbb{P}(V(t) \leq x) &= \Phi\left(\frac{x - \mathbb{E}(V(t))}{\sqrt{\mathbb{V}\mathrm{ar}(V(t))}}\right) \\
\mathbb{C}\mathrm{ov}(V(u), V(s)) &= \mathbb{E}\left((V(u) - \mathbb{E}(V(u))(V(s) - \mathbb{E}(V(s))\right) = \mathbb{E}\left(\frac{e^{-\beta u}\sigma}{\sqrt{2\beta}} B\left(e^{2\beta u} - 1\right) \frac{e^{-\beta s}\sigma}{\sqrt{2\beta}} B\left(e^{2\beta s} - 1\right)\right) \\
&= \frac{\sigma^2 \exp\left(-\beta(u+s)\right)}{2\beta} \mathbb{E}\left(B\left(e^{2\beta u} - 1\right) B\left(e^{2\beta s} - 1\right)\right) \\
&= \frac{\sigma^2 \exp\left(-\beta(u+s)\right)}{2\beta}\left(e^{2\beta u} - 1\right) = \sigma^2\left(\frac{\exp(-\beta(s-u)) - \exp(-\beta(s+u))}{2\beta}\right) \\
\mathbb{E}(V(\Delta t)) &= ve^{-\beta \Delta t} = v(1 - \beta\Delta t + o(\Delta t)) \\
\mathbb{E}(V(\Delta t) - V(0)) &= v(1 - \beta\Delta t + o(\Delta t)) - v = -\beta v \Delta t + 0(\Delta t) \\
\mathbb{V}\mathrm{ar}(V(\Delta t)) &= \sigma^2 \frac{1 - e^{-2\beta\Delta t}}{2\beta} = \sigma^2 \frac{2\beta\Delta t + o(\Delta t)}{2\beta} = \sigma^2 \Delta t + o(\Delta t)
\end{aligned}
$$

Position process

Model of velocity rather than position. Then position

$$
S(t) = S(0) + \int_0^t V(t)\mathrm{d}t
$$

As $V(t)$ is continuous the integral is well-defined. For $S(0) = V(0) = 0$ the variance of $S(t)$ is calculated to be

$$
\mathbb{V}\mathrm{ar}(S(t)) = \frac{\sigma^2}{\beta^2}\left[t - \frac{2}{\beta}\left(1 - e^{-\beta t}\right) + \frac{1}{2\beta}\left(1 - e^{-2\beta t}\right)\right]
$$

Limiting/stationary distribution

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{E}(V(t)) &= \lim_{t\to\infty} ve^{-\beta t} = 0 \\
\lim_{t\to\infty} \mathbb{V}\mathrm{ar}(V(t)) &= \lim_{t\to\infty} \sigma^2 \frac{1-e^{-2\beta t}}{2\beta} = \frac{\sigma^2}{2\beta} \\
\lim_{t\to\infty} \mathbb{P}(V(t)\le y) &= \Phi\left(\frac{y\sqrt{2\beta}}{\sigma}\right) \\
V^s(t) &= \sigma \frac{e^{-\beta t}}{\sqrt{2\beta}} B\left(e^{2\beta t}\right) \\
\mathbb{C}\mathrm{ov}(V^s(s), V^s(t)) &= \frac{e^{-\beta(t+s)}}{2\beta} \min\left(e^{2\beta t}, e^{2\beta t}\right) = \frac{e^{-\beta|t-s|}}{2\beta}
\end{aligned}
$$

We define a bivariate normal vector $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$
\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right).
$$

With

$$
\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}
$$

we can express the joint density as

$$
\begin{aligned}
f(x_1, x_2) &= \frac{1}{2\pi\sqrt{\det(\boldsymbol{\Sigma})}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)}}
\end{aligned}
$$

we have the conditional density

$$
f_{X_2|X_1=x_1}(x_2) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}}} = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} e^{-\frac{(x_2-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1))^2}{2\sigma_2^2(1-\rho^2)}}
$$

to get

$$
\mathbb{E}\left(X_2|X_1=x_1\right) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1), \quad \mathbb{E}\left(X_2|X_1\right) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X_1-\mu_1)
$$

or

$$\mathbb{E}\left(X_2-\mu_2|X_1=x_1\right)=\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1),\quad \mathbb{E}\left(X_2-\mu_2|X_1\right)=\rho\frac{\sigma_2}{\sigma_1}(X_1-\mu_1).$$

To ease notation and with no real loss of generality we assume $\mu_1=\mu_2=0$

$$\mathbb{E}\left(X_2|X_1=x_1\right)=\rho\frac{\sigma_2}{\sigma_1}x_1,\quad \mathbb{E}\left(X_2-\mu_2|X_1\right)=\rho\frac{\sigma_2}{\sigma_1}X_1$$

and we rewrite to get

$$\sigma_1^2\mathbb{E}\left(X_2|X_1=x_1\right)=\rho\sigma_1\sigma_2x_1,\quad \sigma_1^2\mathbb{E}\left(X_2|X_1\right)=\rho\sigma_1\sigma_2X_1$$

to finally obtain

$$\sigma_1^2\mathbb{E}\left(X_2|X_1\right)=\mathbb{C}\text{ov}\left(X_1,X_2\right)X_1.$$

We now want to construct a one-dimensional Gaussian process where this relation holds for any pair $(X(t_1),X(t_2))$, so we assume that the covariance function $\Gamma(t_1,t_2)=\mathbb{C}\text{ov}((X(t_1),X(t_2))$ is time homogenous such that $\mathbb{C}\text{ov}((X(t_1),X(t_2))=\Gamma(t_2-t_1)$. Our assumption amounts to $\sigma^2\mathbb{E}\left(X(t)|X(0)\right)=\Gamma(t)X(0)$ or $\Gamma(0)\mathbb{E}\left(X(t)|X(0)\right)=\Gamma(t)X(0)$ We now evaluate $\Gamma(t_1+t_2)$.

$$\begin{aligned}\Gamma(t_1+t_2) &= \mathbb{E}[X(0)X(t_1+t_2)]=\mathbb{E}\left[\mathbb{E}(X(0)X(t_1+t_2)|X(0),X(t_1))\right]\\ &= \mathbb{E}\left[X(0)\mathbb{E}(X(t_1+t_2)|X(0),X(t_1))\right]=\mathbb{E}\left[X(0)\mathbb{E}(X(t_1+t_2)|X(t_1))\right]=\mathbb{E}\left[X(0)\frac{1}{\sigma^2}\Gamma(t_2)X(t_1)\right]\\ &= \frac{1}{\sigma^2}\Gamma(t_1)\Gamma(t_2),\end{aligned}$$

and get the functional equation

$$\sigma^2\Gamma(t_1+t_2)=\Gamma(t_1)\Gamma(t_2),$$

with solution

$$\Gamma(t)=\sigma^2e^{-\alpha|t|}$$

for some α. We have defined a stationary Gaussian process with $X(t)\sim \mathrm{N}(0,\sigma^2)$, i.e. $\mu(t)=0$ and covariance function $\Gamma(t)=\sigma^2e^{-\alpha|t|}$. This process is called the stationary Ornstein-Uhlenbeck process.

The approach is taken from [1] Section 9.6 Page 407.

References

[1] G. R. Grimmett and D. R. Stirzacker. *Probability and Random Processes.* Oxford University Press, third edition, 1995.