Brownian Motion with drift
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BFN/bfn

Reflected Brownian motion

$\{R(t); t \geq 0\}$

$$\begin{aligned}
R(t) &= |B(t)| \\
\mathbb{E}(R(t)) &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{x^2}{2t}} \mathrm{d}x = \sqrt{\frac{2t}{\pi}} \int_0^{\infty} \frac{x}{t} e^{-\frac{x^2}{2t}} \mathrm{d}x = \sqrt{\frac{2t}{\pi}} \\
\mathbb{E}\left(R(t)^2\right) &= \mathbb{E}\left(B(t)^2\right) = t \\
\mathbb{V}\mathrm{ar}(R(t)) &= t\left(1 - \frac{2}{\pi}\right) \\
\mathbb{P}\{R(t) \leq y | R(0) = x\} &= \mathbb{P}\{-y \leq B(t) \leq y | B(0) = x\} = \Phi\left(\frac{y-x}{\sqrt{t}}\right) - \Phi\left(\frac{-y-x}{\sqrt{t}}\right) \\
&= \Phi\left(\frac{y-x}{\sqrt{t}}\right) - \left(1 - \Phi\left(\frac{y+x}{\sqrt{t}}\right)\right) = \Phi\left(\frac{y+x}{\sqrt{t}}\right) + \Phi\left(\frac{y-x}{\sqrt{t}}\right) - 1
\end{aligned}$$

Absorbed Brownian motion

τ time to absorption or first passage time

$$\begin{aligned}
\tau &= \min\{t \geq 0 : B(t) = 0\} \\
A(t) &= B(t) 1\{t \leq \tau\} \\
G_t(x,y) &= \mathbb{P}\{A(t) > y | A(0) = x\} \\
\{\omega \in \Omega : B(t) > y\} &= \{w \in \Omega : B(t) > y, \tau > t\} \cup \{\omega \in \Omega : B(t) > y, \tau \leq t\} \\
\mathbb{P}\{B(t) > y, \tau \leq t | B(0) = x\} &\overset{RP}{=} \mathbb{P}\{B(t) < -y, \tau \leq t | B(0) = x\} \\
&= \mathbb{P}\{B(t) < -y | B(0) = x\} = \Phi\left(\frac{-y-x}{\sqrt{t}}\right) \\
\mathbb{P}\{B(t) > y, \min_{0 \leq u \leq t}\{B(u)\} \leq 0 | B(0) = x\} &\overset{RP}{=} \mathbb{P}\{B(t) < -y, \min_{0 \leq u \leq t}\{B(u)\} \leq 0 | B(0) = x\} \\
&= \mathbb{P}\{B(t) < -y | B(0) = x\} = \Phi\left(\frac{-y-x}{\sqrt{t}}\right) \\
G_t(x,y) &= 1 - \Phi\left(\frac{y-x}{\sqrt{t}}\right) - \Phi\left(\frac{-y-x}{\sqrt{t}}\right) = \Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right) \\
\mathbb{P}\{A(t) = 0 | A(0) = x\} &= \mathbb{P}\{\tau \leq t\} = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)
\end{aligned}$$

Or

$$
\begin{aligned}
\mathbb{P}\{A(t)=0|A(0)=x\} &= 1-\mathbb{P}\{A(t)>0|A(0)=x\} = 1-G_t(x,0) = 1-\left(1-\Phi\left(\frac{0-x}{\sqrt{t}}\right)-\Phi\left(\frac{-0-x}{\sqrt{t}}\right)\right) \\
&= 2\Phi\left(\frac{-x}{\sqrt{t}}\right) = 2\left(1-\Phi\left(\frac{x}{\sqrt{t}}\right)\right)
\end{aligned}
$$

Brownian Meander

$$
\begin{aligned}
\{B^+(t); t\geq 0\} & \\
\mathbb{P}\{B^+(t)>y|B^+(0)=x\} &= \mathbb{P}\{B(t)>y|\tau>t, B(0)=x\} \\
\mathbb{P}\{A|B\cap C\} &= \frac{\mathbb{P}\{A\cap B\cap C\}}{\mathbb{P}\{B\cap C\}} = \frac{\mathbb{P}\{A\cap B|C\}\mathbb{P}\{C\}}{\mathbb{P}\{B|C\}\mathbb{P}\{C\}} = \frac{\mathbb{P}\{A\cap B|C\}}{\mathbb{P}\{B|C\}} \\
\mathbb{P}\{B^+(t)>y|B^+(0)=x\} &= \mathbb{P}\{B(t)>y|\tau>t, B(0)=x\} = \frac{\mathbb{P}\{B(t)>y, \tau>t|B(0)=x\}}{\mathbb{P}\{\tau>t|B(0)=x\}} \\
&= \frac{\Phi\left(\frac{x+y}{\sqrt{t}}\right)-\Phi\left(\frac{y-x}{\sqrt{t}}\right)}{1-2\left(1-\Phi\left(\frac{x}{\sqrt{t}}\right)\right)} = \frac{\Phi\left(\frac{x+y}{\sqrt{t}}\right)-\Phi\left(\frac{y-x}{\sqrt{t}}\right)}{\Phi\left(\frac{x}{\sqrt{t}}\right)-\left(1-\Phi\left(\frac{x}{\sqrt{t}}\right)\right)} = \frac{\Phi\left(\frac{x+y}{\sqrt{t}}\right)-\Phi\left(\frac{y-x}{\sqrt{t}}\right)}{\Phi\left(\frac{x}{\sqrt{t}}\right)-\Phi\left(\frac{-x}{\sqrt{t}}\right)} \\
\lim_{x\to 0}\mathbb{P}\{B^+(t)>y|B^+(0)=0\} &= \lim_{x\to 0}\frac{\Phi\left(\frac{x+y}{\sqrt{t}}\right)-\Phi\left(\frac{y-x}{\sqrt{t}}\right)}{\Phi\left(\frac{x}{\sqrt{t}}\right)-\Phi\left(\frac{-x}{\sqrt{t}}\right)} = \frac{\phi\left(\frac{x+y}{\sqrt{t}}\right)+\phi\left(\frac{y-x}{\sqrt{t}}\right)}{\phi\left(\frac{x}{\sqrt{t}}\right)+\phi\left(\frac{-x}{\sqrt{t}}\right)} = \frac{2\phi\left(\frac{y}{\sqrt{t}}\right)}{2\phi(0)} = e^{-\frac{y^2}{2t}} \\
&\quad \text{(scaled Rayleigh)}
\end{aligned}
$$

Brownian Bridge

$$\begin{aligned}
&\{B^0(t); 0 \le t \le 1\} & & \\
\mathbb{P}\{B^0(t) \in A\} &= \mathbb{P}\{B(t) \in A | B(1) = 0\} \\
\boldsymbol{X} &\sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \quad n\text{-dimensional with } \boldsymbol{\Sigma} \text{ positive definite} \\
f_{\boldsymbol{X}}(\boldsymbol{x}) &= (2\pi)^{-\frac{n}{2}} \sqrt{\det(\boldsymbol{\Sigma})}^{-1} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})} \\
\mathbb{E}(B(t)) &= \mathbb{E}(B(1)) = 0 \\
\mathbb{C}\mathrm{ov}(B(t), B(1)) &= t \\
\mathbb{C}\mathrm{orr}(B(t), B(1)) &= \frac{t}{\sqrt{t} \cdot 1} = \sqrt{t} = \rho \\
(B(t), B(1)) &\sim \mathrm{N}\left(\left\| \begin{array}{c} 0 \\ 0 \end{array} \right\|, \left\| \begin{array}{cc} t & t \\ t & 1 \end{array} \right\| \right) \\
\det(\boldsymbol{\Sigma}) &= t - t^2 = t(1-t) \\
f(x,y) &= \frac{1}{2\pi} \frac{1}{\sqrt{t(1-t)}} e^{-\frac{\left\| \begin{array}{cc} x & y \end{array} \right\| \left\| \begin{array}{cc} 1 & -t \\ -t & t \end{array} \right\| \left\| \begin{array}{c} x \\ y \end{array} \right\|}{2t(1-t)}} = \frac{1}{2\pi} \frac{1}{\sqrt{t(1-t)}} e^{-\frac{x^2 - 2txy + ty^2}{2t(1-t)}} \\
f(x,0) &= \frac{1}{2\pi} \frac{1}{\sqrt{t(1-t)}} e^{-\frac{x^2}{2t(1-t)}} \\
f_{B(1)}(0) &= \frac{1}{\sqrt{2\pi}} \\
f_{B(t)|B(1)=0}(x) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t(1-t)}} e^{-\frac{x^2}{2t(1-t)}}
\end{aligned}$$

Similarly with some additional work

$$\begin{aligned}
\mathbb{E}(B(s)B(t)|B(1)=0) &= s(1-t) \\
\boldsymbol{\Sigma} &= \left\| \begin{array}{ccc} s & s & s \\ s & t & t \\ s & t & 1 \end{array} \right\| \\
\det(\boldsymbol{\Sigma}) &= s\left[\left(t - t^2\right) - (s - st) + (st - st)\right] = s(t-s)(1-t) \\
f(x,y,z) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{s(t-s)(1-t)}} e^{-\frac{\left\| \begin{array}{ccc} x & y & z \end{array} \right\| \left\| \begin{array}{ccc} t(1-t) & -s(1-t) & 0 \\ -s(1-t) & s(1-s) & -s(t-s) \\ 0 & -s(t-s) & s(t-s) \end{array} \right\| \left\| \begin{array}{c} x \\ y \\ z \end{array} \right\|}{2s(t-s)(1-t)}} \\
f(x,y,z) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{s(t-s)(1-t)}} e^{-\frac{1}{2}\left\| \begin{array}{ccc} x & y & z \end{array} \right\| \left\| \begin{array}{ccc} \frac{t}{s(t-s)} & -\frac{1}{t-s} & 0 \\ -\frac{1}{t-s} & \frac{1-s}{(t-s)(1-t)} & -\frac{1}{1-t} \\ 0 & -\frac{1}{1-t} & \frac{1}{1-t} \end{array} \right\| \left\| \begin{array}{c} x \\ y \\ z \end{array} \right\|} \\
f_{(B(s),B(t))|B(1)=0}(x,y) &= \sqrt{2\pi} f(x,y,0) = \frac{1}{2\pi} \frac{1}{\sqrt{s(t-s)(1-t)}} e^{-\frac{1}{2}\left\| \begin{array}{cc} x & y \end{array} \right\| \left\| \begin{array}{cc} \frac{t}{s(t-s)} & -\frac{1}{t-s} \\ -\frac{1}{t-s} & \frac{1-s}{(t-s)(1-t)} \end{array} \right\| \left\| \begin{array}{c} x \\ y \end{array} \right\|}
\end{aligned}$$

Brownian Bridge - empirical distribution function

$$\begin{aligned}
F_{e,n}(x) &= \sum_{i=1}^{n} 1_{X_i \leq x} \\
\mathbb{E}\left(F_{e,n}(x)\right) &= \mathbb{E}\left(\sum_{i=1}^{n} 1_{X_i \leq x}\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left(1_{X_i \leq x}\right) \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}\{X_i \leq x\} = \frac{1}{n}\sum_{i=1}^{n} F(x) = F(x) \\
X_i &\sim \text{Unif}(0;1) \\
X_n(t) &= \frac{\left(\sum_{i=1}^{n} 1_{X_i \leq x}\right) - nt}{\sqrt{n}} \\
\mathbb{E}(X_n(t)) &= 0, \quad \mathbb{V}\text{ar}(X_n(t)) = \frac{nt(1-t)}{n} = t(1-t)
\end{aligned}$$

Asymptotically $X_n(t)$ behaves like the Brownian Briduge. This extends to general $F(x)$ (Donsker's theorem) with application to the Kolmogorov Smirnov test

Brownian motion with drift

Like the usual transformation from standard normal to general normal

$$\begin{aligned}
X(t) &= \mu t + \sigma B(t) \\
p(y,t|x) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{t}} e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}} \\
\Delta X &= X(t+\Delta t) - X(t) \\
\Delta B &= B(t+\Delta t) - B(t) \\
\Delta X &= \mu(t+\Delta t) + \sigma B(t+\Delta) - \mu t - \sigma B(t) = \mu\Delta t + \sigma\Delta B \\
X(t+\Delta t) &= X(t) + \mu\Delta + \sigma\Delta B
\end{aligned}$$

Assumptions during Δt

$$\begin{aligned}
\mathbb{E}(\Delta X) &= \mu\Delta t \\
\mathbb{E}\left((\Delta X)^2\right) &= \mu^2(\Delta t)^2 + \sigma^2\Delta t \\
\mathbb{V}\text{ar}(\Delta X) &= \sigma^2\Delta t \\
\mathbb{E}\left((\Delta X)^x\right) &= o(\Delta t), \quad c > 2 \\
\mathbb{E}\left((\Delta X)^3\right) &= \mathbb{E}\left(((\Delta X - \Delta\mu t) + \Delta\mu t)^3\right) \\
&= \mathbb{E}\left((\Delta X - \mu\Delta t)^3\right) - 3\mu\Delta t\mathbb{E}\left((\Delta X - \mu\Delta t)^2\right) + 3(\mu\Delta t)^2\mathbb{E}(\Delta X - \mu\Delta t) - (\mu\Delta t)^3 \\
&= \mathbb{E}\left((\sigma\Delta B)^3\right) - 3\mu\Delta t\mathbb{E}\left((\sigma\Delta B)^2\right) + 3(\mu\Delta t)^2\mathbb{E}\left(\sigma\Delta B\right) - (\mu\Delta t)^3 = 0(\Delta t), \quad \text{etc.}
\end{aligned}$$

Absorption probabilities/first hitting times

$$\begin{aligned}\tau &= \min\{t \geq 0 : X(t) \in \{a, b\}\} \\ u(x) &= \mathbb{P}\{X(\tau) = b | X(0) = x\}\end{aligned}$$

can be solved by constructing an appropriate martingale. But here first step analysis with some assumptions.

Recall

$$\begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & q & r & p \\ 2 & 0 & 0 & 1 \end{array}$$

$$\begin{aligned}u &= \mathbb{P}\{X_\tau = 2 | X_0 = 1\} = \mathbb{E}\left(\mathbf{1}_{X_\tau = 2} | X_0 = 1\right) \\ u &= \mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_{X_\tau=2} | X_1, X_0 = 1\right)\right] = \mathbb{E}_{X_0=1}\left[\mathbb{E}\left(\mathbf{1}_{X_\tau=2} | X_1\right)\right] = q \cdot 0 + r \cdot u + p \cdot 1\end{aligned}$$

$$\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & p_{10} & p_{11} & p_{12} & p_{13} \\ 2 & p_{20} & p_{21} & p_{22} & p_{23} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\begin{aligned}u_i &= \mathbb{P}\{X_\tau = 3 | X_0 = i\} \\ u_i &= \mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_{X_\tau=3} | X_1, X_0 = i\right)\right] = \mathbb{E}_{X_0=i}\left[\mathbb{E}\left(\mathbf{1}_{X_\tau=3} | X_1\right)\right] \\ u(x) &= \mathbb{P}\{X(\tau) = b | X(0) = x\} = \mathbb{E}(\mathbf{1}_{X(\tau)=b} | X(0) = x) \\ u(x) &= \mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_{X(\tau)=b} | X(\Delta t), X(0) = x\right)\right] = \mathbb{E}_{X(0)=x}\left[\mathbb{E}\left(\mathbf{1}_{X(\tau)=b} | X(\Delta t) = x + \Delta X\right)\right] \\ &= \mathbb{E}\left(\mathbf{1}_{X(\tau)=b} | X(\Delta t) = x + \Delta X\right) = \mathbb{E}\left(\mathbb{P}\{X(\tau) = b | X(\Delta t) = x + \Delta X\}\right) \\ &= \mathbb{E}\left(u(x + \Delta X)\right)\end{aligned}$$

where the second last equation is due to the independent increment property of the process

Assume u twice differentiable

$$\begin{aligned}
u(x) &= \mathbb{E}\left(u(X+\Delta X)\right) = \mathbb{E}\left(u(x)+u'(x)\Delta X+\frac{1}{2}u''(x)(\Delta X)^2+o\left((\Delta X)^2\right)\right) \\
&= \mathbb{E}\left(u(X+\Delta X)\right) = \mathbb{E}\left(u(x)+u'(x)\Delta X+\frac{1}{2}u''(x)(\Delta X)^2+o(\Delta t)\right) \\
&= u(x)+u'(x)\mathbb{E}(\Delta X)+\frac{1}{2}u''(x)\mathbb{E}\left((\Delta X)^2\right)+o(\Delta t) = u(x)+u'(x)\mu\Delta t+\frac{1}{2}u''(x)\sigma^2\Delta t+o(\Delta t) \\
0 &= u'(x)\mu+\frac{1}{2}u''(x)\sigma^2 \\
u''(x) &= -\frac{2\mu}{\sigma^2}u'(x) \\
u(x) &= Ae^{-\frac{2\mu}{\sigma^2}x}+B \\
u(a) &= 0, \quad u(b)=1 \\
0 &= Ae^{-\frac{2\mu}{\sigma^2}a}+B \\
1 &= Ae^{-\frac{2\mu}{\sigma^2}b}+B \\
1 &= A\left(e^{-\frac{2\mu}{\sigma^2}b}-e^{-\frac{2\mu}{\sigma^2}a}\right) \\
u(x) &= \frac{e^{-\frac{2\mu}{\sigma^2}x}-e^{-\frac{2\mu}{\sigma^2}a}}{e^{-\frac{2\mu}{\sigma^2}b}-e^{-\frac{2\mu}{\sigma^2}a}}
\end{aligned}$$

$$\begin{aligned}
v(x) &= \mathbb{E}(\tau|X(0)=x) \\
v(x) &= \Delta t+\mathbb{E}(v(X+\Delta X)), \quad \text{etc.} \\
v(a) &= 0, \quad v(b)=0 \\
v(x) &= \frac{1}{\mu}(u(x)(b-a)-(x-a))
\end{aligned}$$

Back to $u(x)$

$$u(0) = \frac{1-e^{-\frac{2\mu}{\sigma^2}a}}{e^{-\frac{2\mu}{\sigma^2}b}-e^{-\frac{2\mu}{\sigma^2}a}}$$

For $\mu<0$

$$\begin{aligned}
u(0) &= \frac{1-e^{-\frac{2\mu}{\sigma^2}a}}{e^{-\frac{2\mu}{\sigma^2}b}-e^{-\frac{2\mu}{\sigma^2}a}} \\
e^{-\frac{2\mu}{\sigma^2}a} &\overset{a\to-\infty}{\to} 0 \\
u(0) &\overset{a\to-\infty}{\to} e^{2\frac{\mu}{\sigma^2}b} = e^{-2\frac{|\mu|}{\sigma^2}b}
\end{aligned}$$

This is the probability of ever reaching b, equivalently the distribution of the maximum of $X(t)$

$$\mathbb{P}\{\max_{0\geq t} X(t)>x\} = e^{-2\frac{|\mu|}{\sigma^2}x}$$

exponential

Geometric Brownian motion

$$
\begin{aligned}
X(t) &= X(0) + \mu t + \sigma B(t) \\
Z(t) &= \exp^{X(t)} = z \exp^{\mu t + \sigma B(t)} \\
\mathbb{E}(Z(t)) &= z e^{\mu t} \mathbb{E}\left(e^{\sigma B(t)}\right) \\
\mathbb{E}\left(e^{\sigma B(t)}\right) &= \int_{-\infty}^{\infty} e^{\sigma x} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{x^2}{2t}} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\sigma t x}{2t}} \mathrm{d}x = e^{\frac{1}{2}\sigma^2 t} \int_{-\infty}^{\infty} e^{-\frac{(x-\sigma t)^2}{2t}} \mathrm{d}x = e^{\frac{1}{2}\sigma^2 t} \\
\mathbb{E}(Z(t)) &= z e^{(\mu + \frac{1}{2}\sigma^2)t} \quad \left(z e^{\alpha t}\right)
\end{aligned}
$$

Introduce $\alpha = \mu + \frac{1}{2}\sigma^2$, (so $\mu = \alpha - \frac{1}{2}\sigma^2$)

$$
\begin{aligned}
X(t) &= X(0) + \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B(t) \\
Z(t) &= z e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B(t)} \\
\mathbb{V}\mathrm{ar}(Z(t)) &= z^2 e^{2\alpha} \left(e^{\sigma^2 t} - 1\right)
\end{aligned}
$$

Like N and LN, B and GM, i.e. Theorem 8.3