Brownian Motion
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BFN/bfn

Simple random walk

$$\begin{aligned}
X_n &= X_{n-1} + 2I - 1, \quad ,\mathbb{E}(I) = p \\
\mathbb{P}\{X_n = j | X_0 = i\} &= p_{i,j}^{(n)} \\
p_{i,j}^{(n)} &= p_{i,j+1}^{(n-1)} q + p_{i,j-1}^{(n-1)} p \\
p_{i,j}^{(n)} - p_{i,j}^{(n-1)} &= \left(p_{i,j+1}^{(n-1)} - p_{i,j}^{(n-1)}\right) q + \left(p_{i,j-1}^{(n-1)} - p_{i,j}^{(n-1)}\right) p \\
&= q\left(p_{i,j+1}^{(n-1)} - p_{i,j}^{(n-1)}\right) - p\left(p_{i,j}^{(n-1)} - p_{i,j-1}^{(n-1)}\right)
\end{aligned}$$

For $p = q = \frac{1}{2}$

$$\begin{aligned}
p_{i,j}^{(n)} - p_{i,j}^{(n-1)} &= \frac{1}{2}\left[\left(p_{i,j+1}^{(n-1)} - p_{i,j}^{(n-1)}\right) - \left(p_{i,j}^{(n-1)} - p_{i,j-1}^{(n-1)}\right)\right] \\
\Delta_n p_{i,j}^{(n)} &= \Delta_j^2 p_{i,j}^{(n-1)}
\end{aligned}$$

with proper limiting choices where now $X(t)$ continuous

$$p(y,t|x) = \mathbb{P}\{X(t) = y | X(0) = x\}$$

$$\frac{\partial}{\partial t} p(y,t|x) = \frac{1}{2}\frac{\partial^2}{\partial t^2} p(y,t|x)$$

Characteristic function

$$\mathbb{E}\left(e^{i\theta X}\right) = \int_{-\infty}^{\infty} e^{i\theta x} f(x) \mathrm{d}x$$

$$\begin{aligned}
X &\sim \mathrm{N}(0,\sigma^2) \\
\mathbb{E}\left(e^{i\theta X}\right) &= \int_{-\infty}^{\infty} e^{i\theta x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dtx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-i\theta\sigma^2)^2+\theta^2\sigma^4}{2\sigma^2}} = e^{-\frac{\theta^2\sigma^2}{2}}
\end{aligned}$$

Define

$$M(\theta,t) = \int_{-\infty}^{\infty} e^{i\theta y} p(y,t|x) \mathrm{d}y = \mathbb{E}\left(e^{i\theta X(t)}\right)$$

the characteristic function related to $p(y,t|x)$ seen as a function of y. We can use the chararcteristic function by solving the PDE by transformation to find

$$\begin{aligned} M'(\theta,t) &= -\frac{\theta^2}{2}M(\theta,t) \\ M(\theta,t) &= e^{-\frac{\theta^2 t}{2}} \\ p(y,t|x) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{(y-x)^2}{2t}} \end{aligned}$$

We can prove that

$$p(y,t|x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{(y-x)^2}{2t}}$$

is a solution to result by verification by taking the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{(y-x)^2}{2t}} &= -\frac{1}{\sqrt{2\pi}}\frac{1}{2t\sqrt{t}}e^{-\frac{(y-x)^2}{2t}} + \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{(y-x)^2}{2t}}\frac{(y-x)^2}{2t^2} \\ \frac{\partial}{\partial y}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-x)^2}{2t}} &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{(y-x)^2}{2t}}(-1)\frac{2(y-x)}{2t} \\ \frac{\partial^2}{\partial y^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-x)^2}{2t}} &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{(y-x)^2}{2t}}\frac{4(y-x)^2}{4t^2} - \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-\frac{(y-x)^2}{2t}}\frac{2}{2t} \end{aligned}$$

Independent increment discrete process

Poisson process with rate λ $\{N(t); t \geq 0\}$

1. $N(s+t) - N(s) \sim \text{Pois}(0, \lambda t)$
2. $N(t_4) - N(t_3)$ independent of $N(t_2) - N(t_1)$ for $t_1 < t_2 < t_3 < t_4$, generalising to independence among n non-overlapping intervals
3. $N(0) = 0$ (if nothing else stated)

We have $\mathbb{V}\text{ar}(N(t)) = \lambda t$.

Independent increment contionuous process

Brownian motion with diffusion coefficient σ^2 is the stochastic process $\{B(t); t \geq 0\}$

1. $B(s+t) - B(s) \sim \mathrm{N}(0, \sigma^2 t)$
2. $B(t_4) - B(t_3)$ independent of $B(t_2) - B(t_1)$ for $t_1 < t_2 < t_3 < t_4$, generalising to independence among n non-overlapping intervals
3. $B(t)$ is continuous in t, $B(0) = 0$ (if nothing else stated)

$$\mathbb{C}\mathrm{ov}(B(s), B(t)) = \mathbb{E}(B(s)B(t)) = \mathbb{E}(B(s)(B(s) + B(t) - B(s))) = \mathbb{E}(B(s)B(s)) + \mathbb{E}(B(s)(B(t) - B(s))) = s$$

or if $t < s$ we get t, $\mathbb{C}\mathrm{ov}(B(s), B(t)) = \min(s, t)$

Invariance Principle

$$S_n = S_{n-1} + Z_n = \sum_{i=1}^{n} Z_i, \quad Z_i \text{ i.i.d}$$

Assume $\mathbb{E}(Z_i) = 0, \mathbb{E}\left(Z_i^2\right) = 1$

$$\mathbb{P}\left\{\frac{S_n}{\sqrt{n}} \le x\right\} \cong \Phi(x)$$

Now define

$$B_n(t) = \frac{S_{[nt]}}{\sqrt{n}}$$

a piecewise constant function of t

$$\begin{aligned}
S_{[nt]} &= S_k, \quad \text{for } k \le [nt] < k+1 \\
S_{[nt]} &= S_k, \quad \text{for } k \le nt < k+1 \\
S_{[nt]} &= S_k, \quad \text{for } \frac{k}{n} \le t < \frac{k}{n} + \frac{1}{n} \\
B_n(t) &= \frac{\sqrt{[nt]}}{\sqrt{n}} \frac{S_k}{\sqrt{k}} \\
\mathbb{V}\mathrm{ar}(B_n(t)) &\cong t
\end{aligned}$$

For symmetric random walk $\mathbb{P}\{S_n \text{ reaches } b \text{ before } b\} = \frac{a}{a+b}$, so we argue this must be the same for Brownian motion due to the invariance principle. But then this should generalise to arbitrary distributions of Z_i.

Gaussian processes

$$\begin{aligned} X &\sim \mathrm{N}(\mu, \sigma^2) \\ f_X(x) &= \frac{1}{\sqrt{2pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \end{aligned}$$

Random vector

$$\begin{aligned} \boldsymbol{X} &= (X_1, X_2, \ldots, X_n) \\ Y &= \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n = \boldsymbol{\alpha X} \end{aligned}$$

Suppose $Y \sim \mathrm{N}(\boldsymbol{\alpha}\mu, \boldsymbol{\alpha\Sigma\alpha}')$ for some $\boldsymbol{\mu}$ and positive semidefinite $\boldsymbol{\Sigma}$ and all $\boldsymbol{\alpha} \in \mathbb{R}^n$ then

$$\begin{aligned} \boldsymbol{X} &\overset{\text{def}}{\sim} \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \\ f_{\boldsymbol{X}}(\boldsymbol{x}) &= \frac{(2\pi)^{-\frac{n}{2}}}{\det(\boldsymbol{\Sigma})} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}, \quad \text{assuming } \boldsymbol{\Sigma} \text{ positive definite} \end{aligned}$$

If we have stochastic process $\{X(t), t \in T\}$ such that $(X(t_1), X(t_2), \ldots, X(t_n))$ is Gaussian for all n then $X(t)$ is a Gaussian process with $\mu(t) = X(t), \Gamma(s,t) = \mathbb{C}\mathrm{ov}(X(s), X(t))$.

A matrix $\boldsymbol{\Gamma}$ is positive semidefinite if $\boldsymbol{a}'\boldsymbol{\Gamma a} \geq 0\ \forall \boldsymbol{a}$. Correspondingly the function $\Gamma(s,t)$ is positive semidefinte if $\sum_{i=1}^n \sum_{j=1}^n a_i \Gamma(t_i, t_j) a_j \geq 0\ \forall n \geq 0$ and arbitrary real $a_i s$. Brownian motion is a (the prime example of) Gaussian process.

Correspondingly. For functions $\mu(t)$ and $\Gamma(s,t)$ positive semidefinite then there exists a Gaussian process. We will see examples of the use of this next week.

$\{Z_i(t); t \geq 0\}, i = 1, \ldots, N$ independent random functions/stochastic processes

$$\begin{aligned} \mathbb{E}(Z_i(t)) &= \mu(t) \\ \mathbb{C}\mathrm{ov}(Z_i(s), Z_i(t)) &= \Gamma(s,t) \\ X_N(t) &= \frac{\sum_{i=1}^N (Z_i(t) - \mu(t))}{\sqrt{N}} \end{aligned}$$

in the limit $X_N(t)$ is a gaussion process with $(\mu(t), \Gamma(s,t))$.

Maximum and first hitting time - reflection principle

$$\begin{aligned}
M(t) &= \max_{0\le u\le t} B(u) \\
\mathbb{P}\{M(t) > x\} &= \mathbb{P}\left\{\max_{0\le u\le t} B(u) > x\right\}, \quad \text{survival function} \\
&= \mathbb{P}\left\{\max_{0\le u\le t} B(u) > x, B(t) > x\right\} + \mathbb{P}\left\{\max_{0\le u\le t} B(u) > x, B(t) > x\right\} \\
&\quad + \mathbb{P}\left\{\max_{0\le u\le t} B(u) > x, B(t) = x\right\} \\
&= \mathbb{P}\left\{B(t) > x\right\} + \mathbb{P}\left\{\max_{0\le u\le t} B(u) > x, B(t) > x\right\} \\
&= \left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right) + \mathbb{P}\left\{\max_{0\le u\le t} B(u) > x, B(t) > x\right\}
\end{aligned}$$

We now argue that

$$\mathbb{P}\left\{\max_{0\le u\le t} B(u) > x, B(t) > x\right\} = \mathbb{P}\left\{\max_{0\le u\le t} B(u) > x, B(t) > x\right\}$$

$$\begin{aligned}
\tau_x &= \min\{u \in [0;t] : B(u) = x\} \\
B^*(u) &= \begin{cases} B(u) & u \le \tau_x \\ x - (B(u) - x) & \tau_x < u \end{cases}
\end{aligned}$$

Somewhat repetitive (don't need to do on blackboard)

$$\begin{aligned}
\{\omega \in \Omega : B(t) > x\} &\subset \{\omega \in \Omega : M(t) > x\} \\
\mathbb{P}\{M(t) > x \wedge B(t) < x\} &= \mathbb{P}\{M(t) > x \wedge B(t) \ge x\} \\
&= \mathbb{P}\{B(t) > x\} = 1 - \Phi\left(\frac{x}{\sqrt{t}}\right) \\
\mathbb{P}\{M(t) > x\} &= 2\left[1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right] \\
\mathbb{P}\{\tau_x \le t\} &= \mathbb{P}\{M(t) > x\} = 2\left[1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right] \\
&= 2\int_x^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{t}} e^{-\frac{u^2}{2t}} \,\mathrm{d}u = \sqrt{\frac{2}{\pi}} \int_{\frac{s}{\sqrt{t}}}^\infty e^{-\frac{u^2}{2}} \,\mathrm{d}u \\
f_{\tau_x}(t) &= \frac{1}{\sqrt{2\pi}} \frac{x}{t\sqrt{t}} e^{-\frac{x^2}{2t}}
\end{aligned}$$

Probability of hitting 0

$$
\begin{aligned}
v(t,t+s) &= \mathbb{P}\{\exists u \in [s,t] : B(u)=0\} = \frac{2}{\pi}\text{Arctan}\left(\sqrt{\frac{s}{t}}\right) = \frac{2}{\pi}\text{Arccos}\left(\sqrt{\frac{t}{t+s}}\right) \\
v(t,t+s) &= \int_{-\infty}^{\infty} \mathbb{P}\{\tau_x \leq s\} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{x^2}{2t}}\,\mathrm{d}x = 2\int_0^{\infty} \mathbb{P}\{\tau_x \leq s\} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{x^2}{2t}}\,\mathrm{d}x
\end{aligned}
$$