Distributions of phase type

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Phase Type distributions

Today:

- Phase type distribuions
  - Definition
  - Basic properties
  - Closure properties
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Next week

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Two weeks from now

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Definition of Phase type distributions

Finite state space Markov chain
One absorbing state
All other states transient
Transition matrix/generator

$$\begin{bmatrix} S & \boldsymbol{s} \\ \mathbf{0} & \{0,1\} \end{bmatrix}$$

Initial distribution amongs states $(\boldsymbol{\alpha}, \alpha_p)$.
Representation $(\boldsymbol{\alpha}, S)$ (not unique)

Probability functions

	Discrete case	continuous case
Density	$\alpha T^{x-1} \boldsymbol{t}$	$\alpha e^{Tx} \boldsymbol{t}$
Survival function	$\alpha T^{x} \boldsymbol{e}$	$\alpha e^{Tx} \boldsymbol{e}$
Distribution function	$1 - \alpha T^{x} \boldsymbol{e}$	$1 - \alpha e^{Tx} \boldsymbol{e}$

Addition of PH distributed random variables

$X \sim \text{PH}(\boldsymbol{\alpha}, S),\ Y \sim \text{PH}(\boldsymbol{\beta}, T)$
$Z = X + Y$

$$\begin{bmatrix} S & \boldsymbol{s}\boldsymbol{\beta} \\ 0 & T \end{bmatrix} = L$$

thus $Z \sim \text{PH}(\gamma, L)$ with $\gamma = (\boldsymbol{\alpha}, \alpha_{p_1+1}\boldsymbol{\beta})$

Minimum of PH Distributed random variables

$X \sim \text{PH}(\alpha, S)$, $Y \sim \text{PH}(\beta, T)$
$Z = \min(X, Y)$
State space of dimension $p_1 \times p_2$
Combination of two independent chains

Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1k}B \\ a_{21}B & a_{22}B & \dots & a_{2k}B \\ \vdots & \dots & \vdots & \vdots \\ a_{\ell 1}B & a_{\ell 2}B & \dots & a_{\ell k}B \end{bmatrix} \tag{1}$$

Consider the matrices A, B, and I given by:

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 3 & 5 & 11 \end{bmatrix}, \quad B = \begin{bmatrix} 13 & 4 \\ 0 & 17 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad .$$

Then $A \otimes B$, $A \otimes I$ and $I \otimes B$ is given by

$$A \otimes B = \begin{bmatrix} 2\cdot 13 & 2\cdot 4 & 7\cdot 13 & 7\cdot 4 & 1\cdot 13 & 1\cdot 4 \\ 2\cdot 0 & 2\cdot 17 & 7\cdot 0 & 7\cdot 17 & 1\cdot 0 & 1\cdot 17 \\ 3\cdot 13 & 3\cdot 4 & 5\cdot 13 & 5\cdot 4 & 11\cdot 13 & 11\cdot 4 \\ 3\cdot 0 & 3\cdot 17 & 5\cdot 0 & 5\cdot 17 & 11\cdot 0 & 11\cdot 17 \end{bmatrix}$$

$$A \otimes I = \begin{bmatrix} 2 & 0 & 0 & 7 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 7 & 0 & 0 & 1 \\ 3 & 0 & 0 & 5 & 0 & 0 & 11 & 0 & 0 \\ 0 & 3 & 0 & 0 & 5 & 0 & 0 & 11 & 0 \\ 0 & 0 & 3 & 0 & 0 & 5 & 0 & 0 & 11 \end{bmatrix}$$

$$I \otimes B = \begin{bmatrix} 13 & 4 & 0 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13 & 4 & 0 & 0 \\ 0 & 0 & 0 & 17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13 & 4 \\ 0 & 0 & 0 & 0 & 0 & 17 \end{bmatrix}.$$

$Z \sim \mathrm{PH}(\gamma, L)$

$$L = \begin{cases} S \otimes T & \text{discrete case} \\ S \otimes I + I \otimes T & \text{continuous case} \end{cases}$$

$\boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$

Maximum of PH Distributed random variables

In the continuous case we get

$$L = \begin{bmatrix} S \otimes I + I \otimes T & I \otimes \boldsymbol{t} & \boldsymbol{s} \otimes I \\ 0 & S & 0 \\ 0 & 0 & T \end{bmatrix}$$

with obvious modifications for the transition probability matrix in the discrete case.
The initial probability distribution is
$\gamma = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}, \boldsymbol{\alpha} \cdot \beta_{p_2+1}, \alpha_{p_1+1} \cdot \boldsymbol{\beta})$.

Random sum of continuous PH distributed random variables

$X \sim \text{PH}(\boldsymbol{\alpha}, S)$, $Y_i \sim \text{PH}(\boldsymbol{\beta}, T)$ where X is a discrete random variable and Y_i are continuous random variables.

$$Z = \sum_{i=0}^{X} Y_i$$

$Z \sim \text{PH}(\gamma, L)$ with

$$L = T \otimes I + S \otimes \boldsymbol{t}\boldsymbol{\beta}$$

$$\gamma = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$$

Expectation

Discrete case:

$$\sum_{i=0}^{\infty} S^i = (I - T)^{-1} = U$$

Continuous case

$$\int_0^{\infty} e^{Sx} \mathrm{d}x = (-S)^{-1} = U$$

The (i, j)th element of U is the expected time spent in stat j before absorption, conditioned on initialization in state i.

$$\mathbb{E}(X) = \alpha U \boldsymbol{e}$$

The PH renewal process

A finite state irreducible Markov chain.

$$Q = S + (1-\alpha_p)^{-1}\boldsymbol{s}\boldsymbol{\alpha}$$

$$\pi Q = \begin{cases} \boldsymbol{e} & \text{discrete case} \\ \boldsymbol{0} & \text{continuous case} \end{cases}$$

Renewal density

$$h(x) = \boldsymbol{\alpha} e^{Qx}\boldsymbol{t}$$

Renewal function

$$H(x) =$$

Residual life time and age distribution

(π, S)

Distribution of spread

Moment generating functions

$$\mathbb{E}\left(e^{\theta X}\right) = \begin{cases} \alpha_{p+1} + \boldsymbol{\alpha}(I - e^{\theta} S)^{-1}\boldsymbol{t} & \text{discrete case} \\ \alpha_{p+1} + \boldsymbol{\alpha}(I - \theta(-S)^{-1})^{-1}\boldsymbol{e} & \text{continuous case} \end{cases}$$

For the discrete case it is more common to use the generating function

$$\mathbb{E}\left(z^{X}\right) = \alpha_{p+1} + \boldsymbol{\alpha}(I - zS)^{-1}\boldsymbol{t}$$

Moments

By differentation of the moment generating functions we get

$$\mathbb{E}\left(X^i\right) = \alpha i! U^i \boldsymbol{e} \qquad \text{continuous case}$$

$$\mathbb{E}\left(X(X-1)\ldots(X-i+1)\right) = \alpha i! U^i \boldsymbol{e} \qquad \text{discrete case}$$

Exercises