Distributions of phase type

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Phase Type distributions

Today:

- Phase type distribuions
  - Definition
  - Basic properties
  - Closure properties

Next week

- Phase type distributions
- Conditional Expectation, Martingales

Two weeks from now

- Brownian motion

Definition of Phase type distributions

Finite state space Markov chain
One absorbing state
All other states transient
Transition matrix/generator

$$\begin{bmatrix} \boldsymbol{S} & \boldsymbol{s} \\ \boldsymbol{0} & \{0,1\} \end{bmatrix}$$

Initial distribution amongs states $(\boldsymbol{\alpha}, \alpha_p)$.
Representation $(\boldsymbol{\alpha}, \boldsymbol{S})$ (not unique)

Probability functions

	Discrete case	continuous case
Density	$\boldsymbol{\alpha}\boldsymbol{S}^{x-1}\boldsymbol{s}$	$\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}$
Survival function	$\boldsymbol{\alpha}\boldsymbol{S}^{x}\boldsymbol{1}$	$\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{1}$
Distribution function	$1-\boldsymbol{\alpha}\boldsymbol{S}^{x}\boldsymbol{1}$	$1-\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{1}$

Addition of PH distributed random variables

$X \sim \text{PH}(\alpha, \boldsymbol{S})$, $Y \sim \text{PH}(\beta, \boldsymbol{T})$
$Z = X + Y$

$$\begin{bmatrix} \boldsymbol{S} & \boldsymbol{s}\beta \\ \boldsymbol{0} & \boldsymbol{T} \end{bmatrix} = \boldsymbol{L}$$

thus $Z \sim \text{PH}(\gamma, \boldsymbol{L})$ with $\gamma = (\alpha, \alpha_{p_1+1}\beta)$

Minimum of PH Distributed random variables

$X \sim \text{PH}(\alpha, \boldsymbol{S})$, $Y \sim \text{PH}(\beta, \boldsymbol{T})$
$Z = \min(X, Y)$
State space of dimension $p_1 \times p_2$
Combination of two independent chains

Kronecker product

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \dots & a_{1k}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \dots & a_{2k}\boldsymbol{B} \\ \vdots & \dots & \vdots & \vdots \\ a_{\ell 1}\boldsymbol{B} & a_{\ell 2}\boldsymbol{B} & \dots & a_{\ell k}\boldsymbol{B} \end{bmatrix} \quad (1)$$

Consider the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{I}$ given by:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 7 & 1 \\ 3 & 5 & 11 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 13 & 4 \\ 0 & 17 \end{bmatrix}, \quad \boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\boldsymbol{A} \otimes \boldsymbol{B}$, $\boldsymbol{A} \otimes \boldsymbol{I}$ and $\boldsymbol{I} \otimes \boldsymbol{B}$ is given by

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} 2\cdot 13 & 2\cdot 4 & 7\cdot 13 & 7\cdot 4 & 1\cdot 13 & 1\cdot 4 \\ 2\cdot 0 & 2\cdot 17 & 7\cdot 0 & 7\cdot 17 & 1\cdot 0 & 1\cdot 17 \\ 3\cdot 13 & 3\cdot 4 & 5\cdot 13 & 5\cdot 4 & 11\cdot 13 & 11\cdot 4 \\ 3\cdot 0 & 3\cdot 17 & 5\cdot 0 & 5\cdot 17 & 11\cdot 0 & 11\cdot 17 \end{bmatrix}$$

$$\boldsymbol{A} \otimes \boldsymbol{I} = \begin{bmatrix} 2 & 0 & 0 & 7 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 7 & 0 & 0 & 1 \\ 3 & 0 & 0 & 5 & 0 & 0 & 11 & 0 & 0 \\ 0 & 3 & 0 & 0 & 5 & 0 & 0 & 11 & 0 \\ 0 & 0 & 3 & 0 & 0 & 5 & 0 & 0 & 11 \end{bmatrix}$$

$$\boldsymbol{I} \otimes \boldsymbol{B} = \begin{bmatrix} 13 & 4 & 0 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13 & 4 & 0 & 0 \\ 0 & 0 & 0 & 17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13 & 4 \\ 0 & 0 & 0 & 0 & 0 & 17 \end{bmatrix}.$$

Minimum of PH Distributed Random Variables

$Z = \min(X, Y)$ $Z \sim \text{PH}(\gamma, \boldsymbol{L})$

$$\boldsymbol{L} = \begin{cases} \boldsymbol{S} \otimes \boldsymbol{T} & \text{discrete case} \\ \boldsymbol{S} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T} & \text{continuous case} \end{cases}$$

$\gamma = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$

Maximum of PH Distributed random variables

$Z = \max(X, Y)$
In the continuous case we get

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{S} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{T} & \boldsymbol{I} \otimes \boldsymbol{t} & \boldsymbol{s} \otimes \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T} \end{bmatrix}$$

with obvious modifications for the transition probability matrix in the discrete case.
The initial probability distribution is
$\gamma = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}, \boldsymbol{\alpha} \cdot \beta_{p_2+1}, \alpha_{p_1+1} \cdot \boldsymbol{\beta})$.

Random sum of continuous PH distributed random variables

$X \sim \text{PH}(\alpha, \boldsymbol{S})$, $Y_i \sim \text{PH}(\beta, \boldsymbol{T})$ where X is a discrete random variable and Y_i are continuous random variables.

$$Z = \sum_{i=0}^{X} Y_i$$

$Z \sim \text{PH}(\gamma, \boldsymbol{L})$ with

$$\boldsymbol{L} = \boldsymbol{T} \otimes \boldsymbol{I} + \boldsymbol{S} \otimes \boldsymbol{t}\beta$$

$\gamma = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$

Expectation

Discrete case:

$$\sum_{i=0}^{\infty} \boldsymbol{S}^i = (\boldsymbol{I} - \boldsymbol{S})^{-1} = \boldsymbol{U}$$

Continuous case

$$\int_0^{\infty} e^{\boldsymbol{S}x} dx = (-\boldsymbol{S})^{-1} = \boldsymbol{U}$$

The (i, j)th element of $\boldsymbol{U}$ is the expected time spent in stat j before absorption, conditioned on initialization in state i.

$$\mathbb{E}(X) = \alpha \boldsymbol{U} \boldsymbol{1}$$

The PH renewal process

A finite state irreducible Markov chain.

$$\boldsymbol{Q} = \boldsymbol{S} + (1-\alpha_{p+1})^{-1}\boldsymbol{s}\boldsymbol{\alpha}$$

Renewal density (assuming $\alpha_{p+1} = 0$)

$$h(x) = \boldsymbol{\alpha} e^{\boldsymbol{Q}x}\boldsymbol{s}$$

Renewal function (assuming $\alpha_{p+1} = 0$)

$$\begin{aligned} H(x) &= \frac{x\boldsymbol{\beta}\boldsymbol{\nu}}{\boldsymbol{\alpha}\boldsymbol{S}^{-2}\boldsymbol{s}} - \boldsymbol{\beta}\left(\boldsymbol{I} - e^{(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha})x}\right)(\boldsymbol{S}+\boldsymbol{s}\boldsymbol{\alpha}-\boldsymbol{\nu}\boldsymbol{\pi})^{-1}\boldsymbol{s} \\ &= \frac{x\boldsymbol{\beta}\boldsymbol{\nu}}{\boldsymbol{\alpha}\boldsymbol{S}^{-2}\boldsymbol{s}} - \boldsymbol{\beta}\left(\boldsymbol{I} - e^{\boldsymbol{Q}x}\right)(\boldsymbol{Q}-\boldsymbol{\nu}\boldsymbol{\pi})^{-1}\boldsymbol{s} \end{aligned}$$

Residual life time and age distribution

$$\boldsymbol{\pi}\boldsymbol{Q} = \begin{cases} \boldsymbol{1} & \text{discrete case} \\ \boldsymbol{0} & \text{continuous case} \end{cases}$$

$$\boldsymbol{\pi} \stackrel{\sim}{=} \boldsymbol{\alpha}\boldsymbol{U}$$

$$\boldsymbol{\pi} = \boldsymbol{\alpha}\boldsymbol{U}(\boldsymbol{\alpha}\boldsymbol{U}\boldsymbol{1})^{-1}$$

Residual life time and age in a stationary PH renewal process is PH($\boldsymbol{\pi}, \boldsymbol{S}$) distributed

Moment generating functions

$$\mathbb{E}\left(e^{\theta X}\right) = \begin{cases} \alpha_{p+1} + \boldsymbol{\alpha}(\boldsymbol{I} - e^{\theta}\boldsymbol{S})^{-1}\boldsymbol{s} & \text{discrete case} \\ \alpha_{p+1} + \boldsymbol{\alpha}(\boldsymbol{I} - \theta(-\boldsymbol{S})^{-1})^{-1}\boldsymbol{1} & \text{continuous case} \end{cases}$$

For the discrete case it is more common to use the generating function

$$\mathbb{E}\left(z^X\right) = \alpha_{p+1} + \boldsymbol{\alpha}(\boldsymbol{I} - z\boldsymbol{S})^{-1}\boldsymbol{s}$$

Moments

By differentation of the moment generating functions we get

$$\mathbb{E}\left(X^i\right) = \boldsymbol{\alpha} i! \boldsymbol{U}^i\boldsymbol{1} \qquad \text{continuous case}$$

$$\mathbb{E}\left(X(X-1)\dots(X-i+1)\right) = \boldsymbol{\alpha} i! \boldsymbol{U}^i\boldsymbol{S}^{i-1}\boldsymbol{1} \qquad \text{discrete case}$$