# Renewal Processes

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## Renewal Processes

*Today:*

- Renewal phenomena

*Next week*

- Markov Decision Processes

*Three weeks from now*

- Brownian Motion



## A Poisson proces

Sequence $X_i$, where $X_i \sim \exp(\lambda_i)$ or $X_i \sim \text{PH}((1), [-\lambda])$.
$W_n = \sum_{i=1}^{n} X_i$

$$N(t) = \max_{n \geq 0} \{W_n \leq t\} = \max_{n \geq 0} \left\{ \sum_{i=n} X_i \leq t \right\}$$

Let us consider a sequence, where $X_i \sim \text{PH}(\boldsymbol{\alpha}, \boldsymbol{S})$.



## Underlying Markov Jump Process

Let $J_i(t)$ be the (absorbing) Markov Jump Process related to $X_i$.
Define $J(t) = J_i(t - \sum_{j=1}^{N(t)} \tau_i)$

$$\mathbb{P}\left(J(t+\Delta) = j | J(t) = i\right) = S_{ij} + s_i \alpha_j$$

Such that

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}$$

is the generator for the continued phase proces - $J(t)$ Note the similarity with the expression for a sum of two phase-type distributed variables

## Distribution of $N(t)$

For $X_i \sim \exp(\lambda)$ $\mathbb{P}(N(t)=n) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$ What if $X_i \sim \mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{S})$ Generator up to finite $n$

$$\boldsymbol{A}_n = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{s\alpha} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S} & \boldsymbol{s\alpha} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{S} & \boldsymbol{s\alpha} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{S} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots\ddots\ddots\vdots & \vdots & \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{S} & \boldsymbol{s\alpha} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{S} \end{pmatrix}$$

A quasi-birth process - to calculate $\mathbb{P}(N(t)=n)$ we would need the matrix-exponential of an $(n+1)p$ dimensional square matrix $\mathbb{P}(N(t)>n) = \mathbb{P}(W_n \leq t)$, $W_n$ is an "Erlang-type" PH variable



## Renewal function

$X_i \sim \exp(\lambda)$ then $M(t) = \mathbb{E}(N(t)) = \lambda t$ Probability of having a point in $[t; t+\mathrm{d}t($ $\mathbb{P}(\exists n : W_n \in [t; t+\mathrm{d}t()$ The probability of a point is the probability that $J(t)$ has an instantaneous visit to an absorbing state ($J(t)$ shifts from some $J_n()$ to $J_{n+1}$

$$\begin{aligned} \mathbb{P}(N(t+\mathrm{d}t) - N(t) = 1 | J(t) = i) &= s_i \mathrm{d}t + o(\mathrm{d}t) \\ \mathbb{P}(J(t) = i) &= \boldsymbol{\alpha} e^{\boldsymbol{A}t} \boldsymbol{1}_i \\ \mathbb{P}(N(t+\mathrm{d}t) - N(t) = 1) &= \boldsymbol{\alpha} e^{\boldsymbol{A}t} \boldsymbol{s} \mathrm{d}t + o(\mathrm{d}t) \\ M(t) = \mathbb{E}(N(t) &= \int_0^t \boldsymbol{\alpha} e^{\boldsymbol{A}u} \boldsymbol{s} \mathrm{d}u = \boldsymbol{\alpha} \int_0^t e^{\boldsymbol{A}u} \mathrm{d}u \boldsymbol{s} \end{aligned}$$

The generator $\boldsymbol{A}$ is singular ($\boldsymbol{A1} = \boldsymbol{0}$, $\boldsymbol{\pi A} = \boldsymbol{0}$)



## Calculation of $\int_0^t e^{\boldsymbol{A}u} \mathrm{d}u$

First we note that $\boldsymbol{1\pi} - \boldsymbol{A}$ is non-singular

$$\begin{aligned} \int_0^t e^{\boldsymbol{A}u} \mathrm{d}u &= \int_0^t (\boldsymbol{1\pi} - \boldsymbol{A})^{-1} (\boldsymbol{1\pi} - \boldsymbol{A}) e^{\boldsymbol{A}u} \mathrm{d}u \\ &= (\boldsymbol{1\pi} - \boldsymbol{A})^{-1} \left[ \int_0^t \boldsymbol{1\pi} e^{\boldsymbol{A}u} \mathrm{d}u - \int_0^t \boldsymbol{A} e^{\boldsymbol{A}u} \mathrm{d}u \right] \end{aligned}$$

$$\begin{aligned} \int_0^t \boldsymbol{1\pi} e^{\boldsymbol{A}u} \mathrm{d}u &= \int_0^t \boldsymbol{1\pi} \sum_{n=0}^{\infty} \frac{(\boldsymbol{A}u)^n}{n!} \mathrm{d}u = \int_0^t \boldsymbol{1} \sum_{n=0}^{\infty} \boldsymbol{\pi} \frac{(\boldsymbol{A}u)^n}{n!} \mathrm{d}u \\ &= \int_0^t \boldsymbol{1\pi} \mathrm{d}u = t\boldsymbol{1\pi} \end{aligned}$$

$$\int_0^t \boldsymbol{A} e^{\boldsymbol{A}u} \mathrm{d}u = e^{\boldsymbol{A}t} - \boldsymbol{I}$$



## Back to $M(t)$

We have $(\boldsymbol{1\pi} - \boldsymbol{A}) \boldsymbol{1} = 1$ and $\boldsymbol{\pi} (\boldsymbol{1\pi} - \boldsymbol{A}) = \boldsymbol{\pi}$, so

$$\begin{aligned} M(t) &= \boldsymbol{\alpha} (\boldsymbol{1\pi} - \boldsymbol{A})^{-1} \left[ t\boldsymbol{1\pi} - \left( e^{\boldsymbol{A}t} - \boldsymbol{I} \right) \right] \boldsymbol{s} \\ &= \boldsymbol{\pi s} t + \boldsymbol{\alpha} (\boldsymbol{1\pi} - \boldsymbol{A})^{-1} \boldsymbol{s} - \boldsymbol{\alpha} (\boldsymbol{1\pi} - \boldsymbol{A})^{-1} e^{\boldsymbol{A}t} \boldsymbol{s} \end{aligned}$$

We have $\boldsymbol{\alpha} (\boldsymbol{1\pi} - \boldsymbol{A})^{-1} e^{\boldsymbol{A}t} \boldsymbol{s} \to \boldsymbol{\alpha} (\boldsymbol{1\pi} - \boldsymbol{A})^{-1} \boldsymbol{1\pi s} = \boldsymbol{\pi s}$

## Renewal Processes

$$
\begin{aligned}
F(x) &= \mathbb{P}\{X \le x\} \\
W_n &= X_1 + \cdots + X_n \\
N(t) &= \max\{n : W_n \le t\} \\
\mathbb{E}(N(t)) &= M(t) \;\; \text{Renewal function}
\end{aligned}
$$



## Age, Residual Life, and Total Life (Spread)

$$
\begin{aligned}
\gamma_t &= W_{N(t)+1} - t(\text{ excess or residual life time}) \\
\delta_t &= t - W_{N(t)}(\text{ current life or age}) \\
\beta_t &= \delta_t + \gamma_t(\text{ total life or spread}
\end{aligned}
$$



## Topics in renewal theory

Elementary renewal theorem $\frac{M(t)}{t} \to \frac{1}{\mu}$

$\mathbb{E}\left(W_{N(t)+1}\right) = \mu(1 + M(t))$

$M(t) = \sum_{n=1}^{\infty} F_n(t), \qquad F_n(t) = \int_0^t F_{n-1}(t-x)\mathrm{d}F(t)$

Renewal equation $A(t) = a(t) + \int_0^t A(t-u)\mathrm{d}F(u)$

Solution to renewal equation
$A(t) = \int_0^t a(t-u)\mathrm{d}M(u) \to \frac{1}{\mu}\int_0^{\infty} a(t)\mathrm{d}t$

Limiting distribution of residual life time
$\lim_{t\to\infty} \mathbb{P}\{\gamma_t \le x\} = \frac{1}{\mu}\int_0^x (1 - F(u))\mathrm{d}u$

Limiting distribution of joint distribution of age and residual life time $\lim_{t\to\infty} \mathbb{P}\{\gamma_t \ge x, \delta_t \ge y\} = \frac{1}{\mu}\int_{x+y}^{\infty}(1 - F(z))\mathrm{d}z$

Limitng distribution of total life time (spread)
$\lim_{t\to\infty} \mathbb{P}\{\beta_t \le x\} = \frac{\int_0^x t\mathrm{d}F(t)}{\mu}$



## Continuous Renewal Theory (7.6)

$$
\begin{aligned}
\mathbb{P}\{W_n \le x\} &= F_n(x) \;\; \text{with} \\
F_n(x) &= \int_0^x F_{n-1}(x-y)\mathrm{d}F(y)
\end{aligned}
$$

The expression for $F_n(x)$ is generally quite complicated
Renewal equation

$$v(x) = a(x) + \int_0^x v(x-u)\mathrm{d}F(u)$$

$$v(x) = \int_0^x a(x-u)\mathrm{d}M(u)$$

## Expression for $M(t)$

$$\begin{aligned}
\mathbb{P}\{N(t) \geq k\} &= \mathbb{P}\{W_k \leq t\} = F_k(t) \\
\mathbb{P}\{N(t) = k\} &= \mathbb{P}\{W_k \leq t, W_{k+1} > t\} = F_k(t) - F_{k+1}(t) \\
M(t) &= \mathbb{E}(N(t)) = \sum_{k=1}^{\infty} k\mathbb{P}\{N(t) = k\} \\
&= \sum_{k=0}^{\infty} \mathbb{P}\{N(t) > k\} = \sum_{k=1}^{\infty} \mathbb{P}\{N(t) \geq k\} \\
&= \sum_{k=1}^{\infty} F_k(t)
\end{aligned}$$



## $\mathbb{E}[W_{N(t)+1}]$

$$\begin{aligned}
\mathbb{E}[W_{N(t)+1}] &= \mathbb{E}\left[\sum_{j=1}^{N(t)+1} X_j\right] = \mathbb{E}[X_1] + \mathbb{E}\left[\sum_{j=1}^{N(t)+1} X_j\right] \\
&= \mu + \sum_{j=2}^{\infty} \mathbb{E}\left[X_j \mathbf{1}(X_1 + \cdots + X_{j-1} \leq t)\right]
\end{aligned}$$

$X_j$ and $\mathbf{1}(X_1 + \cdots + X_{j-1} \leq t)$ are independent

$$= \mu + \sum_{j=2}^{\infty} \mathbb{E}\left[X_j\right] \mathbb{E}\left[\mathbf{1}(X_1 + \cdots + X_{j-1} \leq t)\right] = \mu + \mu \sum_{j=2}^{\infty} F_{j-1}(t)$$

$$\mathbb{E}[W_{N(t)+1}] = \exp[X_1]\mathbb{E}[N(t) + 1] = \mu(M(t) + 1)$$



## Poisson Process as a Renewal Process

$$\begin{aligned}
F_n(x) &= \sum_{i=n}^{\infty} \frac{(\lambda x)^n}{n!} e^{-\lambda x} = 1 - \sum_{i=0}^{n} \frac{(\lambda x)^n}{n!} e^{-\lambda x} \\
M(t) &= \lambda t
\end{aligned}$$

Excess life, Current life, mean total life



## The Elementary Renewal Theorem

$$\lim_{t\to\infty} \frac{M(t)}{t} = \lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} = \frac{1}{\mu}$$

The constant in the linear asymptote

$$\lim_{t\to\infty} \left[M(t) - \frac{\mu}{t}\right] = \frac{\sigma^2 - \mu^2}{2\mu^2}$$

Example with gamma distribution Page 367

## Asymptotic Distribution of $N(t)$

When $\mathbb{E}[X_k] = \mu$ and $\mathbb{V}\mathrm{ar}[X_k] = \sigma^2$ both finite

$$\lim_{t\to\infty} \mathbb{P}\left\{ \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \mathrm{d}y$$



## Limiting Distribution of Age and Excess Life

$$\lim_{t\to\infty} \mathbb{P}\{\gamma_t \leq x\} = \frac{1}{\mu} \int_0^x (1 - F(y)) \mathrm{d}y = H(x)$$

$$\mathbb{P}\{\gamma_t > x, \delta_t > y\} = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(z)) \mathrm{d}z$$



## Size biased distributions

$$f_i(t) = \frac{t^i f(t)}{\mathbb{E}\left(X^i\right)}$$

The limiting distribution og $\beta(t)$



## Delayed Renewal Process

Distribution of $X_1$ different

## Stationary Renewal Process

Delayed renewal distribution where
$\mathbb{P}\{X_1 \leq x\} = G_s(x) = \mu^{-1}\int_0^x (1-F(y))\mathrm{d}y$

$$\begin{aligned} M_S(t) &= \frac{t}{\mu} \\ Prob\{\gamma_t \leq x\} &= G_s(x) \end{aligned}$$



## Additional Reading

S. Karlin, H. M. Taylor: “A First Course in Stochastic Processes” Chapter 5 pp.167-228
William Feller: “An introduction to probability theory and its applications. Vol. II.” Chapter XI pp. 346-371
Ronald W. Wolff: “Stochastic Modeling and the Theory of Queues” Chapter 2 52-130
D. R. Cox: “Renewal Processes”
Søren Asmussen: “Applied Probability and Queues” Chapter V pp.138-168
Darryl Daley and Vere-Jones: “An Introduction to the Theory of Point Processes” Chapter 4 pp. 66-110



## Discrete Renewal Theory (7.6)

$\mathbb{P}\{X = k\} = p_k$

$$\begin{aligned} M(n) &= \sum_{k=0}^{n} p_k[1 + M(n-k)] \\ &= F(n) + \sum_{k=0}^{n} p_k M(n-k) \end{aligned}$$

A renewal equation. In general

$$v_n = b_n + \sum_{k=0}^{n} p_k v_{n-k}$$

The solution is unique, which we can see by solving recursively

$$\begin{aligned} v_0 &= \frac{b_0}{1-p_0} \\ v_1 &= \frac{b_1 + p_1 v_0}{1-p_0} \end{aligned}$$



## Discrete Renewal Theory (7.6) (cont.)

Let $u_n$ be the renewal density ($p_0 = 0$) i.e. the probability of having an event at time $n$

$$u_n = \delta_n + \sum_{k=0}^{n} p_k u_{n-k}$$

**Lemma**
7.1 Page 381 If $\{v_n\}$ satisfies $v_n = b_n + \sum_{k=0}^{n} p_k v_{n-k}$ and $u_n$ satisfies $u_n = \delta_n + \sum_{k=0}^{n} p_k u_{n-k}$ then

$$v_n = \sum_{k=0}^{n} b_{n-k} u_k$$

□

# The Discrete Renewal Theorem

**Theorem**

*7.1 Page 383* Suppose that $0 < p_1 < 1$ and that $\{u_n\}$ and $\{v_n\}$ are the solutions to the renewal equations $v_n = b_n + \sum_{k=0}^n p_k v_{n-k}$ and $u_n = \delta_n + \sum_{k=0}^n p_k u_{n-k}$, respectively. Then

**(a)** $\lim_{n\to\infty} u_n = \frac{1}{\sum_{k=0}^\infty kp_k}$

**(b)** if $\sum_{k=0}^\infty |b_k| < \infty$ then $\lim_{n\to\infty} v_n = \frac{\sum_{k=0}^\infty b_k}{\sum_{k=0}^\infty kp_k}$

□